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Apostol Vol 1, 2

11.6 Power series.

For $a, a_n \in \mathbb{C}$, consider the series

$\sum_{n=0}^{\infty} a_n (z-a)^n$. $z \in \mathbb{C}$. ~~for~~ converges for some z, not other z.

For simplicity, $a=0$.

~~Thm 11.6. Assume that $\sum_n a_n z_1^n$ is convergent for some $z = z_1$.~~

(1) Then for any z, $|z| < |z_1|$ $\sum a_n z^n$ is ~~con~~ absolutely converge

(2) for any $R < |z_1|$ the series $\sum a_n z^n$ is uniformly convergent for $|z| < R$.

proof) If $\sum_n a_n z_1^n$ is convergent, in particular $|a_n z_1^n| < 1$ for large n.

Let us take $R < |z_1|$, and $z \in \mathbb{C}$, s.t. $|z| < R$.

$|a_n z^n| = \left|a_n z_1^n \cdot \frac{z^n}{z_1^n}\right| \leq 1 \cdot \frac{|z|^n}{|z_1|^n} < t^n$, where $t = \frac{R}{|z_1|} < 1$.

Since $t < 1$, $\sum_n t^n$ is convergent. By Weierstrass' M-test, (thm 11.5)

this converges uniformly and absolutely.

For $|z| < |z_1|$, one can find $|z| < R < |z_1|$.

Example. $a_n = \frac{1}{n}$, $\sum \frac{1}{n} z^n$ convergent if $|z| < 1$, $z = -1$, divergent if $|z| > 1$.

Thm. 11.7. If $\sum a_n z^n$ is convergent for z_1, not convergent for z_2.

Then there is $r > 0$ s.t. $\sum a_n z^n$ is convergent for $|z| < r$, not conver for $|z| > r$.

proof) By thm 11.6, $\sum a_n z^n$ is convergent for z, $|z| < |z_1|$.

Let A be the set of positive numbers s.t. if $z \in \mathbb{C}$, $|z| \in A$,

then $\sum a_n z^n$ is convergent. By the assumption (z_2), A is bounded.

Let r be the least upper bound of A.

For z, $|z| < r$, there is $a \in A$, $|z| < a$, and by Thm 11.6,

$\sum a_n z^n$ is convergent.

For z_1, $|z_1| > r$, if $\sum a_n z_1^n$ were convergent, then

$\sum a_n z^n$ is convergent for all $r < |z| < |z_1|$, and this

contradicts the definition of r. Therefore for $|z_1| > r$, $\sum a_n z_1^n$

is not convergent.

Example $\sum \frac{1}{n} z^n$.

Exercises.

Real functions and real series.

Today, from now, $a_n \in \mathbb{R}$, x

If the series $\sum a_n x^n$ converges for some $x \in \mathbb{R}$, we can define a function by
$f(x) = \sum a_n x^n$

Thm. 11.8 Assume that, for $x \in (a-r, a+r)$
$$f(x) = \sum a_n (x-a)^n \text{ is convergent.}$$
Then $f(x)$ is continuous and
$$\int_a^x f(t)dt = \sum \frac{1}{n+1} a_n (x-a)^{n+1}$$

proof). ~~By Thm~~ $u_n(x) = a_n(x-a)^n$.

By Thm 11.6, $f(x) = \sum u_n(x)$ is uniformly convergent.

By Thm 11.2, f is continuous and by 11.4, we can integrate termwise: $\int_a^x f(t)dt = \sum \int_a^x a_n(t-a)^n dt = \sum \frac{1}{n+1} a_n (x-a)^{n+1}$

Thm 11.9. If $f(x) = \sum a_n(x-a)^n$ for $x \in (a-r, a+r)$, then
$$f'(x) = \sum n a_n (x-a)^{n-1}.$$

proof). We assume $a=0$ for simplicity.

For x, $|x|<r$, we take x_1, $|x|<|x_1|<r$.

$\sum_{n=1} n a_n x^{n-1} = \sum a_n x_1^{n-1} \left(\frac{x}{x_1}\right)^n n$ and $\left(\frac{x}{x_1}\right)^n n$ is bounded. (Ex. 10.4 4)

By 11.5. $\sum_{n=1}^{\infty} n a_n x^{n-1}$ is convergent.

$g(x) = \sum_{n=1}^{\infty} n a_n x^n$. By 11.8, $\int_0^x g(t)dt = \sum_{n=1}^{\infty} a_n x^n = f(x) - a_0$

Or. $f'(x) = g(x)$,

Example (1) $\frac{1}{x+1} = \sum (-1)^n x^n \quad |x|<1$ (2) $\frac{1}{x^2+1} = \sum (-1)^n x^{2n}$

$\log(x+1) = \sum \frac{(-1)^n}{n+1} x^{n+1}$ $\arctan x = \sum \frac{(-1)^n}{2n+1} x^{2n+1}$

$f(x) = \sum a_n x^n$. $f^{(k)}(0) = k! \cdot a_k$.

Thm 11.10 If $f(x) = \sum a_n x^n = \sum b_n x^n$, then $a_n = b_n = \frac{f^{(n)}(0)}{n!}$

Taylor's series

<u>If</u> $f(x) = \sum a_n (x-a)^n$ then $a_n = \frac{f^{(n)}(a)}{n!}$

What if $f(x)$ is ∞-times differentiable and we develop $\underbrace{\sum \frac{f^{(n)}(a)}{n!}(x-a)^n}_{\text{Taylor's series for } f(x).}$ does that coincide with $f(x)$?

<u>Not always.</u> Ex. 11.13 24

$$E_n(x) := f(x) - \sum_{k=0}^{n} \frac{f^{(k)}}{k!}(x-a)^n$$

Taylor's series converge to $f(x)$ if $E_n(x) \to 0$.

Thm 7.6 $E_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt$.

Thm. 11.11 If there is $A \geq 0$ s.t. $|f^{(n)}(x)| \leq A^n$ for $t \in (a-r, a+r)$ then $E_n(x) \to 0$ as $n \to \infty$.

proof). $\left|\int_a^x (x-t)^n f^{(n+1)}(t)\,dt\right| \leq \int_a^x |x-t|^n |f^{(n+1)}(t)|\,dt$

(if $x \geq a$) $\leq A^{n+1}(-1)^n \int_a^x (t-x)^n\,dt$

$= A^{n+1}(-1)^n \frac{1}{n+1}\left[(t-x)^{n+1}\right]_a^x = \frac{A^{n+1}}{n+1}(x-a)^{n+1}$

For any $x \in (a-r, a+r)$, we have $\left|\int_a^x (x-t)^n f^{(n+1)}(t)\,dt\right| \leq A^{n+1} |x-a|^{n+1}$

$|E_n(x)| \leq \frac{1}{(n+1)!} A^{n+1}|x-a|^{n+1} = \frac{B^{n+1}}{(n+1)!} \to 0$ (as $n \to \infty$).

$B = A|x-a|$.

<u>Example.</u> $f(x) = \sin x$. $f^{(1)}(x) = \cos x$, $f^{(2)}(x) = -\sin x$, $f^{(3)}(x) = -\cos x$, $f^{(4)}(x) = \sin x$

$|f^{(n)}(x)| \leq 1$, Thm 11.11 applies with $a=0$.

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

similarly, $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$f(x) = e^x$ on $x \in [-T, T]$.

$f^{(n)}(x) = e^x$ $|f^{(n)}(x)| \leq e^T$. Thm 11.11 applies

$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Thm 11.12. If $f^{(n)}(x) \geq 0$, then $E_n(x) \to 0$.

We omit the proof.

11.14 Differential equation (equations for functions).

Let $y(x)$ be a function of x.

Some differential equations can be solved by power series.

Problem Find $y(x)$ s.t $(1-x^2)y''(x) = -2y(x)$.

Solution Step 1. Assume that $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

y must satisfy $(1-x^2)y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n.$

$$= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} n(n-1)a_n x^n$$

$$= \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - n(n-1)a_n) x^n$$

$$= -2y(x) = -2\sum_{n=0}^{\infty} a_n x^n.$$

Step 2. By Thm 11.10, ~~an must~~ a_n must satisfy. (if $\sum b_n x^n = \sum c_n x^n$ then $b_n = c_n$.)

$$-2a_n = (n+2)(n+1)a_{n+2} - n(n-1)a_n.$$

$$(n+2)(n+1)a_{n+2} = [n(n-1)-2]a_n = (n+1)(n-2)a_n.$$

$$a_{n+2} = \frac{n-2}{n+2}a_n.$$

Step 3. $-a_0 = a_2$. $a_4 = 0 = a_6 = \cdots$

$a_3 = \frac{1}{3}a_1$, $a_5 = \frac{1}{5 \cdot 3}a_1$, $a_7 = \frac{3}{7 \cdot 5}a_5 = \frac{1}{7 \cdot 5}a_1, \cdots$

$$a_{2n+1} = \frac{1}{(2n+1)(2n-1)}a_1.$$

Step 4. $y(x) = a_0 - a_0x^2 + \sum \frac{a_1}{(2n+1)(2n-1)}x^{2n+1}$

This is convergent, so this y is a solution for any $a_0, a_1 \in$ ... $|x|<1$.

Another strategy $a_n = \frac{y^{(n)}(0)}{n!}$ $(1-x^2)y''(x) = -2y(x) \cdots (*)$

Put $x=0$ $2a_2 = -2a_0$.

Differentiate $(*)$ by x. $(1-x^2)y^{(3)}(x) - 2y''(x) = -2y'(x)$.

Put $x=0$ and we get an equation for a_3...

11.15. Binominal series.

$\alpha \in \mathbb{R}$. $\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$

Thm $(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$ for $|x|<1$.

proof). By ratio Test, the RHS converges.

Put $f(x) = (1+x)^\alpha$. Then $f'(x) = \alpha\frac{f(x)}{x+1}$. $f(0)=1$.

Put $g(x) = \sum \binom{\alpha}{n} x^n$. $(x+1)g'(x) = \sum n\binom{\alpha}{n}x^{n-1}(x+1)$

$g(0)=1$. $\Rightarrow f(x) = g(x)$ (Thm 8.3) $= \sum [n\binom{\alpha}{n} + (n+1)\binom{\alpha}{n+1}]x^n = \alpha g(x)$.

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8.1. Scalar and vector fields

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$x \in \mathbb{R}^n$ $\quad x = (x_1, x_2, \dots x_n)$, $\quad x_k \in \mathbb{R}$

$x \cdot y = \sum_{k=1}^{n} x_k y_k \in \mathbb{R}$ $\quad \|x\| = \sqrt{x \cdot x} = \sqrt{\sum_{k=1}^{n} x_k^2}$

Triangle inequality $\|x+y\| \le \|x\| + \|y\|$

Cauchy-Schwarz inequality $|x \cdot y| \le \|x\| \cdot \|y\|$

$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is called a "field". $m=1$: scalar field. $m>1$: vector field

Practical examples: $T : \mathbb{R}^3 \longrightarrow \mathbb{R}$ temperature of each point

$V : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ wind velocity.

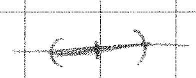
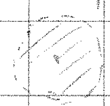
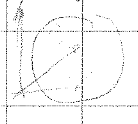
$\mathbb{E} : \mathbb{R}^3 \to \mathbb{R}^3$ electric field.

$f(x)$ and $f(x_1, \dots x_n)$ are the same thing. $f : \mathbb{R}^n \to \mathbb{R}^m$

8.2 Open balls and open sets

$a \in \mathbb{R}^n$, $r > 0$. $\quad B(a; r) = \{x \in \mathbb{R}^n : \|x - a\| < r\}$

the open n-ball with radius r, center a.

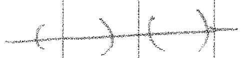
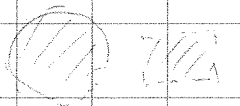
$\mathbb{R}^1$  $\mathbb{R}^2$  $\mathbb{R}^3$ 

Let $S \subset \mathbb{R}^n$, $a \in S$.

Def a is called an interior point if there is $r > 0$ s.t. $B(a; r) \subset S$.

$\operatorname{int} S := \{x \in S : x \text{ is an interior point}\}$.

S is open if $\operatorname{int} S = S$.

Example $\mathbb{R}^1$  $\mathbb{R}^2$  $\mathbb{R}^n$. $B(a; r)$

Def $S \subset \mathbb{R}^n$, $x \notin S$. x is an exterior point of S if there is $r > 0$, s.t. $B(x; r) \cap S = \phi$.

Ext S: the set of exterior points of S.

$\partial S = \mathbb{R}^n \setminus (\operatorname{int} S \cup \operatorname{ext} S)$ the boundary of S

K is a closed set if $\mathbb{R}^n \setminus K$ is open.

8.4. Limits

Let $S \subset \mathbb{R}^n$, $f : S \to \mathbb{R}^m$ (vector field)

$\mathbb{a} \in \mathbb{R}^n$, $\mathbb{b} \in \mathbb{R}^m$

If $\lim_{\|x - \mathbb{a}\| \to 0} \|f(x) - \mathbb{b}\| = 0$, then we write $\lim_{x \to \mathbb{a}} f(x) = \mathbb{b}$.

f is continuous at $\mathbb{a}$ if $\lim_{x \to \mathbb{a}} f(x) = f(\mathbb{a})$.

<u>Thm 8.1</u> If $\lim_{x \to \mathbb{a}} f(x) = \mathbb{b}$, $\lim_{x \to \mathbb{a}} g(x) = \mathbb{c}$, then

(a) $\lim_{x \to \mathbb{a}} [f(x) + g(x)] = \mathbb{b} + \mathbb{c}$.

(b) $\lim_{x \to \mathbb{a}} \lambda f(x) = \lambda \mathbb{b}$, where $\lambda \in \mathbb{R}$.

(c) $\lim_{x \to \mathbb{a}} f(x) \cdot g(x) = \mathbb{b} \cdot \mathbb{c}$.

(d) $\lim_{x \to \mathbb{a}} \|f(x)\| = \|\mathbb{b}\|$.

proof) We do only (c) and (d).

(c) $$\|f(x) \cdot g(x) - \mathbb{b} \cdot \mathbb{c}\| = \|(f(x) - \mathbb{b}) \cdot (g(x) - \mathbb{c}) + (f(x) - \mathbb{b}) \cdot \mathbb{c} + \mathbb{b} \cdot (g(x) - \mathbb{c})\|$$

$$\leq \|f(x) - \mathbb{b}\| \cdot \|g(x) - \mathbb{c}\| + \|f(x) - \mathbb{b}\| \|\mathbb{c}\| + \|\mathbb{b}\| \|g(x) - \mathbb{c}\|$$

$$\to 0 \quad (\text{as } x \to \mathbb{a}).$$

(d) $\lim_{x \to \mathbb{a}} \|f(x)\|^2 = f(x) \cdot f(x) \Rightarrow \mathbb{b} \cdot \mathbb{b} = \|\mathbb{b}\|^2$ by (c).

<u>Example</u> $f : \mathbb{R}^n \to \mathbb{R}^m$

$f(x) = (f_1(x), f_2(x), \cdots f_m(x))$. $f_k(x)$'s are scalar fields.

If f is continuous, f_k's are continuous, because

$f_k(x) = f(x) \cdot e_k$, where $e_k = (0, 0, \cdots \underset{k\text{-th}}{1}, 0, \cdots 0)$ and Thm 8.1 (c)

<u>Thm 8.2</u> Let f, g be functions $g : \underset{\mathbb{R}^k}{S} \to \mathbb{R}^m$, $f : \underset{\mathbb{R}^m}{T} \to \mathbb{R}^n$

and $g(S) \subset T$. $f \circ g(x) = f(g(x))$. $\quad f \circ g : S \to \mathbb{R}^n$.

If f is continuous at $g(\mathbb{a})$ and g is continuous at $\mathbb{a}$,

then $f \circ g$ is continuous at $\mathbb{a}$.

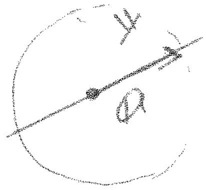
proof) $\lim_{x \to \mathbb{a}} \|f(g(x)) - f(g(\mathbb{a}))\| = \lim_{y \to g(\mathbb{a})} \|f(y) - f(g(\mathbb{a}))\| = 0$.

<u>Example.</u> $P(x_1, x_2) = x_1^2 + 4x_1x_2 + x_2^3$ or any polynomial.

$\sin(x_1^2 x_2)$

8.6. Derivatives of scalar fields

$S \subset \mathbb{R}^n$ an open set, $f : S \longrightarrow \mathbb{R}$ $\quad a \in B(a; r) \subset S$



$$\frac{f(a + h y) - f(a)}{h}$$

$$f'(a; y) := \lim_{h \to 0} \frac{f(a + h y) - f(a)}{h}$$

For fixed y, we define $g(t) = f(a + t y)$

<u>Thm 8.3</u> $g'(t)$ exists if and only if $f'(a + t y; y)$ exists and $g'(t) = f'(a + t y, y)$.

proof) By definition, $\dfrac{g(t+h) - g(t)}{h} = \dfrac{f(a + t y + h y) - f(a + t y)}{h}$

<u>Example</u> $f(x) = \|x\|^2$. Compute $f'(a; y)$.

$g(t) = \|x + t y\|^2 = \|x\|^2 + 2t\, x \cdot y + t^2 \|y\|^2$

$g'(t) = 2 x \cdot y + 2t \|y\|^2$. $f'(a; y) = g'(0) = 2 x \cdot y$.

<u>Thm 8.4</u> Assume that $f'(a + t y, y)$ exists for $0 \le t \le 1$. Then there exists θ s.t. $0 \le \theta \le 1$ and $f(a+y) - f(a) = f'(a + \theta y, y)$.

proof) Apply the mean value theorem to $g(t) = f(a + t y)$

$\Rightarrow$ There is θ, $0 \le \theta \le t$ s.t. $g(1) - g(0) = g'(\theta) = f'(a + \theta y, y)$

$f(a+y) - f(\bar{a})$.

8.7 Partial derivatives.

Let $e_k = (0, 0, \cdots 0, 1, 0, \cdots 0)$ (k-th.)

$\frac{\partial}{\partial x_k} f(a) := D_k f(a) = f'(a, e_k)$

$\mathbb{R}^3 \ni x = (x, y, z)$

$D_1 f(x) = \frac{\partial}{\partial x_1} f$

$D_2 f(x) = \frac{\partial}{\partial y} f$

If $D_k f$ exists, one can also consider $D_\ell D_k f$.

In general, $D_k D_\ell f \neq D_\ell D_k f$.

<u>Example</u> $f(x, y) = x^2 + 3yx + y^4$ $\quad \frac{\partial}{\partial x} f(x, y) = 2x + 3y$, $\frac{\partial}{\partial y} f(x, y) = 3x + 3y^3$

<u>Bad example</u> $f(x, y) = \dfrac{x y^2}{x^2 + y^4}$ $\quad f(0, y) = 0$. $\frac{\partial}{\partial y} f(0, 0) = 0$.

$f(x, 0) = 0$, $\frac{\partial}{\partial x} f(0, 0) = 0$.

But, if $x = t^2$, $y = t$, $f(t^2, t) = \dfrac{t^4}{2t^4} = \dfrac{1}{2}$.

8.11 Total derivative.

$\mathbb{R}^1$. $f(a+h) = f(a) + hf'(a) + hE(a,h)$.

$E(a,h) \to 0$ (as $h \to 0$) if f is differentiable.

Def $S \subset \mathbb{R}^n$ $f: S \to \mathbb{R}$. We say that f is differentiable at $a \in S$ if there is $T_a \in \mathbb{R}^n$ and $E(a, v)$ s.t.

$f(a+v) = f(a) + T_a \cdot v + \|v\| E(a, v)$

for $\|v\| < r$ and $E(a, v) \to 0$ as $\|v\| \to 0$.

T_a is called the total derivative of f at a.

Thm 8.5 If f is differentiable, then $T_a = \left(\frac{\partial}{\partial x_1} f(a), \cdots \frac{\partial}{\partial x_n} f(a)\right)$ and $f'(a; y) = T_a \cdot y$.

proof) As f is differentiable, $f(a+v) = f(a) + T_a \cdot v + \|v\| E(a, v)$

We take $v = hy$. $f'(a;y) = \lim_{h\to 0} \frac{f(a+hy) - f(a)}{h} = \lim_{h \to 0} \frac{T_a \cdot (hy) + h\|y\| E(a,h}{h}$

$= \lim_{h\to 0} T_a \cdot y + \|y\| \cdot E(a, hy) = T_a \cdot y$.

Especially, if $y = e_k$ $\frac{\partial}{\partial x_k} f(a) = f'(a; e_k) = T_a \cdot e_k$

$\Rightarrow T_a = \left(\frac{\partial}{\partial x_1} f(a), \cdots \frac{\partial}{\partial x_n} f(a)\right)$.

$\left(\frac{\partial}{\partial x_1} f(a), \cdots \frac{\partial}{\partial x_n} f(a)\right)$ is called the gradient of f at a. $\nabla f(a)$.

Thm 8.6 If f is differential, then f is continuous.

proof). $\|f(a+v) - f(a)\| = |T_a \cdot v + \|v\| E(a, v)|$

$\leq \|T_a\| \|v\| + \|v\| |E(a,v)| \to 0$ as $\|v\| \to 0$.

Thm 8.7 Assume that $\frac{\partial f}{\partial x_1}, \cdots \frac{\partial f}{\partial x_n}$ exist in $B(a;r)$ and are continuous at a. then f is differentiable at a.

proof). We have to prove that $f(a+v) - f(a) = \nabla f \cdot v + \|v\| E(a, v)$ and $|E(a,v)| \to 0$ as $\|v\| \to 0$.

$v = (v_1, \cdots v_n)$

$f(a+v) - f(a) = \sum_{k=1}^{n} f(a+u_k) - f(a+u_{k-1})$ where $u_k = (v_1, \cdots v_k, 0, 0, \cdots 0)$, $u_0 = (0, 0, \cdots 0)$.

$= \sum_{k=1}^{n} f'(a + u_{k-1} + \theta_k v_k e_k ; e_k)$. $u_k - u_{k-1} = v_k e_k$.

$= \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(a) v_k + \sum_{k=1}^{n} v_k \left(\frac{\partial f}{\partial x_k}(a + u_{k+1} + \theta_k v_k e_k) - \frac{\partial f}{\partial x_k}(a)\right)$

$= \nabla f \cdot v + \|v\| E(a, v)$, $E(a,v) = \sum_{k=1}^{n} \frac{v_k}{\|v\|} \left(\frac{\partial f}{\partial x_k}(a + u_{k-1} + \theta_k v_k e_k) - \frac{\partial f}{\partial x_k}(a)\right)$

9.1 Partial differential equations (PDE's)

Examples $\frac{\partial^2 f}{\partial x^2}(x,y)+\frac{\partial^2 f}{\partial y^2}(x,y)=0$ (Laplace's eq)

$\frac{\partial^2 f}{\partial t^2}(x,t)-c^2\frac{\partial^2 f}{\partial x^2}(x,t)=0$ (Wave eq). c: const

$\frac{\partial f}{\partial t}=k\frac{\partial^2 f}{\partial x^2}$ (heat eq). k: const.

In general, PDE's have many solutions.

Consider $\frac{\partial f}{\partial x}(x,y)=0$. $f(x,y)=g(y)$ is a solution for any function $g(y)$.

To fix a particular solution, we need to specify a "boundary condition"

9.2 First order linear PDE.

Consider $3\frac{\partial f}{\partial x}+2\frac{\partial f}{\partial y}=0$ and find all its solutions.

$(3e_x+2e_y)\cdot\nabla f=0$ $(e_x=(1,0),\ e_y=(0,1))\Longrightarrow f'(x;3e_x+2e_y)=0$

f is constant on $2x-3y=c$. $f(x,y)=g(2x-3y)$ is a solution.

Indeed, $\frac{\partial f}{\partial x}=2\cdot g'(2x-3y)$, $\frac{\partial f}{\partial y}=-3g'(2x-3y)\Longrightarrow 3\frac{\partial f}{\partial x}+2\frac{\partial f}{\partial y}=0$.

Conversely, any solution is of this form. Indeed, introduce u,v by

$u=2y-x$, $v=2x-3y$, or, $x=3u+2v$, $y=2u+v$

Define $h(u,v)=f(3u+2v,2u+v)$.

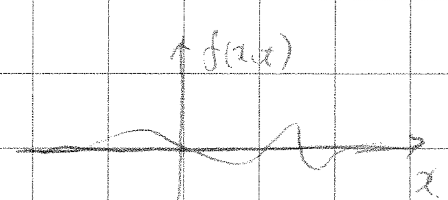
$\frac{\partial h}{\partial u}=3\frac{\partial f}{\partial x}(3u+2v,2u+v)+2\frac{\partial f}{\partial y}(3u+2v,2u+v)=0$. $h(u,v)=g(v)=g(2x-3y)$

<u>Thm 9.1</u> Let g be a differential function on $\mathbb{R}$, $a,b\in\mathbb{R}$, $(a,b)\neq(0,0)$.

Define $f(x,y)=g(bx-ay)$. Then f satisfies $a\frac{\partial f}{\partial x}+b\frac{\partial f}{\partial y}=0\cdots(*)$

Conversely, every solution of $(*)$ is of the form $g(bx-ay)$ for some g.

9.4 One dimensional wave equation.



x: coodinate on a string t: time

$f(x,t)$: displacement of the string.

When $f(x,t)$ is small, it should satisfy

$\frac{\partial^2 f}{\partial t^2}=c^2\frac{\partial^2 f}{\partial x^2}$, where c is a constant which depends on the string.

c.f. $m\frac{d^2 r}{dt^2}=F$

Thm 9.2 Let F be a twice differentiable function. G a differentiable function

Then $f(x,t) = \dfrac{F(x+ct)+F(x-ct)}{2} + \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} G(s)\,ds$

satisfies $\dfrac{\partial^2 f}{\partial t^2} = c^2 \dfrac{\partial^2 f}{\partial x^2}$, $f(x,0) = F(x)$, $\dfrac{\partial f}{\partial t}(x,0) = G(x)$ ... (*)

Conversely, any solution of (*) is of the form above, if $\dfrac{\partial^2}{\partial t \partial x} f = \dfrac{\partial^2}{\partial x \partial t} f$.

proof).

$$\frac{\partial f}{\partial x} = \frac{F'(x+ct)+F'(x-ct)}{2} + \frac{1}{2c}\left[G(x+ct) - G(x-ct)\right]$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{F''(x+ct)+F''(x-ct)}{2} + \frac{1}{2c}\left[G'(x+ct) - G'(x-ct)\right]$$

$$\frac{\partial f}{\partial t} = \frac{cF'(x+ct)-cF'(x-ct)}{2} + \frac{1}{2c}\left[c\cdot G(x+ct) - (-c)G'(x-ct)\right]$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{c^2F''(x+ct)+c^2F''(x-ct)}{2} + \frac{c}{2}\left[G'(x+ct) - G'(x-ct)\right] \Rightarrow \frac{\partial^2 f}{\partial t^2} = c\frac{\partial^2 f}{\partial x^2}$$

Conversely, assume that f satisfies (*).

Introduce $u = x+ct$, $v = x-ct$. $x = \dfrac{u+v}{2}$, $t = \dfrac{u-v}{2c}$

$$g(u,v) = f(x,t) = f\left(\frac{u+v}{2}, \frac{u-v}{2c}\right).$$

$$\frac{\partial g}{\partial u}(u,v) = \frac{1}{2}\frac{\partial f}{\partial x}\left(\frac{u+v}{2}, \frac{u-v}{2c}\right) + \frac{1}{2c}\frac{\partial f}{\partial t}\left(\frac{u+v}{2}, \frac{u-v}{2c}\right) \quad \text{(chain rule)}.$$

$$\frac{\partial^2 g}{\partial v \partial u}(u,v) = \frac{1}{4}\frac{\partial^2 f}{\partial x^2} - \frac{1}{4c}\frac{\partial^2 f}{\partial t \partial x} + \frac{1}{4c}\frac{\partial^2 f}{\partial x \partial t} - \frac{1}{4c^2}\frac{\partial^2 f}{\partial t^2} = 0.$$

$\Rightarrow \dfrac{\partial g}{\partial u} = \varphi(u)$ $\quad g(u,v) = \varphi_1(u) + \varphi_2(v)$. $\quad \varphi_1'(u) = \varphi(u)$

$f(x,t) = \varphi_1(x+ct) + \varphi_2(x-ct)$.

$f(x,0) = \varphi_1(x) + \varphi_2(x) = F(x)$, $\Rightarrow \varphi_1'(x) + \varphi_2'(x) = F'(x)$.

$\dfrac{\partial f}{\partial t}(x,t) = c\varphi_1(x+ct) - c\varphi_2(x-ct)$ $\quad \dfrac{\partial f}{\partial t}(x,0) = c\varphi_1'(x) - c\varphi_2'(x) = G(x)$

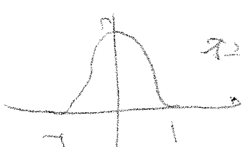
$\varphi_1'(x) = \dfrac{1}{2}F'(x) + \dfrac{1}{2c}G(x)$, $\quad \varphi_2'(x) = \dfrac{1}{2}F'(x) - \dfrac{1}{2c}G(x)$.

$$\varphi_1(x) - \varphi_1(0) = \frac{F(x)-F(0)}{2} + \frac{1}{2c}\int_0^x G(s)\,ds.$$

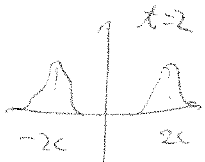
$$\varphi_2(x) - \varphi_2(0) = \frac{F(x)-F(0)}{2} - \frac{1}{2c}\int_0^x G(s)\,ds$$

$$f(x,t) = \varphi_1(x+ct) + \varphi_2(x-ct) = \frac{F(x+ct)+F(x-ct) - 2F(0) + 2\varphi_1(0) - 2\varphi_2(0)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\,ds$$

Example $F(x) = \begin{cases} 1+\cos \pi x & |x|<1 \\ 0 & |x| \geq 1 \end{cases}$, $G(x) = 0$



$$f(x,t) = \frac{F(x+ct)+F(x-ct)}{2}$$



A similar equation on $\mathbb{R}^4$ is satisfied by the Electromagnetic field.

9.6 Implicit functions

$x^2+y^2+z^2=1$ Defines the unit sphere.

$z = \pm\sqrt{1-x^2-y^2}$ $\quad z = f(x,y) = \sqrt{1-x^2-y^2}$ The sphere is the graph of f.

If $F(x,y,z)$ is a function, $F(x,y,z)=0$ may define a surface,

but it is not always possible to write $z = f(x,y)$ explicitly.

$F(x,y,z) = y^2 + xz + z^2 - e^z - 4 = 0$, $\quad f(x,y) = ??$

But we can say something about $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$

We assume that there is $f(x,y)$ s.t. $F(x,y,f(x,y))=0$.

and say that f is implicitly defined by F.

If we think $g(x,y) = F(x,y,f(x,y)) = 0$ as a function of (x,y),

then $\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = 0$

Formally, $g(x,y) = F(u_1(x,y), u_2(x,y), u_3(x,y))$, where

$u_1(x,y) = x$, $u_2(x,y) = y$, $u_3(x,y) = f(x,y)$.

By the chain rule, $0 = \frac{\partial g}{\partial x} = 1 \cdot \frac{\partial F}{\partial x}(x,y,f(x,y)) + 0 \cdot \frac{\partial F}{\partial y} + \frac{\partial f}{\partial x}(x,y)\frac{\partial F}{\partial z}(x,y,f(x,y$

therefore,
$$\frac{\partial f}{\partial x}(x,y) = -\frac{\frac{\partial F}{\partial x}(x,y,f(x,y))}{\frac{\partial F}{\partial z}(x,y,f(x,y))}$$

Similary,
$$\frac{\partial f}{\partial y} = -\frac{\frac{\partial F}{\partial y}(x,y,f(x,y))}{\frac{\partial F}{\partial z}(x,y,f(x,y))}.$$

<u>Example</u>. $F(x,y,z) = y^2 + xz + z^2 - e^z - C$. Assume the existence of f.

Find the value of C s.t. $f(0,e)=2$ and compute $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ at $(0,e)$

solution. $0 = e^2 + 0 + f(0,e)^2 - e^{f(0,e)} - C = e^2 + 4 - e^2 - C \Longrightarrow C = 4$

$$\frac{\partial f}{\partial x}(x,y) = -\frac{f(x,y)}{x+2f(x,y)-e^{f(x,y)}} \quad \left(\text{because } \frac{\partial F}{\partial z} = x+2z-e^z,\ \frac{\partial F}{\partial x} = z\right)$$

$$\frac{\partial f}{\partial x}(0,e) = \frac{2}{e^2-4}. \quad \frac{\partial f}{\partial y}(x,y) = -\frac{2y}{x+2f(x,y)-e^{f(x,y)}} \quad \frac{\partial f}{\partial y}(0,e) = \frac{2e}{e^2-4}$$

Similarly, if $F(x_1,\dots x_n)$ defines $x_n = f(x_1,\dots x_{n-1})$ implicitly,

then
$$\frac{\partial f}{\partial x_k} = -\frac{\frac{\partial F}{\partial x_k}(x_1,\dots x_{n-1}, f(x_1,\dots x_{n-1}))}{\frac{\partial F}{\partial x_n}(x_1,\dots x_{n-1}, f(x_1,\dots x_{n-1}))} \quad (\text{Thm. } 9.3).$$

Let us consider two surfaces $F(x,y,z)=0$, $G(x,y,z)=0$ and assume that their intersection is a curve $x=X(z)$, $y=Y(z)$.
namely, $\underbrace{F(X(z),Y(z),z)}_{f(z)}=0$, $\underbrace{G(X(z),Y(z),z)}_{g(z)}=0$.

$$0=f'(z)=X'(z)\frac{\partial F}{\partial x}(X(z),Y(z),z)+Y'(z)\frac{\partial F}{\partial y}(X(z),Y(z),z)+\frac{\partial F}{\partial z}(X(z),Y(z),z)$$

$$\Rightarrow X'\frac{\partial F}{\partial x}+Y'\frac{\partial F}{\partial y}=-\frac{\partial F}{\partial z}$$

Similarly, $X'\frac{\partial G}{\partial x}+Y'\frac{\partial G}{\partial y}=-\frac{\partial G}{\partial z}$ $\Rightarrow \begin{pmatrix}X'\\ Y'\end{pmatrix}=\begin{pmatrix}\frac{\partial F}{\partial x} & \frac{\partial F}{\partial y}\\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y}\end{pmatrix}^{-1}\cdot\begin{pmatrix}-\frac{\partial F}{\partial z}\\ -\frac{\partial G}{\partial z}\end{pmatrix}$.

at $(X(z),Y(z),z)$.

9.7. Examples

2. $g(x,y)=0$ defines $Y(x)$. Let $f(x,y)$ be a function.
$h(x)=f(x,Y(x))$ is a function of x.

$$h'(x)=\frac{\partial f}{\partial x}(x,Y(x))+Y'(x)\frac{\partial f}{\partial y}(x,Y(x))=\frac{\partial f}{\partial x}(x,Y(x))-\frac{\frac{\partial g}{\partial x}(x,Y(x))\cdot\frac{\partial f}{\partial y}(x,Y(x))}{\frac{\partial g}{\partial y}(x,Y(x))}$$

3. $2x=v^2-u^2$, $y=uv$ defines $u(x,y)$, $v(x,y)$
(explicity, $2x+\left(\frac{y}{v}\right)^2=v^2$, $\left(\frac{y}{u}\right)^2-u^2=2x$)

By differentiating w.r.t. x, $2=2v\frac{\partial v}{\partial x}-2u\frac{\partial u}{\partial x}$, $0=\frac{\partial u}{\partial x}v+u\frac{\partial v}{\partial x}$

$$\Rightarrow 1=v\cdot\left(-\frac{\partial u}{\partial x}\cdot\frac{v}{u}\right)-u\frac{\partial u}{\partial x}\Rightarrow\frac{\partial u}{\partial x}=-\frac{u}{u^2+v^2},$$

similarly, $\frac{\partial v}{\partial x}=\frac{v}{u^2+v^2}$,

w.r.t y. $0=2v\cdot\frac{\partial v}{\partial y}-2u\frac{\partial u}{\partial y}$, $1=\frac{\partial u}{\partial y}v+u\frac{\partial v}{\partial y}$

$$\Rightarrow 1=\frac{\partial u}{\partial y}v+u\cdot\left(\frac{u}{v}\cdot\frac{\partial u}{\partial y}\right)\Rightarrow\frac{\partial u}{\partial y}=\frac{v}{u^2+v^2},\quad\frac{\partial v}{\partial y}=\frac{u}{u^2+v^2}.$$

4. u is defined by $F(u+x,yu)=u$. $F(X,Y)$
Let $u=g(x,y)\Rightarrow g(x,y)=F(x+g(x,y),yg(x,y))$.
Differentiating by x, $\frac{\partial g}{\partial x}=\left(1+\frac{\partial g}{\partial x}\right)\frac{\partial F}{\partial X}(\cdots,\cdots)+y\frac{\partial g}{\partial x}\cdot\frac{\partial F}{\partial Y}(\cdots,\cdots)$

$$\Rightarrow\frac{\partial g}{\partial x}=\frac{-\frac{\partial F}{\partial X}(x+g(x,y),\,yg(x,y))}{\frac{\partial F}{\partial X}(\cdots,\cdots)+y\frac{\partial F}{\partial Y}(\cdots,\cdots)-1}.$$

5. $x=u+v$, $y=uv^2$ defines $u(x,y)$, $v(x,y)$.
Compute $\frac{\partial v}{\partial x}$.
$y=(x-v)v^2$. $\frac{\partial y}{\partial x}=0=\left(1-\frac{\partial v}{\partial x}\right)v^2+(x-v)\cdot 2v\cdot\frac{\partial v}{\partial x}\Rightarrow\frac{\partial v}{\partial x}=\frac{v}{3v-2x}$

Similarly, $\frac{\partial v}{\partial y}=\frac{1}{2xv-3v^2}$.

9.11 Stationary point.

When $\nabla f(\mathbf{a})=0$, min? max? saddle?

Thm 9.5 Let $A=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn}\end{pmatrix}$ a real symmetric matrix.

$Q(\mathbf{y}) = \mathbf{y} A \mathbf{y}^t$.

(a) $Q(\mathbf{y})>0$ for all $\mathbf{y}\neq \mathbf{0}=(0,0,\cdots 0) \iff$ all eigenvalues of A are positive

(b) $\quad\quad <0 \quad\quad \iff \quad\quad$ negative

proof). By Thm 5.11. there is an orthogonal matrix C s.t. $C^tAC=\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$

$Q(\mathbf{y}) = \mathbf{y}CC^tACC^t\mathbf{y}^t = \mathbf{v}\cdot C^tAC\cdot \mathbf{v}^t = \sum_{i=1}^{n} \lambda_i v_i^2$, where $\mathbf{v}=\mathbf{y}C$.

If all $\lambda_j>0$, then $Q(\mathbf{y})>0$ for $\mathbf{v}\neq\mathbf{0}$.

If there are $\lambda_k>0$. $\lambda_\ell<0$, then take $\mathbf{v}_k=(0,\cdots 0,\underset{k\text{-th}}{1},0\cdots 0)$. $\mathbf{y}_k=\mathbf{v}C^t$. $Q(\mathbf{y}_k)=\lambda_k>0$.

$\mathbf{v}_\ell=(0,\cdots 0,\underset{\ell\text{-th}}{1},0,\cdots 0)$ $\mathbf{y}_\ell=\mathbf{v}C^t$. $Q(\mathbf{y}_\ell)=\lambda_\ell<0$.

Thm 9.6. Let f be a scalar field with continuous second partial derivatives on $B(\mathbf{a};r)$

Assume that $\nabla f(\mathbf{a})=0$.

(a) If all the eigenvalues λ_j of $H(\mathbf{a})$ are positive, then f has a rel. minimum at $\mathbf{a}$.

(b) $\quad$ negative, $\quad$ maximum

(c). If some $\lambda_k>0$, $\lambda_\ell<0$, then f has a saddle point at $\mathbf{a}$.

proof). (a). Let $Q(\mathbf{y})=\mathbf{y}H(\mathbf{a})\mathbf{y}^t$.

Let h be the smallest eigenvalue. $h>0$. Diagonalize $H(\mathbf{a})$ by C.

We ~~have~~ set $\mathbf{y}C=\mathbf{v}$. $\|\mathbf{v}\|^2=\|\mathbf{y}\|^2$ and

$\mathbf{y}H(\mathbf{a})\mathbf{y}^t = \mathbf{v}\cdot C^tH(\mathbf{a})C\cdot\mathbf{v}^t = \sum_{j=1}^{n}\lambda_j v_j^2 \geq \sum_{j=1}^{n} h v_j^2 = h\|\mathbf{v}\|^2 = h\|\mathbf{y}\|^2$.

By Thm 9.4. $f(\mathbf{a}+\mathbf{y}) = f(\mathbf{a}) + \frac{1}{2}\mathbf{y}H(\mathbf{a})\mathbf{y}^t + \|\mathbf{y}\|^2 E_2(\mathbf{a},\mathbf{y})$.

and $E_2(\mathbf{a},\mathbf{y})\to 0$ as $\|\mathbf{y}\|\to 0$. There is $r_1<r$ s.t $|E_2(\mathbf{a},\mathbf{y})|<\frac{h}{2}$ for $\mathbf{y}\in B(\mathbf{a},r_1)$

$f(\mathbf{a}+\mathbf{y}) = f(\mathbf{a}) + \frac{1}{2}\mathbf{y}H(\mathbf{a})\mathbf{y}^t + \|\mathbf{y}\|^2E_2(\mathbf{a},\mathbf{y}) > f(\mathbf{a}) + \frac{h}{2}\|\mathbf{y}\|^2 - \frac{h}{2}\|\mathbf{y}\|^2 \geq f(\mathbf{a})$.

$\Rightarrow f$ has a min at $\mathbf{a}$.

(b) Similar.

(c) Let $\lambda_k>0$. $\lambda_\ell<0$ and $\mathbf{y}_k$. $\mathbf{y}_\ell$ the corresponding eigenvectors of $H(\mathbf{a})$.

then $\mathbf{y}_kH(\mathbf{a})\mathbf{y}_k^t = \lambda_k\|\mathbf{y}_k\|^2>0$, $\mathbf{y}_\ell H(\mathbf{a})\mathbf{y}_\ell^t = \lambda_\ell\|\mathbf{y}_\ell\|^2<0$.

As in (a), $f(\mathbf{a}+\mathbf{y}_k)>f(\mathbf{a})$. $f(\mathbf{a}+\mathbf{y}_\ell)<f(\mathbf{a})$. $\Rightarrow \mathbf{a}$ is a saddle.

Example $f(x,y)=xy$. $\nabla f=(y,x)$. $H(x,y)=\begin{pmatrix}0&1\\1&0\end{pmatrix}$.

eigenvalues are $1,-1$. $\Rightarrow (0,0)$, with $\nabla f(0,0)=(0,0)$ is a saddle.

9.9 Maxima, minima and saddle points.

Let $S \subset \mathbb{R}^n$ be open, $f: S \to \mathbb{R}$ and $a \in S$.

If $f(a) \underset{\leq}{\geq} f(x)$ for all $x \in S$, $f(a)$ is said to be the absolute $\begin{smallmatrix}\text{maximum}\\ \text{minimum}\end{smallmatrix}$ of f.

If $f(a) \underset{\leq}{\geq} f(x)$ for some $B(a;r)$, then $f(a)$ is a relative $\begin{smallmatrix}\text{maximum}\\ \text{minimum}\end{smallmatrix}$.

$\left.\begin{smallmatrix}\text{maximum}\\ \text{minimum}\end{smallmatrix}\right\}$ extremum.

<u>Thm.</u> If f is differentiable and has a relative $\begin{smallmatrix}\min\\ \max\end{smallmatrix}$ at a, then $\nabla f(a) = 0$.

proof). Consider $g(u) = f(a+uy)$. $g'(0) = 0 \Rightarrow f'(a;y) = 0$ for any $y \Rightarrow \nabla f(a) = 0$

But $\nabla f(a) = 0$ does not imply that f takes a rel. $\begin{smallmatrix}\min\\ \max\end{smallmatrix}$ at a.

(Even in $\mathbb{R}$, consider $f(x) = x^3$ at $x = 0$. ).

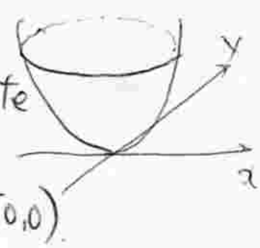
<u>Def</u>. If $\nabla f(a) = 0$, a is said to be a stationary point.

If $\nabla f(a) = 0$ and f has neither max nor min at a, then a is a saddle point.

<u>Examples</u> 2. $f(x,y) = x^2 + y^2$

$f(0,0) = 0$ is the absolute minimum.

$\nabla f = (2x, 2y)$. $\nabla f(0,0) = (0,0)$



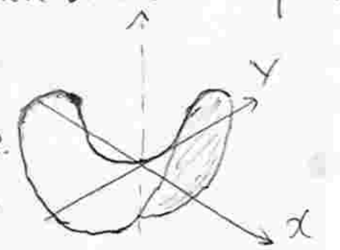
3 $f(x,y) = xy$

$f(0,0) = 0$ is a saddle.

$x > 0, y > 0 \Rightarrow f(x,y) > 0$.

$x > 0, y < 0 \Rightarrow f(x,y) < 0$.

$\nabla f = (y, x)$, $\nabla f(0,0) = (0,0)$



More examples on the textbook.

9.10 Second-order Taylor formula.

f: differentiable. First-order: $f(a+y) = f(a) + \nabla f(a) \cdot y + \|y\| E_1(a,y)$.

and $E_1(a,y) \to 0$ as $\|y\| \to 0$.

Let f have continuous second partial derivatives. $D_{ij} f = \dfrac{\partial^2 f}{\partial x_i \partial x_j}$

Define the Hessian $H(x) = \begin{pmatrix} D_{11}f(x) & D_{12}f(x) & \cdots & D_{1n}f(x) \\ \vdots & & & \\ D_{n1}f(x) & \cdots & - & D_{nn}f(x) \end{pmatrix}$. $(n \times n)$-matrix

For $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, $yH(x)y^t = \sum_{i,j=1}^{n} D_{ij} f(x) y_i y_j$

<u>Thm 9.4</u> Let f be a scalar field with continuous second partial derivatives on $B(a;r)$.

Then, for y s.t. $a + y \in B(a;r)$, there is $0 < c < 1$ s.t.

$f(a+y) = f(a) + \nabla f(a) \cdot y + \frac{1}{2!} y H(a+cy) y^t$ and.

$f(a+y) = f(a) + \nabla f(a) \cdot y + \frac{1}{2!} y H(a) y^t + \|y\|^2 E_2(a,y)$. $E_2(a,y) \to 0$ as $\|y\| \to 0$.

proof). Define $g(u) = f(a+uy)$. Apply the Taylor formula. $g(1) = g(0) + g'(0) + \frac{1}{2!} g''(c)$.

By chain rule, $g'(u) = \nabla f(a+uy) \cdot y$ because $g(u) = f(a_1 + uy_1, \dots, a_n + uy_n)$.

$g''(u) = yH(a+uy)y^t$ because $g'(u) = \sum_j D_j f(a_1 + uy_1, \dots, a_n + uy_n) \cdot y_j$

$E_2(a,y) = \frac{1}{2} y\left(H(a+cy) - H(a)\right)y^t / \|y\|^2$

$|E_2(a,y)| \leq \frac{1}{2} \sum_{i,j=1}^{n} \frac{y_i y_j}{\|y\|^2} \left| D_{ij}f(a+cy) - D_{ij}f(a) \right| \to 0$ by continuity of $D_{ij}f$

9.14 Extrema with constraints.

f: scalar field. We restrict f to the set $g(x_1, \dots x_n) = 0$.

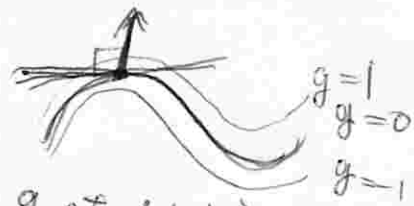
Can we find relative min / max ?

Lagrange's multiplier method

f: scalar field on $\mathbb{R}^n$ and gets restricted to $g_1(x_1, \dots x_n), \dots g_m(x_1, \dots x_n)$.

$m < n$. Then there are $\lambda_1, \dots \lambda_m \in \mathbb{R}$ for an extremum a of f

s.t. $\nabla f(a) = \lambda_1 \nabla g_1(a) + \dots + \lambda_m \nabla g_m(a)$



Let $n=2$, consider f on $g(x, y) = 0$.

If f has an extremum at (x_1, y_1), then

f should be stationary along the tangent of g at (x_1, y_1).

$\Rightarrow \nabla f(x_1, y_1)$ is orthogonal to the tangent of g at (x_1, y_1).

$\Rightarrow \nabla f(x_1, y_1) = \lambda \nabla g(x_1, y_1)$ for some λ.

Example 1 $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ $g(x, y, z) = 0$.

Similar to above, $\nabla f(x_1, y_1, z_1) = \lambda \nabla g(x_1, y_1, z_1)$ if

(x_1, y_1, z_1) is an extremal of f restricted to $g = 0$.

(3+1 equations for 3+1 variables).

Example 2. Determine extrema of f on $g_1(x, y, z) = 0$, $g_2(x, y, z) = 0$.

$g_1 = g_2 = 0$ defines a curve, C. As above, if a is

an extremum, then $\nabla f(a)$, is orthogonal to the tangent

of C at a. So are $\nabla g_1(a)$, $\nabla g_2(a)$.

If $\nabla g_1(a)$ and $\nabla g_2(a)$ are linearly independent, then

they span the orthogonal complement. $\Rightarrow \nabla f(a) = \lambda_1 \nabla g_1(a) + \lambda_2 \nabla g_2(a)$

Exercise. 9.15 1. Find the extreme of $f(x, y) = xy$ on $x + y = 1$.

Solution 1 $\nabla f(x, y) = (y, x)$. $g(x, y) = x + y - 1$. $\nabla g(x, y) = (1, 1)$.

$(y, x) = \lambda(1, 1)$. $x + y - 1 = 0 \Rightarrow x = y = \lambda$, $2x - 1 = 0$

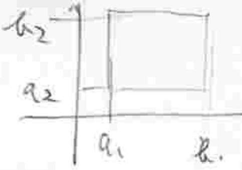
$\Rightarrow (x, y) = (\frac{1}{2}, \frac{1}{2})$. $f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4}$.

Solution 2 $y = 1 - x = Y(x)$. $f(x, Y(x)) = x(1 - x) = x - x^2 = \frac{1}{4} - (\frac{1}{2} - x)^2$

$x = \frac{1}{2}$, $y = \frac{1}{2}$ is a max. $f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4}$.

9.16. Extreme value theorem.

$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$



<u>Lemma</u> Let x_k be a bounded sequence. Then there is a subsequence N_k $k=1,2,\cdots$, $N_k < N_{k+1}$ s.t. $\{x_{N_k}\}_{k=1,2,\cdots}$ is convergent.

proof). $x_k \in [-C, C]$ for some C. As $\{x_k\}$ is infinite, one of $[-C, 0]$ and $[0, C]$ contains infinite subsequence. Take the first element x_{N_1}. If $[0, C]$ contains infinite subsequence, then consider $[0, \frac{C}{2}]$ and $[\frac{C}{2}, C]$. One of them contains an infinite subsequence. Take x_{N_2}. Repeat.

$Y_k = \sup_{l \geq k} x_{N_l}$, $Z_k = \inf_{l \geq k} x_{N_l}$. Y_k, Z_k are monotone and bounded.

$Y_k \to Y$, $Z_k \to Z$. $|Y_k - Z_k| \leq \frac{C}{2^k} \Rightarrow Y = Z$.

$Z_k \leq x_{N_k} \leq Y_k \Rightarrow x_{N_k}$ is convergent to $Y = Z$.

<u>Lemma</u>. Let x_k be a sequence of points in R.
Then there is a subsequence x_{N_k} which is convergent.

proof). Apply the previous Lemma to components of $x_{N_k} = ((x_{N_k})_1, (x_{N_k})_2, \cdots (x_{N_k})_n)$ from the 1st to n-th.

<u>Thm. 9.8</u>. Let f be a continuous function on R.
Then f is bounded, i.e., there is $C > 0$ s.t. $|f(x)| < C$.

proof). Assume the contrary. then, for each N, there is x_N s.t. $|f(x_N)| \geq N$.
By Lemma. there is a subsequence x_{N_k}. and $x_{N_k} \to a \in R$.
By continuity of f at a, $|f(x)| \leq f(a) + \varepsilon$ for $x \in B(a; \delta)$.
this contradicts $|f(x_{N_k})| > N_k$.

<u>Thm. 9.9</u> Let f be continuous on R, then there are c, d s.t.
$f(c) = \sup_{x \in R} f(x)$, $f(d) = \inf_{x \in R} f(x)$

proof). There is x_N s.t. $f(x_N) \to \sup f(x)$. By Lemma. $x_{N_k} \to c$.
By continuity, $f(c) = \lim f(x_k) = \sup f(x)$.

<u>Thm 9.10</u>. Let f be continuous on R. Then for any $\varepsilon > 0$ there is δ s.t. for any $x \in R$, $y, z \in B(x; \delta) \cap R$, $|f(y) - f(z)| < \varepsilon$.

proof). Assume the contrary. There is ε: for $\frac{1}{2^k}$, there are $x_k, y_k, z_k \in B(x_k, \frac{1}{2^k})$ $|f(y_k) - f(z_k)| \geq \varepsilon$. By Lemma $x_{N_k} \to x$. By continuity. $|f(y) - f(z)| < \varepsilon$ for $B(x; r) \supset B(x_{N_k}; \frac{1}{2^{N_k}})$. Contradiction.

10.2 Line integrals

$\alpha(t)$: vector-valued function $[a,b] \to \mathbb{R}^n$ continuous. $\alpha(t) = (\alpha_1(t), \cdots \alpha_n(t))$

α is said to be smooth if the components α_k are differentiable on $[a,b]$ (at a, b, we take the right/left derivative).

piecewise smooth if $[a,b] = [a,a_1] \cup [a_1,a_2] \cup \cdots \cup [a_k, b]$ and α is smooth on each subinterval.

Let α be a piecewise smooth path on $[a,b]$ and f a vector field. If f is continuous, we define the line integral of f along α by $\int_a^b f(\alpha(t)) \cdot \alpha'(t)\,dt$ and denote $\int f \cdot d\alpha$.

<u>Example</u>. $f(x,y) = (\sqrt{y}, x^3+y)$, $\alpha(t) = (t^2, t^3)$ $t \in [0,1]$

$\alpha'(t) = (2t, 3t^2)$. $\int f \cdot d\alpha = \int_0^1 (\sqrt{t^3}, (t^2)^3 + t^3) \cdot (2t, 3t^2)\,dt$

$= \int_0^1 (2t^{\frac{5}{2}} + 3t^8 + 3t^5)\,dt = \left[\frac{4}{7}t^{\frac{7}{2}} + \frac{t^9}{3} + \frac{t^6}{2}\right]_0^1 = \frac{59}{42}$

It holds that $\int (cf + dg) \cdot d\alpha = c\int f \cdot d\alpha + d\int g \cdot d\alpha$.

if $\alpha(t) = \begin{cases} \alpha_1(t) & \text{on } [a,c] \\ \alpha_2(t) & \text{on } [c,b] \end{cases}$ then $\int f \cdot d\alpha = \int f \cdot d\alpha_1 + \int f \cdot d\alpha_2$.

What happens if $\alpha(t)$ on $[a,b]$ and $\beta(t)$ on $[c,d]$ represent the same curve?

We say $\alpha(t)$ and $\beta(t)$ are equivalent if there is a differentiable function $u : [c,d] \to [a,b]$ s.t. $\alpha(u(t)) = \beta(t)$.

If $u(c) = a$, $u(d) = b$, $u'(t) > 0$, then α and β are in the same direction

$u(c) = b$, $u(d) = a$, $u'(t) < 0$, then α and β are in the opposite direction

<u>Thm 10.1</u> Let f be a continuous vector field, α, β equivalent piecewise smooth paths. Then $\int f \cdot d\alpha = \int f \cdot d\beta$ (the same direction).

$\int f \cdot d\alpha = -\int f \cdot d\beta$ (the opposite direction)

proof). We may assume that α and β are smooth, by decomposing the integrals into subintervals where α and β are smooth.

There is a differentiable u s.t. $\beta(t) = \alpha(u(t))$. $\beta'(t) = u'(t) \cdot \alpha'(u(t))$.

$\int f \cdot d\beta = \int_c^d f(\beta(t)) \cdot \beta'(t)\,dt = \int_c^d \underbrace{\cancel{f(\beta(t))}}_{f(\alpha(u(t)))} \cdot u'(t)\alpha'(u(t))\,dt = \begin{cases} \int_a^b f(\alpha(t))\,\alpha'(t)\,dt \\ \int_b^a f(\alpha(t))\,\alpha'(t)\,dt \end{cases}$

Exercise 10.5

1. $f(x,y) = (x^2 - 2xy, y^2 - 2xy)$. $\alpha(t) = (t, t^2)$ on $[-1,1]$.

$\int f(\alpha(t)) \cdot \alpha'(t)\,dt = \int_{-1}^1 (t^2 - 2t^3, t^4 - 2t^3) \cdot (1, 2t)\,dt = \int_{-1}^1 (t^2 - 2t^3 + 2t^5 - 4t^4)\,dt$

$= \left[\frac{t^3}{3} - \frac{t^4}{2} + \frac{t^6}{3} - \frac{4}{5}t^5\right]_{-1}^1 = -\frac{14}{15}$.

Example. $h(x,y)$ the height of a mountain.

$f(x,y) = \nabla h(x,y)$ the gradient field.

Walk along the path α on $[0,1]$. $h(\alpha(t))$ is the height at time t.

$\frac{d}{dt} h(\alpha(t)) = \nabla h(\alpha(t)) \cdot \alpha'(t)$.

$\int f \cdot d\alpha = \int f(\alpha(t)) \cdot \alpha'(t)\,dt = \int \frac{d}{dt} h(\alpha(t))\,dt = [h(\alpha(1)) - h(\alpha(0))]$.

10-6. Work in mechanics.

Consider the gravitational field around the earth surface $f(x,y,z) = (0,0,mg)$ with m : the mass of a paricle ; $g = 9.8\ m/s^2$.

Move a particle from $a = (a_1, a_2, a_3)$ to $b = (b_1, b_2, b_3)$ along a path $\alpha(t)$, $[0,1]$. The work is $\int f \cdot d\alpha(t) = \int f\cdot(\alpha(t))\alpha'(t)\,dt = \int_0^1 mg\,\alpha_3'(t)\,dt$

$= mg[\alpha_3(t)]_0^1 = mg(b_3 - a_3)$.

The work depends only on the height $\Rightarrow$ potential energy.

Let f be a force field. A particle is moving in f, with velocity $v(t)$ at time t. Define the kinetic energy by $\frac{1}{2}m\|v(t)\|^2$.

Let $r(t)$ be the position of the particle. $v(t) = r'(t)$.

$\int_0^1 f(r(t)) \cdot r'(t)\,dt = \int_0^1 m\,v'(t)\cdot v(t)\,dt = \int_0^1 \frac{m}{2}\frac{d}{dt}(v(t)\cdot v(t))\,dt = \left[\frac{m}{2}\|v(1)\|^2 - \frac{m}{2}\|v(0)\|^2\right]$

(By Newton's law, $f(r(t)) = m\,r''(t) = m\,v'(t)$)

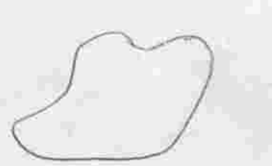
10.7. Line integral with respect to arc length.

$\alpha(t)$: curve. $s(t) = \int_a^t \|\alpha'(u)\|\,du$ length of the arc.

$s'(t) = \|\alpha'(t)\|$.

10-8 Mass of a wire.

Assume that $\alpha(t)$ represents a wire.



and the mass per unit length at $\alpha(t)$ is $\varphi(\alpha(t))$.

$M = \int \varphi(\alpha(t))\,s'(t)\,dt$.

Example the mass of a coil $\alpha(t) = (a\cos t, a\sin t, bt)$ $t \in [0, 2\pi]$.

$\varphi(x,y,z) = x^2 + y^2 + z^2$.

$\alpha'(t) = (-a\sin t, a\cos t, b)$. $\|\alpha'(t)\| = \sqrt{a^2+b^2} = s$

$M = \int_0^{2\pi} (a^2 + b^2t^2)\sqrt{a^2+b^2}\,dt = \sqrt{a^2+b^2}\left(2\pi a^2 + \frac{8\pi^3}{3}b^2\right)$.

10.11 The second fundamental theorem of calculus

φ : real smooth function, φ' continuous on $[a,b]$. $\int_a^b \varphi'(t)\,dt = \varphi(b)-\varphi(a)$

<u>Thm 10.3</u> Let φ be a differentiable scalar field, $\nabla\varphi$ continuous on $S \subset \mathbb{R}^n$. Let α be a path in S. $\alpha(a)=\mathbb{a}$, $\alpha(b)=\mathbb{b}$, piecewise smooth. Then $\int \nabla\varphi\cdot d\alpha = \varphi(\mathbb{b})-\varphi(\mathbb{a})$.

proof) Assume that α is smooth on $[a,b]$. Then

$$\frac{d}{dt}\varphi(\alpha(t)) = \frac{d}{dt}\varphi(\alpha_1(t),\dots\alpha_n(t)) = \alpha_1'(t)\frac{\partial\varphi}{\partial x_1}(\alpha(t)) + \dots \alpha_n'(t)\frac{\partial\varphi}{\partial x_n}(\alpha(t))$$
$$= \alpha'(t)\cdot\nabla\varphi(\alpha(t)).$$

Now. $\int\nabla\varphi\cdot d\alpha = \int_a^b \nabla\varphi(\alpha(t))\cdot\alpha'(t)\,dt = \int_a^b \frac{d}{dt}\varphi(\alpha(t))\,dt = [\varphi(\alpha(b))-\varphi(\alpha(a))]$
$= \varphi(\mathbb{b})-\varphi(\mathbb{a})$.

If α is only piecewise smooth, $[a,b]=[a,a_1]\cup[a_1,a_2]\cup\dots\cup[a_k,b]$ and smooth on each of these subintervals, then

$$\int\nabla\varphi\cdot d\alpha = [\varphi(\alpha(b))-\varphi(\alpha(a_k))]+[\varphi(\alpha(a_k))-\varphi(\alpha(a_{k-1}))]+\dots[\varphi(\alpha(a_1))-\varphi(\alpha(a))]$$
$$= \varphi(\alpha(b))-\varphi(\alpha(a))$$

<u>Example 2.</u> Put the center of the earth at $(0,0,0)$, with mass M. Take a (small) particle with mass m. The force field of gravitation is given by

$$\mathbb{f}(\mathbb{a}) = \frac{-GmM\,\mathbb{a}}{\|\mathbb{a}\|^3}$$

We can take the potential as $\varphi(\mathbb{a}) = GmM\frac{1}{\|\mathbb{a}\|} = \frac{GmM}{\sqrt{a_1^2+a_2^2+a_3^2}}$

Indeed, $\nabla\varphi(\mathbb{a}) = \left(-\frac{GmM\cdot 2a_1}{2\sqrt{a_1^2+a_2^2+a_3^2}^{\,3}}, -\frac{GmM\cdot 2a_2}{2\sqrt{a_1^2+a_2^2+a_3^2}^{\,3}}, -\frac{GmM\cdot 2a_3}{2\sqrt{a_1^2+a_2^2+a_3^2}^{\,3}}\right)$
$= -\frac{GmM\,\mathbb{a}}{\|\mathbb{a}\|^3}$.

<u>Example 3</u> Let us assume that the force field is given by a potential φ. $\mathbb{f}(\mathbb{x}) = \nabla\varphi(\mathbb{x})$.

A particle travels along the path α. $\alpha(a)=\mathbb{a}$, $\alpha(b)=\mathbb{b}$.

We have shown that

$\begin{cases} \text{the work done by the force field is } \int\nabla\varphi\cdot d\alpha = \varphi(\mathbb{b})-\varphi(\mathbb{a}) \\ \text{the change in the kinetic energy is } \frac{1}{2}m\|v(b)\|^2-\frac{1}{2}m\|v(a)\|^2 = \int\nabla\varphi\cdot d\alpha \end{cases}$

If we define $V(\mathbb{x}) = -\varphi(\mathbb{x})$, then.

$$\frac{1}{2}m\|v(b)\|^2 + V(\alpha(b)) = \frac{1}{2}m\|v(b)\|^2 - \varphi(\alpha(b)) = \frac{1}{2}m\|v(a)\|^2 + V(\alpha(a))$$

The conservation of energy.

10.14 The first fundamental theorem of calculus.

f: a continuous function. $\varphi(x) = \int_a^x f(t)\,dt$ $\quad \varphi'(x) = f(x)$.

For a vector field $\mathbf{f}$, we want to define $\varphi(\mathbf{x}) = \int \mathbf{f}\cdot d\boldsymbol{\alpha}$ for a path $\boldsymbol{\alpha}$ s.t. $\boldsymbol{\alpha}(b) = \mathbf{x}$, $\boldsymbol{\alpha}(a) = \mathbf{a}$.

But this may depend on $\boldsymbol{\alpha}$.

Thm 10.4 Let $\mathbf{f}$ be a continuous vector field on $S \subset \mathbb{R}^n$, where S is connected. Assume that, given $\mathbf{x}, \mathbf{a} \in S$, the line integral $\int \mathbf{f}\cdot d\boldsymbol{\alpha}$ along a path $\boldsymbol{\alpha}$ s.t. $\boldsymbol{\alpha}(b) = \mathbf{x}$, $\boldsymbol{\alpha}(a) = \mathbf{a}$, does not depend on $\boldsymbol{\alpha}$.

We fix $\mathbf{a} \in S$ and define $\varphi(\mathbf{x}) = \int \mathbf{f}\cdot d\boldsymbol{\alpha}$ for some path $\boldsymbol{\alpha}$, $\boldsymbol{\alpha}(b) = \mathbf{x}$, $\boldsymbol{\alpha}(a) = \mathbf{a}$.

Then φ is differentiable and $\nabla\varphi = \mathbf{f}$.

proof). Note that $\varphi(\mathbf{x}+h\mathbf{e}_k) - \varphi(\mathbf{x}) = \int_{\mathbf{a}}^{\mathbf{x}+h\mathbf{e}_k} \mathbf{f}\cdot d\boldsymbol{\alpha}_h - \int_{\mathbf{a}}^{\mathbf{x}} \mathbf{f}\cdot d\boldsymbol{\alpha}_0$ where $\boldsymbol{\alpha}_h$ is a path $\boldsymbol{\alpha}_h(b+h') = \mathbf{x}+h'\mathbf{e}_k$

$= \int \mathbf{f}\cdot d\boldsymbol{\beta}_h$ where $\boldsymbol{\beta}_h(0) = \mathbf{x}$, $\boldsymbol{\beta}_h(h') = \mathbf{x}+h'\mathbf{e}_k$.

$\boldsymbol{\beta}_h'(h') = \mathbf{e}_k$

Therefore, $\frac{\partial\varphi}{\partial x_k}(\mathbf{x}) = \lim_{h\to 0} \frac{\varphi(\mathbf{x}+h\mathbf{e}_k) - \varphi(\mathbf{x})}{h} = \int_0^h \frac{\mathbf{f}(\boldsymbol{\beta}(h'))\cdot\mathbf{e}_k}{h}\,dh' = \mathbf{f}(\mathbf{x})\cdot\mathbf{e}_k = f_k(\mathbf{x})$.

$\Rightarrow \nabla\varphi(\mathbf{x}) = \mathbf{f}(\mathbf{x})$.

For any $\mathbf{y}$, $\varphi(\mathbf{x}+\mathbf{y}) - \varphi(\mathbf{x}) = \int_0^1 \mathbf{f}(\boldsymbol{\alpha}(t))\cdot\boldsymbol{\alpha}'(t)\,dt$. $\boldsymbol{\alpha}(t) = \mathbf{x} + t\mathbf{y}$.

$= \int_0^1 \mathbf{f}(\boldsymbol{\alpha}(t))\cdot\mathbf{y}\,dt = \mathbf{f}(\mathbf{x})\cdot\mathbf{y} + \int_0^1 (\mathbf{f}(\boldsymbol{\alpha}(t)) - \mathbf{f}(\mathbf{x}))\cdot\mathbf{y}\,dt$

and $\|\mathbf{f}(\mathbf{x}+t\mathbf{y}) - \mathbf{f}(\mathbf{x})\| \to 0$ as $\|\mathbf{y}\| \to 0$.

Def. A closed path $\boldsymbol{\alpha}$ is a path s.t. $\boldsymbol{\alpha}(a) = \boldsymbol{\alpha}(b)$.

Any closed path $\boldsymbol{\alpha}$ can be written as $\boldsymbol{\alpha}(t) = \begin{cases} \boldsymbol{\alpha}_1(t) & t \in [a, c] \\ \boldsymbol{\alpha}_2(t) & t\,[c, b] \end{cases}$

Thm 10.5. The following are equivalent for $\mathbf{f}$.

① There is φ s.t. $\mathbf{f} = \nabla\varphi$.

② $\int \mathbf{f}\cdot d\boldsymbol{\alpha}$ does not depend on $\boldsymbol{\alpha}$, so long as $\boldsymbol{\alpha}(b) = \mathbf{x}$, $\boldsymbol{\alpha}(a) = \mathbf{a}$.

③ For any closed path $\boldsymbol{\alpha}$, $\int \mathbf{f}\cdot d\boldsymbol{\alpha} = 0$.

proof). ①$\Leftrightarrow$② is OK. To show ②$\Leftrightarrow$③, closed path passing $\mathbf{a}$ and $\mathbf{x}$ $\Leftrightarrow$ two paths $\boldsymbol{\alpha}, \boldsymbol{\beta}$.
$\boldsymbol{\alpha}(a) = \mathbf{a}$, $\boldsymbol{\alpha}(b) = \mathbf{x}$
$\boldsymbol{\beta}(a) = \mathbf{a}$, $\boldsymbol{\beta}(b) = \mathbf{x}$.

Thm 10.6. Let $\mathbf{f}$ be a smooth vector field.

If $\mathbf{f} = \nabla\varphi$, then $\frac{\partial f_l}{\partial x_k} = \frac{\partial f_k}{\partial x_l}$.

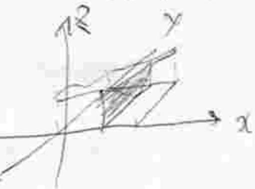
proof) $\frac{\partial f_l}{\partial x_k} = D_k D_l \varphi = D_l D_k \varphi = \frac{\partial f_k}{\partial x_l}$.

Example. $\mathbf{f}(x, y) = (3x^2 y, x^3 y)$. $\frac{\partial f_2}{\partial x} = 3x^2 y$, $\frac{\partial f_1}{\partial y} = 3x^2$. $\Rightarrow$ No φ s.t. $\mathbf{f} = \nabla\varphi$.

Type I region: $S=\{(x,y): a\le x\le b,\ \varphi_1(x)\le y\le \varphi_2(x)\}$

11.13 Area and volume.

Thm 11.19: $\iint_S dx\,dy=\int_a^b\int_{\varphi_1(x)}^{\varphi_2(x)} dy\,dx=\int_a^b(\varphi_2(x)-\varphi_1(x))dx=a(S)$



$V=\{(x,y,z): (x,y)\in S,\ f(x,y)\le z\le g(x,y)\}$

$v(V)=\int_a^b\int_{\varphi_1(x)}^{\varphi_2(x)}(g(x,y)-f(x,y))dy\,dx=\iint_S(g(x,y)-f(x,y))dx\,dy$

11.14. Example 1. Compute the volume of $V=\{(x,y,z): \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1\}$.

We can take $S=\{(x,y): \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\}$, $V=\{(x,y,z): (x,y)\in S, -c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\le z\le c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\}$

$V=\iint_S 2c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dx\,dy=8c\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy\,dx$

Put $A=\sqrt{1-\frac{x^2}{a^2}}$ $\int_0^{bA}\sqrt{A^2-\frac{y^2}{b^2}}\,dy=A\int_0^{bA}\sqrt{1-\frac{y^2}{A^2b^2}}\,dy$ $\boxed{y=Ab\sin t \quad \frac{dy}{dt}=Ab\cos t}$

$=A^2b\int_0^{\frac{\pi}{2}}\cos^2 t\,dt=\frac{\pi}{4}A^2b=\frac{\pi}{4}b\left(1-\frac{x^2}{a^2}\right)$

$V=8c\cdot\frac{\pi}{4}b\int_0^a\left(1-\frac{x^2}{a^2}\right)dx=\frac{4}{3}\pi abc.$

11.16. Further applications.

S: a plate. $f(x,y)$: density at (x,y).

$m(S)=\iint_S f(x,y)dx\,dy$ mass, $a(S)=\iint_S dx\,dy$ area.

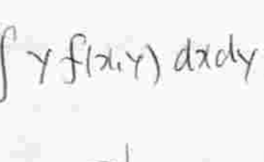
$\frac{m(S)}{a(S)}$: average density.

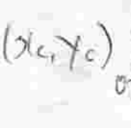
$\bar{x}=\frac{1}{a(S)}\iint x\,dx\,dy$. $\bar{y}=\frac{1}{a(S)}\iint y\,dx\,dy$. $(\bar{x},\bar{y})$: centroid.

$x_c=\frac{1}{m(S)}\iint x f(x,y)dx\,dy$, $y_c=\frac{1}{m(S)}\iint y f(x,y)dx\,dy$. (x_c,y_c) center of mass.

11.17 Theorem of Pappus

$Q:=\{(x,y): 0\le g(x)\le y\le f(x),\ a\le x\le b\}$





Rotate Q around the x-axis to obtain a solid.

The volume is $\int_a^b \pi(f(x)^2-g(x)^2)dx$.

$\bar{y}=\frac{1}{a(Q)}\iint_Q y\,dx\,dy=\frac{1}{a(Q)}\int_a^b\int_{f(x)}^{g(x)} y\,dy\,dx=\frac{1}{a(Q)}\int_a^b\frac{1}{2}(f(x)^2-g(x)^2)dx$

$\Rightarrow v(S)=2\pi a(Q)\cdot\bar{y}$.

"Thm" Let A and B disjoint plates, C_A, C_B the centers of mass

Then $C_{A\cup B}=\frac{m(A)C_A+m(B)C_B}{m(A)+m(B)}$

11.19 Green's theorem.

<u>Thm 11.10</u>. Let $f(x,y) = (P(x,y), Q(x,y))$ be a continuously differentiable vector field on S. Let R a type I/II region in S, C its boundary $\alpha(t)$.

Then $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx dy = \int_C f \cdot d\alpha$  anti-clockwise. smooth.

connected

This holds even if R is a union of finite type I/II regions. with boundary a curve α.

proof) $R = \{(x,y) : a \le x \le b,\ \varphi_1(x) \le y \le \varphi_2(x)\}$

Assume $Q=0$. $\iint_R -\frac{\partial P}{\partial y} dx dy = -\int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial P}{\partial y} dy dx = \int_a^b [P(x,\varphi_1(x)) - P(x,\varphi_2(x))] dx$

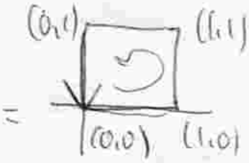
$\alpha_1(t) = (t, \varphi_1(t))$, $\alpha_2(t) = (t, \varphi_2(t))$.

$\int f \cdot d\alpha = \int f \cdot d\alpha_1 - \int f \cdot d\alpha_2 = \int_a^b [P(x,\varphi_1(x)) - P(x,\varphi_2(x))] dx$.

If R is a union,  the theorem holds on each component. $\sum \int_{C_k} f \cdot d\alpha_k = \int f \cdot d\alpha$.

A type II region is a union of type I regions. theorem holds for $P=0$, $Q \neq 0$.
$\Rightarrow$ general Q, P.

11.20 applications.

$f(x,y) = (5 - xy - y^2, -2xy + x^2)$ $\quad \alpha(t) =$ 

$\int f \cdot d\alpha = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx dy = \iint 3x\, dx dy = \frac{3}{2}$.

$a(R) = \iint_R dx dy = \int f \cdot d\alpha$ where $f = (0, x)$, or $(-y, 0)$, or $\frac{1}{2}(-y, x)$.

11.21. Gradient.

<u>Def</u>. A path is a ~~simple~~ step-polygon if each component is parallel to the x-axis or the y-axis.
A region S is simply connected if two step-polygons α_1, α_2 s.t. $\alpha_1(a) = \alpha_2(a)$, $\alpha_1(b) = \alpha_2(b)$ can be always continuously transformed to each other within step-polygons in S.

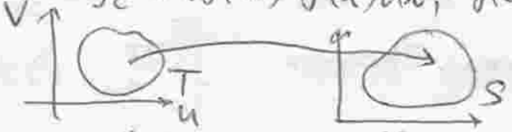
<u>Thm 11.11</u> Let $f(x,y) = (P(x,y), Q(x,y))$, continuously differentiable on S, S simply connected. Then $f = \nabla\varphi \iff \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$.

proof) We know $\Rightarrow$.
Take two step polygons α_1, α_2. $\alpha_1(a) = \alpha_2(a)$. $\alpha_1(b) = \alpha_2(b)$.
If $\int f d\alpha_1 = \int f \cdot d\alpha_2$, then by the proof of Thm 10.4. it is OK.
By simple connectedness, we may assume that α_1, α_2 are close.
The rectangles formed by α_1, α_2 are in S. $\int f \cdot d\alpha_1 - \int f \cdot d\alpha_2 = \sum_k \pm \iint_{R_k} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx dy = 0$

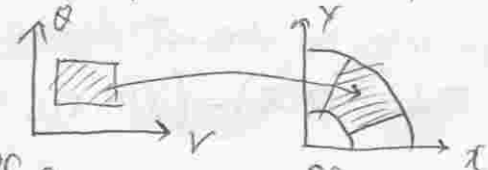
11.26 Change of variables

In one-dimension, $\int_a^b f(x)dx = \int_c^d f(g(t))\, g'(t)dt$, $g(c)=a$, $g(d)=b$

In two-dimension, $x = X(u,v)$
$\qquad\qquad\qquad\quad Y = Y(u,v)$

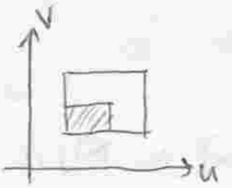


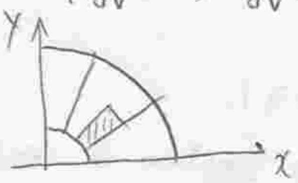
Example. $x = X(r,\theta) = r\cos\theta$
$\qquad\qquad Y = Y(r,\theta) = r\sin\theta.$

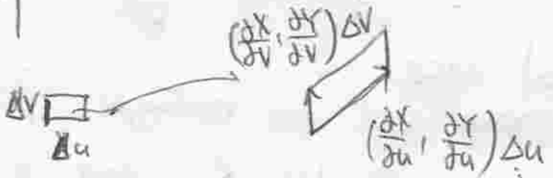


In a good situation, we have $\iint_S f(x,y)\,dx\,dy = \iint_T f(X(u,v), Y(u,v))|J(u,v)|du\,dv$

where $J(u,v) = \begin{vmatrix} \frac{\partial X}{\partial u}(u,v) & \frac{\partial Y}{\partial u}(u,v) \\ \frac{\partial X}{\partial v}(u,v) & \frac{\partial Y}{\partial v}(u,v) \end{vmatrix}$ Jacobian determinant.







Area $\Delta u\cdot\Delta V$ $\qquad$ $\Delta u\, \Delta V \cdot \begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial Y}{\partial u} \\ \frac{\partial X}{\partial v} & \frac{\partial Y}{\partial v} \end{vmatrix}$

11.27 Example.1

$x = X(r,\theta) = r\cos\theta$, $Y = Y(r,\theta) = r\sin\theta$.

$J(r,\theta) = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r.$

Compute the area  $T = \{(r,\theta) : 0 \le r \le a,\ 0 \le \theta \le \frac{\pi}{2}\}$

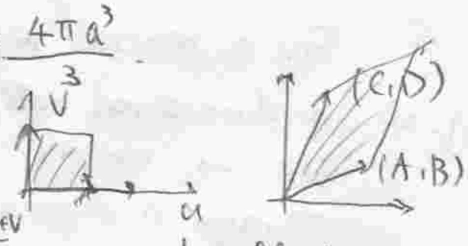
$$\iint_S dx\,dy = \iint_T r\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\int_0^a r\,dr = \frac{\pi}{4}a^2.$$

Volume of a sphere. $S = \{(x,y) : \sqrt{x^2+y^2} \le a\}$ $T = \{(r,\theta) : 0 \le r \le a, 0 \le \theta \le 2\pi\}$

$$\iint_S 2\sqrt{a^2-x^2-y^2}\,dx\,dy = \iint_T 2\sqrt{a^2-r^2}\cdot r\,dr\,d\theta = 2\int_0^{2\pi}\int_0^a r\sqrt{a^2-r^2}\,dr\,d\theta$$

$$= 4\pi\cdot\left[-\frac{1}{3}(a^2-r^2)^{\frac{3}{2}}\right]_0^a = \frac{4\pi a^3}{3}.$$

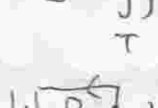
Ex.2. $x = Au + Bv$, $Y = Cu + Dv$. $J = \begin{vmatrix} A & B \\ C & D \end{vmatrix}$



$$\iint_S e^{\frac{x+y}{x-y}}\,dx\,dy. \quad \begin{array}{l} x+y=u \\ x-y=v \end{array} \Rightarrow \begin{array}{l} x=\frac{u+v}{2} \\ Y=\frac{u-v}{2} \end{array} \quad J = -\frac{1}{2}. \quad \iint_T e^{\frac{u}{v}}\,du\,dv$$

11.29 Proof.

$f(x,y)=0$, S is a rectangle. $R \Longleftrightarrow R^*$

We assume that $J(u,v) > 0$ on R^*. X, Y smooth. 

We have to show $\iint_R dx\,dy = \iint_{R^*} J(u,v)\,du\,dv$.

$\iint_R dx\,dy = \int \mathbf{f}\cdot d\alpha$, where $\mathbf{f}(x,y) = (0,x)$ by Green's theorem.

On the other hand, $J(u,v)=\frac{\partial X}{\partial u}\cdot\frac{\partial Y}{\partial v}-\frac{\partial Y}{\partial u}\frac{\partial X}{\partial v}=\frac{\partial}{\partial u}\left(X\frac{\partial Y}{\partial v}\right)-\frac{\partial}{\partial v}\left(X\cdot\frac{\partial Y}{\partial u}\right)$

Put $g(u,v)=\left(X(u,v)\frac{\partial Y}{\partial u}(u,v),\ X(u,v)\frac{\partial Y}{\partial v}(u,v)\right)$

Then by Green's theorem. $\iint_{R^*} J(u,v)\,du\,dv=\int g\cdot d\beta$. where 

As the line integral does not depend on parametrization,

we can take $\alpha(t)=(X(\beta(t)),Y(\beta(t)))$. $\alpha'(t)=\begin{pmatrix}\frac{\partial X}{\partial u}(\beta(t))\beta_1'(t)+\frac{\partial X}{\partial v}(\beta(t))\beta_2'(t)\\ \frac{\partial Y}{\partial u}(\beta(t))\beta_1'(t)+\frac{\partial Y}{\partial v}(\beta(t))\beta_2'(t)\end{pmatrix}$

$$\iint_R dx\,dy=\int f\cdot d\alpha=\int X(\beta(t))\left(\frac{\partial Y}{\partial u}(\beta(t))\cdot\beta_1'(t)+\frac{\partial Y}{\partial v}(\beta(t))\,\beta_2'(t)\right)dt$$

$$=\int g\cdot d\beta=\iint_{R^*}J(u,v)\,du\,dv.$$

11.30 General case.

We prove first for rectangle, $f(x,y)\neq 1$.

For small rectangles R_{ij} and a step function S constant on R_{ij},

$$\iint_{R_{ij}} S(x,y)\,dx\,dy=\iint_{R_{ij}^*} S(X(u,v),Y(u,v))\,|J(u,v)|\,du\,dv.$$

As integral is additive w.r.t. regions, we have

$\iint_R S(x,y)\,dx\,dy=\iint_R S(X,Y)\,|J(u,v)|\,du\,dv$ for step functions.

For integrable f, take step functions s,t such that $s(x,y)\le f(x,y)\le t(x,y)$.

$$\begin{array}{ccccc}\iint_R s(x,y)\,dx\,dy & \le & \iint_R f(x,y)\,dx\,dy & \le & \iint_R t(x,y)\,dx\,dy\\ \| & & & & \|\\ \iint_{R^*} s(X,Y)|J(u,v)|\,du\,dv & \le & \iint_{R^*} f(X,Y)|J(u,v)|\,du\,dv & = & \iint_{R^*} t(X,Y)|J(u,v)|\,du\,dv\end{array}$$

This holds for any test function, hence $\iint_R f(x,y)\,dx\,dy=\iint_{R^*} f(X,Y)|J(u,v)|\,du\,dv$

For a general region, find a larger rectangle.

11.30 n-dimensional integral.

We can consider multiple integral $\iint_S\cdots\int f(x_1,\dots,x_n)\,dx_1dx_2\dots dx_n$.

through n-dimensional step functions, integrable functions.

It can be reduced to $\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} f(x_1,\dots,x_n)\,dx_n\dots dx_1$.

11.31. Change of variables

$$\iint_S\cdots\int f(x_1,\dots,x_n)\,dx_1\dots dx_n=\iint_T\cdots\int f(X_1(u_1,\dots,u_n),\dots,X_n(u_1,\dots,u_n))\,|J(u_1,\dots,u_n)|\,du_1\dots du_n$$

$x_k=X_k(u_1,\dots,u_n)$. $J(u_1,\dots,u_n)=\begin{vmatrix}\frac{\partial X_1}{\partial u_1} & \cdots & \frac{\partial X_n}{\partial u_1}\\ \vdots & & \vdots\\ \frac{\partial X_1}{\partial u_n} & \cdots & \frac{\partial X_n}{\partial u_n}\end{vmatrix}$.

12.1. Parametric representation of a surface.

Consider $x^2+y^2+z^2=1$ (unit sphere) One has $z=\pm\sqrt{1-x^2-y^2}$ or.

$x = X(x,y)$, $y=Y(x,y)$, $z=\sqrt{1-x^2-y^2}=Z(x,y)$.

Alternatively, $x=X(u,v)=\cos u\cos v$, $y=Y(u,v)=\sin u\cos v$,

$z=Z(u,v)=\sin v$.

c.f. Parametrization of the circle $x^2+y^2=1$. $y=\pm\sqrt{1-x^2}$ / $x=\cos t$, $y=\sin t$.

Ex. 2. cone: $x = v\cos\alpha\cos u$, $y=v\cos\alpha\sin u$, $z=v\sin\alpha$

12.2 Vector product.

$x=(x_1,x_2,x_3)$, $y=(y_1,y_2,y_3)$. $x\times y=(x_2y_3-x_3y_2,\ x_3y_1-x_1y_3,\ x_1y_2-x_2y_1)$.

Let $\mathbb{r}(u,v)=(X(u,v),Y(u,v),Z(u,v))$ be a parametrized surface.

$\frac{\partial\mathbb{r}}{\partial u}$, $\frac{\partial\mathbb{r}}{\partial v}$: tangent to the surface.

$$\frac{\partial\mathbb{r}}{\partial u}\times\frac{\partial\mathbb{r}}{\partial v}=\left(\frac{\partial Y}{\partial u}\frac{\partial Z}{\partial v}-\frac{\partial Z}{\partial u}\frac{\partial Y}{\partial v},\ \frac{\partial Z}{\partial u}\frac{\partial X}{\partial v}-\frac{\partial X}{\partial u}\frac{\partial Z}{\partial v},\ \frac{\partial X}{\partial u}\frac{\partial Y}{\partial v}-\frac{\partial Y}{\partial u}\frac{\partial X}{\partial v}\right)$$ is orthogonal to the surface (normal vector).

On a point $(X(u,v),Y(u,v),Z(u,v))=\mathbb{r}(u,v)$, if $\frac{\partial\mathbb{r}}{\partial u}\times\frac{\partial\mathbb{r}}{\partial v}\neq 0$ (if $\frac{\partial\mathbb{r}}{\partial u}$ and $\frac{\partial\mathbb{r}}{\partial v}$ are linearly independent), we say $\mathbb{r}(u,v)$ is a regular point,

Ex. 2. Sphere.

$\mathbb{r}(u,v)=(a\cos u\cos v,\ a\sin u\cos v,\ a\sin v)$.

$\frac{\partial\mathbb{r}}{\partial u}(u,v)=(-a\sin u\cos v,\ a\cos u\cos v,\ 0)$

$\frac{\partial\mathbb{r}}{\partial v}(u,v)=(-a\cos u\sin v,\ -a\sin u\sin v,\ a\cos v)$

$$\frac{\partial\mathbb{r}}{\partial u}\times\frac{\partial\mathbb{r}}{\partial v}=(a^2\cos u\cos^2 v,\ a^2\sin u\cos v,\ a^2(\sin^2 u\cos v\sin v+\cos^2 u\cos v\sin v))$$
$$=a\cos v\cdot\mathbb{r}(u,v).$$

12.5 Area of a surface.

$\left\|\frac{\partial\mathbb{r}}{\partial u}\times\frac{\partial\mathbb{r}}{\partial v}\right\|\Delta u\Delta v$ should correspond to the area of a small piece on the surface.

<u>Def</u>. If S is parametrized by u,v as $\mathbb{r}(u,v)$, $=(X(u,v),Y(u,v),Z(u,v))$ where $(u,v)\in T$. Then the area is defined by

$$a(S)=\iint_T\left\|\frac{\partial\mathbb{r}}{\partial u}\times\frac{\partial\mathbb{r}}{\partial v}\right\|du\,dv$$

If $Z(u,v)=0$, then the surface is on the x-y plane and

$$a(S)=\iint_T\underbrace{\left|\frac{\partial X}{\partial u}\frac{\partial Y}{\partial v}-\frac{\partial Y}{\partial u}\frac{\partial X}{\partial v}\right|}_{J(u,v)}du\,dv=\iint_S dx\,dy$$ by change of variable.

Ex. 1 Hemisphere. $\mathbf{r}(u,v) = (a\cos u\cos v,\ a\sin u\cos v,\ a\sin v)$

$\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v} = a\cos v\cdot\mathbf{r}$ $\quad \left\|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right\| = a^2\cos v$

$\iint_T a^2\cos v\,dv\,du = a^2\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\cos v\,dv\,du = 2\pi a^2 \Rightarrow$ full sphere $4\pi a^2$.

Another parametrization. $\mathbf{r}(u,v) = (u, v, \sqrt{a^2-u^2-v^2})$

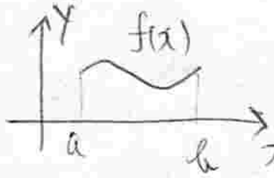
$\frac{\partial \mathbf{r}}{\partial u} = \left(1, 0, \frac{-u}{\sqrt{a^2-u^2-v^2}}\right)$, $\frac{\partial \mathbf{r}}{\partial v} = \left(0, 1, \frac{-v}{\sqrt{a^2-u^2-v^2}}\right)$.

$\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v} = \left(\frac{-u}{\sqrt{a^2-u^2-v^2}}, \frac{-v}{\sqrt{a^2-u^2-v^2}}, 1\right)$ $\quad \left\|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right\| = \frac{a}{\sqrt{a^2-u^2-v^2}}$

$\iint_T \frac{a}{\sqrt{a^2-u^2-v^2}}\,du\,dv = \int_0^{2\pi}\int_0^a \frac{a}{\sqrt{a^2-r^2}}\,r\,dr\,d\theta = a\int_0^{2\pi}\left[-\sqrt{a^2-r^2}\right]_0^a d\theta = 2\pi a^2.$

(improper integral).

Ex. 2. Pappus' theorem.



rotate around x-axis $\Longrightarrow$ surface. $\quad \alpha(x) = (x, f(x), 0),\ \alpha'(x) = (1, f'(x), 0)$

$s(x) = \int_a^x \sqrt{1+f'(x)^2}\,dx$

$L = \int_a^b \sqrt{1+f'(x)^2}\,dx$, $\bar{x} = \frac{1}{L}\int_a^b x\sqrt{1+f'(x)^2}\,dx$ centroid.

$\mathbf{r}(u,\theta) = (u\cos\theta, u\sin\theta, f(u))$.

$\frac{\partial \mathbf{r}}{\partial u} = (\cos\theta, \sin\theta, f'(u))$, $\frac{\partial \mathbf{r}}{\partial \theta} = (-u\sin\theta, u\cos\theta, 0)$,

$\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial \theta} = (-uf'(u)\cos\theta, -uf'(u)\sin\theta, u)$. $\left\|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial \theta}\right\| = u\sqrt{1+f'(u)^2}$.

$\iint_T \left\|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial \theta}\right\| du\,d\theta = \int_0^{2\pi}\int_a^b u\sqrt{1+f'(u)^2}\,du = 2\pi L\bar{x}.$

12-7 Surface integral.

Let S be a surface, parametrized by smooth $\mathbf{r}(u,v)$ on T.

Let f be a scalar field on S. The surface integral of f over S is

$$\iint_S f\,dS := \iint_T f(\mathbf{r}(u,v))\left\|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right\| du\,dv.$$

Example 1. $f = 1$. This gives the area of S

$$a(S) = \iint_T \left\|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right\| du\,dv$$

Example 2. If a curved plate is put on S, density $f(\mathbf{r}(u,v))$,

Mass $M = \iint_T f(\mathbf{r}(u,v))\left\|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right\| du\,dv$.

The centre of mass $x_c = \frac{1}{M}\iint_T X(u,v) f(\mathbf{r}(u,v))\left\|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right\| du\,dv$, y_c, z_c.

12.8. Change of parametrization

S: surface, $\mathbf{r}(u,v)$: a parametrization, $(u,v)\in A$

$\mathbf{G}$: change of variables $\mathbf{G}(s,t) = (U(s,t), V(s,t))$: $B \longrightarrow A$

$\mathbf{R}(s,t) := \mathbf{r}(\mathbf{G}(s,t))$: new parametrization of S on B

Thm 12.1 Let $\mathbf{r}$ and $\mathbf{R}$ be two smooth parametrizations of S related by a smooth map $\mathbf{G}$ as above. Then we have

$$\frac{\partial \mathbf{R}}{\partial s}\times\frac{\partial \mathbf{R}}{\partial t}(s,t) = \frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}(U(s,t),V(s,t))\cdot\left(\frac{\partial U}{\partial s}\frac{\partial V}{\partial t}-\frac{\partial V}{\partial s}\frac{\partial U}{\partial t}\right)$$

proof) $\dfrac{\partial \mathbf{R}}{\partial s} = \dfrac{\partial \mathbf{r}}{\partial u}\dfrac{\partial U}{\partial s}+\dfrac{\partial \mathbf{r}}{\partial v}\dfrac{\partial V}{\partial s}$, $\dfrac{\partial \mathbf{R}}{\partial t} = \dfrac{\partial \mathbf{r}}{\partial u}\dfrac{\partial U}{\partial t}+\dfrac{\partial \mathbf{r}}{\partial v}\dfrac{\partial V}{\partial t}$ (chain rule)

and $\dfrac{\partial \mathbf{R}}{\partial s}\times\dfrac{\partial \mathbf{R}}{\partial t} = \dfrac{\partial \mathbf{r}}{\partial u}\times\dfrac{\partial \mathbf{r}}{\partial v}\left(\dfrac{\partial U}{\partial s}\dfrac{\partial V}{\partial t}-\dfrac{\partial V}{\partial s}\dfrac{\partial U}{\partial t}\right)$.

Recall that a surface integral of a function f is

$$\iint_S f\cdot dS := \iint_A f(\mathbf{r}(u,v))\left\|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right\| du\,dv \overset{?}{=} \iint_B f(\mathbf{R}(s,t))\left\|\frac{\partial \mathbf{R}}{\partial s}\times\frac{\partial \mathbf{R}}{\partial t}\right\| ds\,dt.$$

$(U(s,t), V(s,t))$

Thm 12.2 Let $\mathbf{r}, \mathbf{R}, \mathbf{G}$ smooth as above and f a function on S. If the first integral exists, then so does the second and

$$\iint_A f(\mathbf{r}(u,v))\left\|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right\| du\,dv = \iint_B f(\mathbf{R}(s,t))\left\|\frac{\partial \mathbf{R}}{\partial s}\times\frac{\partial \mathbf{R}}{\partial t}\right\| ds\,dt.$$

proof) By change of variables for double integral, we have

$$\iint_A f(\mathbf{r}(u,v))\left\|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right\| du\,dv = \iint_B f(U(s,t),V(s,t))\left\|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial t}(U(s,t),V(s,t))\right\|\cdot\left|\begin{matrix}\frac{\partial U}{\partial s} & \frac{\partial V}{\partial s}\\ \frac{\partial U}{\partial t} & \frac{\partial U}{\partial t}\end{matrix}\right| ds\,dt$$

$$= \iint_B f(\mathbf{R}(t,s))\left\|\frac{\partial \mathbf{R}}{\partial s}\times\frac{\partial \mathbf{R}}{\partial t}\right\| ds\,dt \quad \text{by Thm 12.1}$$

12.9. Other notations.

Let $\mathbf{r}(u,v)$ be a surface parametrization. $\dfrac{\partial \mathbf{r}}{\partial u}\times\dfrac{\partial \mathbf{r}}{\partial v} = \mathbf{N} \neq 0$.

Set $\mathbf{n}_1 = \dfrac{\mathbf{N}}{\|\mathbf{N}\|}$, $\mathbf{n}_2 = -\mathbf{n}_1$. If $\mathbf{F}$ is a vector field, we can consider

$$\iint_S \mathbf{F}\cdot\mathbf{n}_1\, dS = \iint_A \mathbf{F}(\mathbf{r}(u,v))\cdot\mathbf{n}_1(u,v)\left\|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right\| du\,dv = \iint_A \mathbf{F}(\mathbf{r}(u,v))\cdot\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\, du\,dv$$

If one uses $\mathbf{n}_2$ instead of $\mathbf{n}_1$, one gets -1.

If $\mathbf{F}(\mathbf{r}(u,v)) = (P(\mathbf{r}(u,v)), Q(\mathbf{r}(u,v)), R(\mathbf{r}(u,v)))$ by components,

This is sometimes written as $\int P\, dy\wedge dz + Q\, dz\wedge dx + R\, dx\wedge dy$.

12.10 Exercise (a). $S: x^2+y^2+z^2=1$, $z\geq 0$. $\mathbf{F}(x,y,z) = (x,y,0)$. Compute $\iint \mathbf{F}\cdot\mathbf{n}_1\, dS$.

$\mathbf{r}(u,v) = (\sin u\cos v, \sin u \sin v, \cos u)$. $0\leq u\leq \frac{\pi}{2}$, $0\leq v\leq 2\pi$.

$\dfrac{\partial \mathbf{r}}{\partial u}\times\dfrac{\partial \mathbf{r}}{\partial v} = \sin u\,\mathbf{r}$. $\mathbf{F}\cdot\dfrac{\partial \mathbf{r}}{\partial u}\times\dfrac{\partial \mathbf{r}}{\partial v} = \sin^3 u$. $\iint_T \mathbf{F}\cdot\mathbf{n}_1\, dS = \iint \sin^3 u\, du\,dv = \frac{4}{3}\pi$.

12.11. Stokes' theorem.

Green's theorem. $\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx dy = \int_C \mathbb{f}\cdot d\alpha$, where $\mathbb{f} = (P, Q)$.

Thm 12.3 (Stokes) Assume that S is a surface with boundary Γ, parametrized by $\mathbb{r}(u,v)$ on T with boundary C parametrized by a single curve α, and T is simply connected. Assume that $\mathbb{r}$ and α ~~are smooth~~. have continuous second derivatives, P, Q, R functions, continuously differentiable.

Then we have, for $\mathbb{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ and
$\mathbb{H} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$, $\iint_S \mathbb{H}\cdot \mathbb{n}, dS = \int_\Gamma \mathbb{F}\cdot d\beta$ $\quad \beta(t) = \mathbb{r}(\alpha(t))$.

proof). Recall that $\iint_S \mathbb{H}\cdot\mathbb{n}, dS = \iint_T \mathbb{H}\cdot \frac{\partial \mathbb{r}}{\partial u}\times\frac{\partial \mathbb{r}}{\partial v} du dv$. $\mathbb{r}(u,v) = (X(u,v), Y(u,v), Z(u,v))$

We first assume that $Q = R = 0$. We have to ~~prove~~ compute.

$$\iint_T \left(0, \frac{\partial P}{\partial z}, -\frac{\partial P}{\partial y}\right)\cdot \frac{\partial \mathbb{r}}{\partial u}\times\frac{\partial \mathbb{r}}{\partial v} du dv = \iint_T -\frac{\partial P}{\partial y}\left(\frac{\partial X}{\partial u}\frac{\partial Y}{\partial v} - \frac{\partial Y}{\partial u}\frac{\partial X}{\partial v}\right) + \frac{\partial P}{\partial z}\left(\frac{\partial Z}{\partial u}\frac{\partial X}{\partial v} - \frac{\partial X}{\partial u}\frac{\partial Z}{\partial v}\right) du dv$$

Note that $\frac{\partial}{\partial u}\left(P(\mathbb{r}(u,v))\frac{\partial X}{\partial v}\right) - \frac{\partial}{\partial v}\left(P(\mathbb{r}(u,v))\frac{\partial X}{\partial u}\right) = -\frac{\partial P}{\partial y}\left(\frac{\partial X}{\partial u}\frac{\partial Y}{\partial v} - \frac{\partial Y}{\partial u}\frac{\partial X}{\partial v}\right) + \frac{\partial P}{\partial z}\left(\frac{\partial Z}{\partial u}\frac{\partial X}{\partial v} - \frac{\partial X}{\partial u}\frac{\partial Z}{\partial v}\right)$.

Hence by Green's theorem,

$$= \int_C \left(P(\mathbb{r}(u,v))\frac{\partial X}{\partial u}, P(\mathbb{r}(u,v))\frac{\partial X}{\partial v}\right)\cdot d\alpha = \int (P, 0, 0)\cdot \frac{d}{dt}\mathbb{r}(\alpha(t)) dt = \int_\Gamma (P, 0, 0)\cdot d\beta$$

the ~~one~~ proof works for P, Q, and the theorem is the sum of three cases.

12.12. Curl and divergence.

Let $\mathbb{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ be a vector field.

Define $\operatorname{curl}\mathbb{F} = \left(\frac{\partial Q}{\partial y} - \frac{\partial R}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$.

Thm of Stokes says: $\iint_S \operatorname{curl}\mathbb{F} = \int_\Gamma \mathbb{F}\cdot d\alpha$.

One also writes $\operatorname{curl}\mathbb{F} = \nabla\times\mathbb{F}$.

$\operatorname{div}\mathbb{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.

Thm 12.4 Let $\mathbb{F}$ be a vector field on an open convex set. Then.

$\mathbb{F}$ is a gradient $\iff$ $\operatorname{curl}\mathbb{F} = 0$.

Exercise

1. $\mathbb{F}(x,y,z) = (y^2, xy, xz)$. $S: x^2+y^2+z^2 = 1$, $z \ge 0$. Compute $\iint_S \mathbb{F}\cdot\mathbb{n}, dS$

By Stokes this is $\int \mathbb{F}\cdot d\alpha$. $\alpha = (\cos t, \sin t, 0)$.

$\alpha'(t) = (-\sin t, \cos t, 0)$, $\mathbb{F}(\alpha(t)) = (\cos^2 t, \sin t\cos t, 0)$. $\mathbb{F}\cdot\alpha'(t) = 0$.

12.14 More on curl and divergence.

Let $\mathbb{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ be a vector field.

Recall that $\operatorname{curl}\mathbb{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$, $\operatorname{div}\mathbb{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.

Example 1. $\mathbb{F}(x,y,z) = (x,y,z)$. $\operatorname{curl}\mathbb{F} = (0,0,0)$, $\operatorname{div}\mathbb{F} = 1+1+1 = 3$.

Example 2. $\mathbb{F}(x,y,z) = (xy^2z^2, z^2\sin y, x^2e^y)$.

$\operatorname{curl}\mathbb{F} = (x^2e^y - 2z\sin y, 2xy^2z - 2xe^y, 2xyz^2)$. $\operatorname{div}\mathbb{F} = y^2z^2 + z^2\cos y$

Example 3 $\mathbb{F}(x,y,z) = \operatorname{grad}\varphi = \left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\right)$

$\operatorname{curl}\mathbb{F} = \left(\frac{\partial^2\varphi}{\partial y\partial z} - \frac{\partial^2\varphi}{\partial z\partial y}, \frac{\partial^2\varphi}{\partial z\partial x} - \frac{\partial^2\varphi}{\partial x\partial z}, \frac{\partial^2\varphi}{\partial x\partial y} - \frac{\partial^2\varphi}{\partial y\partial x}\right) = (0,0,0)$

if φ has continuous second derivatives.

$\operatorname{div}\mathbb{F} = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$ called Laplacian of φ. $\nabla^2\varphi$.

Example 4 $\mathbb{F}(x,y,z) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$.

$\operatorname{curl}\mathbb{F} = \left(0, 0, \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}\right) = (0,0,0)$.

$\operatorname{div}\mathbb{F} = \frac{2xy}{(x^2+y^2)^2} + \frac{-2xy}{(x^2+y^2)^2} + 0 = 0$.

Example 5 $\mathbb{F}$ general. $\operatorname{div}(\operatorname{curl}\mathbb{F}) = 0$, $\operatorname{curl}(\operatorname{curl}\mathbb{F}) = \operatorname{grad}(\operatorname{div}\mathbb{F}) - \nabla^2\mathbb{F}$.

$(\nabla\cdot(\nabla\times\mathbb{F}) = 0$, $\nabla\times(\nabla\times\mathbb{F}) = \nabla(\nabla\cdot\mathbb{F}) - \nabla^2\mathbb{F}$.

12.18 Extension of Stokes' theorem.

Green's theorem. $\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy = \int_C \mathbb{f}\cdot d\alpha$ where $\mathbb{f} = (P,Q)$.

and S has the boundary C, a single curve α.

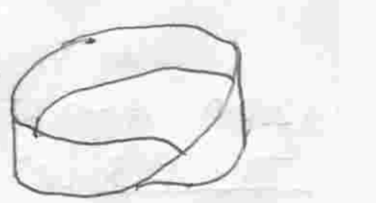
This can be extended to regions with holes:

$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy = \iint_{S_1\cup S_2}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy = \int_{C_1\cup C_2} \mathbb{f}\cdot d\alpha = \int_C \mathbb{f}\cdot d\alpha$.

Stokes' theorem can be also extended to surfaces with holes, if it can be oriented. cylinder

OK.

Mobius band



NG

A sphere

can be decomposed into two hemispheres with normal vectors going outside.

The boundary terms cancel.

$\iint_S \operatorname{curl}\mathbb{F}\, dS = \iint_{S_+\cup S_-} \operatorname{curl}\mathbb{F}\, dS = \iint_{C_1\cup C_2} \mathbb{F}\cdot\alpha = 0$.

12.19 Gauss' theorem.

<u>Def</u>. A solid V is said to be xy-projectable if there are functions f, g such that $V = \{(x,y,z) : g(x,y) \leq z \leq f(x,y), (x,y) \in A\}$ where $A \subset \mathbb{R}^2$.
Similary, yz- and zx-projectable.

<u>Example</u> ~~Sphere~~ V: Ball $x^2+y^2+z^2 \leq 1$. $A = \{(x,y) : x^2+y^2 \leq 1\}$. $V = \{(x,y,z) : -\sqrt{1-x^2-y^2} \leq z \leq \sqrt{1-x^2-y^2}, (x,y) \in A\}$

<u>Thm 12.6 (Gauss)</u> Let V be a solid, xy-, yz-, zx-projectable, or a finite union of projectable solids. Let $\mathbb{n}$ be the unit outer vector on the smooth surface S of V. Then $\iiint_V \mathrm{div}\,\mathbb{F}\,dxdydz = \iint_S \mathbb{F}\cdot\mathbb{n}\,ds$ for a smooth vector field $\mathbb{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$.

proof). Let V be xy-projectable, $V = \{(x,y,z) : g(x,y,z) \leq z \leq f(x,y,z), (x,y) \in A\}$.
The surface S of V consists of $S_1 = \{(x,y,z) : z = f(x,y), (x,y) \in A\}$,
$S_2 = \{(x,y,z) : z = g(x,y), (x,y) \in A\}$
∂A is the boundary of A. $S_3 = \{(x,y,z) : g(x,y) \leq z \leq f(x,y), (x,y) \in \partial A\}$.

Let us first assume that $\mathbb{F} = (0,0,R)$. $\mathrm{div}\,\mathbb{F} = \frac{\partial R}{\partial z}$.

$$\iiint_V \mathrm{div}\,\mathbb{F}\,dxdydz = \iint_A \int_{g(x,y)}^{f(x,y)} \frac{\partial R}{\partial z}\,dz\,dxdy = \iint_A \left[R(x,y,f(x,y)) - R(x,y,g(x,y))\right] dxdy.$$

On S_1, the parametrization is $\mathbb{r}(u,v) = (u,v,f(u,v))$. $\frac{\partial \mathbb{r}}{\partial u} = (1,0,\frac{\partial f}{\partial u})$, $\frac{\partial \mathbb{r}}{\partial v} = (0,1,\frac{\partial \mathbb{r}}{\partial v})$.
hence $\frac{\partial \mathbb{r}}{\partial u} \times \frac{\partial \mathbb{r}}{\partial v} = (-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1)$, $\iint_{S_1} \mathbb{F}\cdot\mathbb{n}\,ds = \iint_A R(u,v,f(u,v))\,dudv$

On S_2, the ~~vecto~~ normal vector is downwards, so $\frac{\partial \mathbb{r}}{\partial v} \times \frac{\partial \mathbb{r}}{\partial u} = (\frac{\partial g}{\partial u}, \frac{\partial g}{\partial u}, -1)$.
and $\iint_{S_2} \mathbb{F}\cdot\mathbb{n}\,ds = -\iint_A R(u,v,g(u,v))\,dudv$.

On S_3, the normal vector is orthogonal to z-axis. $\iint_{S_3} \mathbb{F}\cdot\mathbb{n}\,ds = 0$.
Altogether, $\iiint_V \mathrm{div}\,\mathbb{F}\,dxdydz = \iint_S \mathbb{F}\cdot\mathbb{n}\,ds$.

We can argue similarly for P, Q, then take the sum. A general V is OK.

12.20 Applications.

<u>Thm 12.7</u> Let $V(t)$ be a ball of radius t centred at $\mathbb{a}$, $S(t)$ the boundary. $\mathbb{F}$: smooth vector field. Then $\mathrm{div}\,\mathbb{F}(\mathbb{a}) = \lim_{t\to 0} \frac{1}{\mathrm{vol}\,V(t)} \iint_{S(t)} \mathbb{F}\cdot\mathbb{n}\,dS$.

proof). By Gauss' theorem. $\iint_{S(t)} \mathbb{F}\cdot\mathbb{n}\,ds = \iiint_{V(t)} \mathrm{div}\,\mathbb{F}\,dxdydz$.
As $\mathbb{F}$ is smooth, $|\mathrm{div}\,\mathbb{F}(x) - \mathrm{div}\,\mathbb{F}(\mathbb{a})| \leq E(x,\mathbb{a})$, $E(x,\mathbb{a}) \to 0$ as $x \to \mathbb{a}$.
$\lim_{t\to 0} \frac{1}{\mathrm{vol}\,V(t)} \iint \mathbb{F}\cdot\mathbb{n}\,ds = \mathrm{div}\,\mathbb{F}(\mathbb{a})$.