BSc Engineering Sciences – A. Y. 2017/18

Written exam of the course Mathematical Analysis 2

September 17, 2018

Solve the following problems, motivating in detail the answers.

1. Study the conditional, absolute and uniform convergence of the series

$$\sum_{n=0}^{+\infty} \frac{1}{3^n}(\sqrt{n+1}-\sqrt{n})(2x^2-5)^n \ .$$

Solution. Let us first note that $\sqrt{n+1}-\sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$, hence the series is equal to

$$\sum_{n=0}^{+\infty} \frac{1}{3^n(\sqrt{n+1}+\sqrt{n})}(2x^2-5)^n \ .$$

Let us fix $x \in \mathbb{R}$. By the ratio test with $a_n = \frac{(2x^2-5)^n}{3^n(\sqrt{n+1}+\sqrt{n})}$, we obtain

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|2x^2-5|(\sqrt{n+2}+\sqrt{n+1})}{3(\sqrt{n+1}+\sqrt{n})} \to \frac{|2x^2-5|}{3} \quad (\text{as } n \to \infty).$$

If $|x| < 1$, $\left|\frac{a_{n+1}}{a_n}\right| > 1$ and the series is not absolutely nor conditionally convergent.

If $x = \pm 1$, $2x^2-5 = -3$ and the series becomes

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{\sqrt{n+1}+\sqrt{n}} \ .$$

This is conditionally convergent, as $\frac{1}{\sqrt{n+1}+\sqrt{n}}$ is monotonically decreasing, $\frac{1}{\sqrt{n+1}+\sqrt{n}} \to 0$ as $n \to \infty$, therefore, we can use Leibnitz' criterion.

If $1 < |x| < 2$, $\left|\frac{a_{n+1}}{a_n}\right| < 1$ and the series is absolutely convergent. Moreover, for any $1 < r_1 < r_2 < 2$, the series is uniformly convergent for $r_1 < |x| < r_2$.

If $x = \pm 2$, $2x^2-5 = 3$ and the series becomes

$$\sum_{n=0}^{+\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} \ .$$

This is divergent, as $\frac{1}{\sqrt{n+1}+\sqrt{n}} > \frac{1}{2\sqrt{n+1}}$ and $\sum_1^\infty \frac{1}{\sqrt{n+1}}$ is divergent.

If $|x| > 2$, $\left|\frac{a_{n+1}}{a_n}\right| > 1$ and the series, being a positive terms one, is divergent.

2. Find the extremal values of the function $f(x, y) = 3x - 4y$ on the curve C defined by $3x^2 + 2y^2 = 1$.

Solution. We need to find all the stationary points of the function $f(x, y) = 3x - 4y$ under the condition that $g(x, y) = 3x^2 + 2y^2 - 1 = 0$.

By Lagrange's multiplier method, there is $\lambda \in \mathbb{R}$ such that $\lambda \nabla f(x, y) = \nabla g(x, y)$ at stationary points (x, y). Let us compute these gradients:

$$\nabla f(x, y) = (3,\ -4),$$
$$\nabla g(x, y) = (3x, 2y)\,.$$

From the equation of the multiplier method, for a stationary point (x, y), we have

$$\begin{cases} 3x\lambda & = 3, \\ 2y\lambda & = -4, \end{cases}$$

From these equations, we see that λ cannot be 0. By combining them, we obtain $2x\lambda = -y\lambda$, and hence $2x = -y$ (because $\lambda \neq 0$.

By inserting this reltion to $g(x, y) = 0$, it follows that $3x^2 + 8x^2 = 11x^2 = 1$, or equivalently, $x = \pm\frac{1}{\sqrt{11}}$. Accordingly, $(x, y) = (\frac{1}{\sqrt{11}}, -\frac{2}{\sqrt{11}}), (-\frac{1}{\sqrt{11}}, \frac{2}{\sqrt{11}})$.

The corresponding values of $f(x, y)$ are:

$$f((\tfrac{1}{\sqrt{11}}, -\tfrac{2}{\sqrt{11}})) = -\sqrt{11},$$
$$f((-\tfrac{1}{\sqrt{11}}, \tfrac{2}{\sqrt{11}})) = \sqrt{11}.$$

Therefore, $(\frac{1}{\sqrt{11}}, -\frac{2}{\sqrt{11}})$ is the minumum and $(-\frac{1}{\sqrt{11}}, \frac{2}{\sqrt{11}})$ is the maximum.

3. Determine whether the following vector field on $\mathbb{R}^2$

$$\mathbb{f}(x,y) = \left(\frac{e^x}{e^x+y^2}, \frac{2y}{e^x+y^2}\right)$$

is a gradient of some scalar field. If so, find one of these scalar fields φ such that $\mathbb{f}(x,y) = \nabla\varphi(x,y)$.

Solution. Let us call $\mathbb{f}(x,y) = (f_1(x,y), f_2(x,y))$, where $f_1(x,y) = \frac{e^x}{e^x+y^2}$, $f_2(x,y) = \frac{2y}{e^x+y^2}$. We compute:

$$\begin{aligned}
\frac{\partial f_1}{\partial y}(x,y) &= -\frac{2e^x y}{(e^x+y^2)^2}, \\
\frac{\partial f_2}{\partial x}(x,y) &= -\frac{2e^x y}{(e^x+y^2)^2},
\end{aligned}$$

and we see that they coincide. As $\mathbb{R}^2$ is convex, this implies that $\mathbb{f}$ is a gradient.

To find a concrete potential φ, we can take

$$\begin{aligned}
\varphi(x,y) &= \int_0^x f_1(t,0)dt + \int_0^y f_1(x,t)dt \\
&= x + \left[\log(e^x+t^2)\right]_0^x \\
&= \log(e^x+y^2).
\end{aligned}$$

4. Compute the double integral

$$\iint_T y \left[x \log(1+\sqrt{y}) + \frac{1}{1+x^2} \right] dxdy,$$

where $T = \{(x,y) \in \mathbb{R}^2 \,:\, x^2+y^2 \leq 1,\, y \geq 0\}$.
Solution. The integral region T can be written as

$$T = \{(x,y) \in \mathbb{R}^2 \,:\, -\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2}, 0 \leq y \leq 1\}.$$

Let us decompose the integrand into two parts. The first integral is

$$\begin{aligned}
\iint_T yx \log(1+\sqrt{y}) dxdy &= \int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} yx \log(1+\sqrt{y}) dxdy \\
&= \int_0^1 \log(1+\sqrt{y}) \left[\frac{x^2}{2} \right]_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dy \\
&= \int_0^1 \log(1+\sqrt{y}) \cdot 0 \; dy \\
&= 0.
\end{aligned}$$

As for the second part, we take another representation

$$T = \{(x,y) \in \mathbb{R}^2 \,:\, 0 \leq y \leq \sqrt{1-x^2}, -1 \leq x \leq 1\}.$$

Now we can compute the second part of the integral:

$$\begin{aligned}
\iint_T \frac{y}{1+x^2} dxdy &= \int_{-1}^1 \int_0^{\sqrt{1-x^2}} \frac{y}{1+x^2} dydx \\
&= \int_{-1}^1 \frac{1}{1+x^2} \left[\frac{y^2}{2} \right]_0^{\sqrt{1-x^2}} dx \\
&= \frac{1}{2} \int_{-1}^1 \frac{1-x^2}{1+x^2} dx \\
&= \frac{1}{2} \int_{-1}^1 \left(\frac{2}{1+x^2} - 1 \right) dx \\
&= \frac{1}{2} \left[2 \arctan x - x \right]_{-1}^1 \\
&= \frac{1}{2} \cdot 2 \left(2 \cdot \frac{\pi}{4} - 1 \right) \\
&= \frac{\pi}{2} - 1.
\end{aligned}$$

5. Let $\mathbb{F}(x,y,z) = (xy^2, xz^2, y^2z)$ be a vector field on $\mathbb{R}^3$, S be the surface of the cylinder:

$$S := \{(x,y,z) : 0 \le x^2 + y^2 \le 1, 0 \le z \le 2\},$$

and $\mathbb{n}$ the outgoing normal unit vector on S at each point of S.

Compute the surface integral

$$\iint\limits_S \mathbb{F} \cdot \mathbb{n}\, dS.$$

Solution. Thanks to Gauss' theorem (divergence theorem), this integral is equal to the following volume integral

$$\iiint\limits_V \mathrm{div}\mathbb{F}\, dxdydz$$

where $V = \{(x,y,z) : 0 \le x^2 + y^2 \le 1, 0 \le z \le 2\}$.

Let us compute:

$$\mathrm{div}\mathbb{F} = y^2 + 0 + y^2 = 2y^2.$$

To perform the volume integral, we use the cylindrical coordinate $x = r\cos\theta, y = r\sin\theta, z = z$. The region Q correponding to V in this change of coordinate is $Q = \{(r,\theta,z) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le z \le 2\}$. Recall that the Jacobian determinant is $J(r,\theta,z) = r$, and note that $y^2 = r^2\sin^2\theta = r^2\frac{1-\cos 2\theta}{2}$. Theorefore,

$$\begin{aligned}
\iint\limits_S \mathbb{F} \cdot \mathbb{n}\, dS &= \iiint\limits_V \mathrm{div}\mathbb{F}\, dxdydz \\
&= \iiint\limits_Q 2r^2\sin^2\theta \cdot r dr d\theta dz \\
&= \int_0^2 \int_0^{2\pi} \int_0^1 r^3(1-\cos 2\theta)\, dr d\theta dz \\
&= 2 \cdot 2\pi \cdot \frac{1}{4} \\
&= \pi.
\end{aligned}$$