BSc Engineering Sciences – A. Y. 2017/18
Written exam of the course Mathematical Analysis 2
August 30, 2018

1. Given the power series

$$\sum_{n=1}^{+\infty} \frac{1}{n}\left(1+\frac{1}{n}\right)^{n^2} x^n,$$

determine its radius of convergence r, and study the convergence for $x = \pm r$.
Solution.

By the root test with $a_n = \frac{1}{n}(1+\frac{1}{2})^{n^2} x^n$, we have

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{1}{n}\right)^{\frac{1}{n}} \cdot \left(1+\frac{1}{n}\right)^n |x| \to e|x|,$$

since $\frac{1}{n}\log\frac{1}{n} \to 0$, and hence $(\frac{1}{n})^{\frac{1}{n}} = e^{\frac{1}{n}\log\frac{1}{n}} \to 1$. Therefore, $r = \frac{1}{e}$.

For $x = \frac{1}{e}$, we have to study the series whose generic term is

$$\frac{1}{n}\left(1+\frac{1}{n}\right)^{n^2} e^{-n} = \frac{1}{n} e^{n^2\log(1+\frac{1}{n})-n} = \frac{1}{n} e^{n^2(\frac{1}{n}-\frac{1}{2n^2}+o(\frac{1}{n^2}))-n} \sim \frac{e^{-\frac{1}{2}}}{n},$$

and therefore it is divergent.

Finally, for $x = -\frac{1}{e}$, we have the alternating series

$$\sum_{n=1}^{+\infty} (-1)^n \frac{1}{n}\left(1+\frac{1}{n}\right)^{n^2} e^{-n}.$$

There holds

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{t}e^{t^2\log(1+\frac{1}{t})-t}\right) &= \left[-\frac{1}{t^2} + 2\log\left(1+\frac{1}{t}\right) - \frac{1}{t+1} - \frac{1}{t}\right] e^{t^2\log(1+\frac{1}{t})-t} \\ &= \left[\frac{1}{t(t+1)} - \frac{2}{t^2} + o\left(\frac{1}{t^2}\right)\right] e^{t^2\log(1+\frac{1}{t})-t}\end{aligned}$$

and since clearly $\frac{1}{t(t+1)} - \frac{2}{t^2} < 0$ for $t > 0$ (as this is equivalent to $t^2/2 + t > 0$), we see that the sequence $\frac{1}{n}(1+\frac{1}{n})^{n^2}e^{-n}$ is decreasing for n sufficiently large, and it is infinitesimal (from its asymptotic behavior computed above). Therefore, by Leibniz's rule, the given series converges for $x = -\frac{1}{e}$.

Matriculation: ....

2.

(1) Find all the stationary points of the following scalar field, defined on $\mathbb{R}^2$,

$$f(x,y) = x^2 - 2xy - y + y^3$$

and classify them into relative minima, maxima and saddle points.

(2) Compute the gradient of the following function of (x,y):

$$g(x,y) = xe^{xe^y}.$$

Solution. (1) We have

$$\frac{\partial f}{\partial x}(x,y) = 2x - 2y,$$
$$\frac{\partial f}{\partial y}(x,y) = -2x - 1 + 3y^2.$$

At any stationary point (x,y), one has $\nabla f(x,y) = (0,0)$. This happenes exactly when $2x - 2y = 0$ and $-2x - 1 + 3y^2 = 0$. This is equivalent to $y = x$ and $3y^2 - 2y - 1 = 0$. By solving these equations, the stationary points are $(x,y) = (1,1), (-\frac{1}{3}, -\frac{1}{3})$.

To classify these points, we compute the Hessian matrix $\begin{pmatrix} \frac{\partial^2 f}{\partial^2 x} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial^2 y} \end{pmatrix}$:

$$H(x,y) = \begin{pmatrix} 2 & -2 \\ -2 & 6y \end{pmatrix}.$$

At the point $(x,y) = (1,1)$, we have $H(x,y) = \begin{pmatrix} 2 & -2 \\ -2 & 6 \end{pmatrix}$ and its determinant is 8 and the trace is 8, therefore, it has positive eigenvalues and $(1,1)$ is a relative minumum.

At the point $(x,y) = (-\frac{1}{3}, -\frac{1}{3})$, we have $H(x,y) = \begin{pmatrix} 2 & -2 \\ -2 & -2 \end{pmatrix}$ and its determinant is -8, therefore, it has both positive and negative eigenvalues and $(-\frac{1}{3}, -\frac{1}{3})$ is a suddle.

(2) By chain rule and Leibniz rule, $\frac{\partial g}{\partial x} = e^{xe^y} + xe^y e^{xe^y}$, $\frac{\partial g}{\partial y} = x^2 e^y e^{xe^y}$.

Matriculation: ....

3. Let $\mathbb{f}(x, y) = (e^x, e^y)$. Compute the line integral $\int_C \mathbb{f} \cdot d\boldsymbol{\alpha}$, where C is the segment of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$.

Solution.

By definition of line integral, we need to take a parametrization of the parabola. One such parametrization is $\boldsymbol{\alpha}(t) = (t, t^2), t \in [0, 1]$. With this $\boldsymbol{\alpha}(t)$, we have $\boldsymbol{\alpha}'(t) = (1, 2t)$ and $\mathbb{f}(\boldsymbol{\alpha}(t)) = (e^t, e^{t^2})$.

Now the line integral can be computed:

$$\begin{aligned}
\int_C \mathbb{f} \cdot d\boldsymbol{\alpha} &= \int_0^1 (e^t, e^{t^2}) \cdot (1, 2t) dt \\
&= \int_0^1 (e^t + 2te^{t^2}) dt \\
&= [e^t + e^{t^2}]_0^1 \\
&= (e + e) - (1 + 1) \\
&= 2(e - 1)
\end{aligned}$$

Matriculation: ....

4. Compute the integral

$$\iint_T dxdy\, x\log(1+\sqrt{x^2+y^2})\ ,$$

where $T = \{(x,y) \in \mathbb{R}^2 \,:\, 1 \le x^2+y^2 \le 4,\, x \ge 0\}$.

Solution. Goint to polar coordinates, the region T corresponds to

$$\tilde{T} = \left\{(r,\theta) : 1 \le r \le 2, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right\}.$$

With the Jacobian $J(r,\theta) = r$, the integral becomes

$$\begin{aligned}\iint_T dxdy\, x\log(1+\sqrt{x^2+y^2}) &= \iint_{\tilde{T}} r\cos\theta\log(1+r)\, rdrd\theta \\ &= \int_1^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r\cos\theta\log(1+r)\, rd\theta dr \\ &= 2\int_1^2 r^2\log(1+r)\, dr \\ &= 2\left(\left[\frac{r^3}{3}\log(1+r)\right]_1^2 - \int_1^2 \frac{r^3}{3(1+r)}\right)\, dr,\end{aligned}$$

where in the last step we integrated by parts. Note that

$$\frac{r^3}{3(1+r)} = \frac{(1+r)^2}{3} - (1+r) + 1 - \frac{1}{3(1+r)} = \frac{(1+r)^2}{3} - r - \frac{1}{3(1+r)},$$

and hence

$$\int \frac{r^3}{3(1+r)}\, dr = \frac{(1+r)^3}{9} - \frac{r^2}{2} - \frac{1}{3}\log(1+r) + \text{Const.},$$

Althogether,

$$\begin{aligned}&\iint_T dxdy\, x\log(1+\sqrt{x^2+y^2}) \\ &= 2\left(\left[\frac{r^3}{3}\log(1+r)\right]_1^2 - \left[\frac{(1+r)^3}{9} - \frac{r^2}{2} - \frac{1}{3}\log(1+r)\right]_1^2\right) \\ &= 2\left(\frac{1}{3}(8\log 3 - \log 2) - \left(3 - 2 - \frac{1}{3}\log 3\right) + \left(\frac{8}{9} - \frac{1}{2} - \frac{1}{3}\log 2\right)\right) \\ &= 6\log 3 - \frac{4}{3}\log 2 - \frac{11}{9}.\end{aligned}$$

Matriculation: ....

5. Let $\mathbb{F}(x,y,z) = (e^{zy^2},\ e^{yx^2},\ e^{xz^2})$ be a vector field on $\mathbb{R}^3$, C be the boundary of the square with vertices $(0,0,0),(1,0,0),(1,1,0),(0,1,0)$.

Compute the line integral

$$\int_C \mathbb{F}\cdot d\boldsymbol{\alpha},$$

where $\boldsymbol{\alpha}$ is a parametrization of C going counterclockwise.

Solution.

Let S be a surface which has C as the boundary, and $\mathbb{r}(u,v) = (X(u,v), Y(u,v), Z(u,v))$ be its parametrization such that the inverse-image of $\boldsymbol{\alpha}$ is also going counterclockwise in the uv-plane. Stokes' theorem says, if we let $\mathbb{n} = \frac{\mathbb{N}}{\|\mathbb{N}\|}$ where $\mathbb{N} = \frac{\partial \mathbb{r}}{\partial u} \times \frac{\partial \mathbb{r}}{\partial v}$, then it holds that $\int_C \mathbb{F}\cdot d\boldsymbol{\alpha} = \int_S \operatorname{curl}\mathbb{F}\cdot\mathbb{n}\,dS$.

For the C above, we can take $S = \{(x,y,z) : 0 \le x \le 1, 0 \le y \le 1, z = 0\}$, and a parametrization $X(u,v) = u, Y(u,v) = v, Z(u,v) = 0$. The corresponding region in the uv-plane is $T = \{(u,v) : 0 \le u \le 1, 0 \le v \le 1\}$. It follows that $\frac{\partial \mathbb{r}}{\partial u} = (1,0,0)$, $\frac{\partial \mathbb{r}}{\partial v} = (0,1,0)$ and hence $\frac{\partial \mathbb{r}}{\partial u} \times \frac{\partial \mathbb{r}}{\partial v} = (0,0,1)$. This last vector is already a unit vector.

After some straightforward computations (actually, we only need the z-component), we obtain

$$\operatorname{curl}\mathbb{F} = \begin{pmatrix} 0, \\ y^2 e^{zy^2} - z^2 e^{xz^2}, \\ 2xye^{yx^2} - 2yze^{zy^2} \end{pmatrix}$$

Therefore, on the uv-plane, namely $x = u, y = v, z = 0$,

$$\begin{aligned} &\operatorname{curl}\mathbb{F}(\mathbb{r}(u,v))\cdot\mathbb{n}(u,v) \\ &= (0,0,2uve^{vu^2})\cdot(0,0,1) \\ &= 2uve^{vu^2}. \end{aligned}$$

Altogether, through Stokes' theorem, the given line integral becomes

$$\begin{aligned} \int_C \mathbb{F}\cdot d\boldsymbol{\alpha} &= \int_S \operatorname{curl}\mathbb{F}\cdot\mathbb{n}\,dS \\ &= \iint_T 2uve^{vu^2}\,dudv \\ &= \int_0^1\int_0^1 2uve^{vu^2}\,dudv \\ &= \int_0^1 [e^{vu^2}]_0^1\,dv \\ &= \int_0^1 (e^v - 1)\,dv \\ &= [e^v - v]_0^1 = (e-1) - (1-0) = e - 2. \end{aligned}$$