BSc Engineering Sciences – A. Y. 2017/18
Written exam of the course Mathematical Analysis 2
July 9, 2018

1. (6 points) Find the Taylor series expansion, with initial point $x_0 = 0$, of the function

$$f(x) := \frac{2x}{2x^2 - 3x + 1},$$

determine its radius of convergence r, and study the convergence for $x = \pm r$.
Solution.

We can rewrite $f(x)$ as follows:

$$\begin{aligned} f(x) &= \frac{2x}{2x^2 - 3x + 1} \\ &= \frac{2x}{(2x-1)(x-1)} \\ &= \frac{-2}{2x-1} + \frac{2}{x-1}. \end{aligned}$$

We also know the geometric series expansion $\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$. By applying this to the last expression, we obtain

$$\begin{aligned} f(x) &= \frac{-2}{2x-1} + \frac{2}{x-1} \\ &= 2 \cdot \frac{1}{1-2x} - 2 \cdot \frac{1}{1-x} \\ &= 2\left(\sum_{n=0}^{\infty} (2x)^n - \sum_{n=0}^{\infty} x^n\right) \\ &= 2\left(\sum_{n=0}^{\infty} (2^n - 1)x^n\right) \\ &= \sum_{n=0}^{\infty} 2(2^n - 1)x^n. \end{aligned}$$

By the ratio test with $a_n = 2(2^n - 1)$, we need to check $\frac{a_{n+1}x^{n+1}}{a_n x^n} = \frac{2(2^{n+1}-1)}{2(2^n-1)}x \to 2x$, and the radius of convergence is $\frac{1}{2}$.

As for the convergence at $x = \frac{1}{2}$, the series becomes

$$\sum_{n=0}^{\infty} 2(2^n - 1)(\tfrac{1}{2})^n = 2\textstyle\sum_{n=0}^{\infty}(1 - (\frac{1}{2})^n),$$

where each term $1 - (\frac{1}{2})^n$ is positive, and again by the ratio test it is divergent. At $x = -\frac{1}{2}$, the series is $\sum_{n=0}^{\infty} 2(2^n - 1)(-\frac{1}{2})^n = 2\sum_{n=0}^{\infty}(-1)^n(1 - (\frac{1}{2})^n)$ and this is oscillating (neither convergent nor divergent).

Matriculation: ....

2. (6 points) Find the extremal values of the function $f(x, y) = e^{x^2+y}$ on the circle $x^2+y^2 = 1$.
Solution.

We need to find all the stationary points of the function $f(x, y) = e^{x^2+y}$ (one could also take $x^2 + y$, since they share the same points of minima and maxima, and in this case the computations become simpler) under the condition that $g(x, y) = x^2 + y^2 - 1 = 0$.

By Lagrange's multiplier method, there is $\lambda \in \mathbb{R}$ such that $\lambda\nabla f(x, y) = \nabla g(x, y)$ at stationary points (x, y). Let us compute these gradients:

$$\begin{aligned}\nabla f(x, y) &= (2xe^{x^2+y},\ e^{x^2+y}),\\ \nabla g(x, y) &= (2x,\ 2y)\,.\end{aligned}$$

From the equation of the multiplier method, for a stationary point (x, y), we have

$$\begin{aligned}\lambda 2xe^{x^2+y} &= 2x,\\ \lambda e^{x^2+y} &= 2y.\end{aligned}$$

Let us look at the first equation. If $x = 0$, this is satisfied and by $x^2 + y^2 = 1$, we obtain $y = \pm 1$, and $\lambda = \frac{\pm 1}{e^{\pm 1}}$, and $f(0, 1) = e^1, f(0, -1) = e^{-1}$.

On the other hand, if $x \neq 0$, we see from the first equation that $\lambda e^{x^2+y} = 1$, or equivalently $\lambda = e^{-x^2-y}$. By substituting this to the second equation, we get $1 = 2y$, and hence $y = \frac{1}{2}$. Correspondingly, from $x^2 + y^2 = 1$ it follows that $x = \pm\frac{\sqrt{3}}{2}$. In these cases, $f(\frac{\sqrt{3}}{2}, \frac{1}{2}) = e^{\frac{5}{4}}$ and $f(-\frac{\sqrt{3}}{2}, \frac{1}{2}) = e^{\frac{5}{4}}$.

Altogether, the maxima are $(\frac{\sqrt{3}}{2}, \frac{1}{2}), (-\frac{\sqrt{3}}{2}, \frac{1}{2})$ with the value $e^{\frac{5}{4}}$ and the minimun is $(0, 1)$ with the value e^{-1}.

Matriculation: ....

3. (6 points) Determine whether the following vector field on $\mathbb{R}^2$

$$\mathbb{f}(x,y) = (-y\sin x \cdot \cos(y\cos x),\ \cos x \cdot \cos(y\cos x))$$

is a gradient of some scalar field. If so, find one of these scalar fields φ such that $\mathbb{f}(x,y) = \nabla\varphi(x,y)$.

Solution.

Let us call $\mathbb{f}(x,y) = (f_1(x,y), f_2(x,y))$, where $f_1(x,y) = -y\sin x \cdot \cos(y\cos x)$, $f_2(x,y) = \cos x \cdot \cos(y\cos x))$. We compute:

$$\frac{\partial f_1}{\partial y}(x,y) = -\sin x \cdot \cos(y\cos x) + y\sin x \cdot \cos x \cdot \sin(y\cos x),$$
$$\frac{\partial f_2}{\partial x}(x,y) = -\sin x \cdot \cos(y\cos x) + y\sin x \cdot \cos x \cdot \sin(y\cos x),$$

and we see that they coincide. As $\mathbb{R}^2$ is convex, this implies that $\mathbb{f}$ is a gradient.

To find a concrete potential φ, we can take

$$\begin{aligned}\varphi(x,y) &= \int_0^x f_1(t,0)dt + \int_0^y f_2(x,t)dt \\ &= 0 + [\sin(t\cos x)]_0^y \\ &= \sin(y\cos x).\end{aligned}$$

Matriculation: ....

4. (6 points) Compute the following double integral

$$\iint_T \frac{y}{(x^2+y^2)^2} dxdy,$$

where T is the quadrilateral with vertices $(1,0)$, $(1,\sqrt{3})$, $(3,3\sqrt{3})$, $(3,0)$.

Solution.

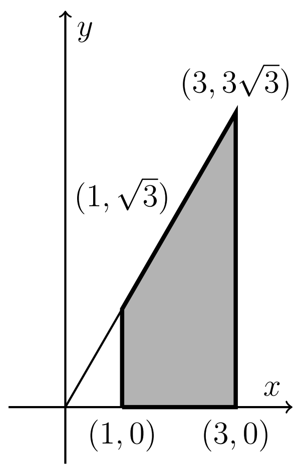
The region T can be written as $T = \{(x,y) : 1 \le x \le 3, 0 \le y \le \sqrt{3}x\}$.

Note: it is also possible to compute the double integral directly in the xy-coordinates.

In the polar coordinates it corresponds to

$$\tilde{T} = \{(r,\theta) : 0 \le \theta \le \frac{\pi}{3}, 1 \le r\cos\theta \le 3\} = \left\{(r,\theta) : 0 \le \theta \le \frac{\pi}{3}, \frac{1}{\cos\theta} \le r \le \frac{3}{\cos\theta}\right\}.$$

Going into the polar coordinates, the integral becomes



$$\begin{aligned}
\iint_T \frac{y}{(x^2+y^2)^2} dxdy &= \iint_{\tilde{T}} \frac{r\sin\theta}{r^4} r dr d\theta = \int_0^{\frac{\pi}{3}} \int_{\frac{1}{\cos\theta}}^{\frac{3}{\cos\theta}} \frac{\sin\theta}{r^2} dr d\theta \\
&= \int_0^{\frac{\pi}{3}} \sin\theta \left[-\frac{1}{r}\right]_{\frac{1}{\cos\theta}}^{\frac{3}{\cos\theta}} d\theta \\
&= \int_0^{\frac{\pi}{3}} \left(-\frac{1}{3}\sin\theta\cos\theta + \sin\theta\cos\theta\right) d\theta \\
&= \frac{1}{3}\int_0^{\frac{\pi}{3}} 2\sin\theta\cos\theta \, d\theta \\
&= \frac{1}{3}\int_0^{\frac{\pi}{3}} \sin(2\theta)\, d\theta \\
&= -\frac{1}{6}[\cos(2\theta)]_0^{\frac{\pi}{3}} \\
&= \frac{1}{6}\left(1+\frac{1}{2}\right) = \frac{1}{4}.
\end{aligned}$$

Matriculation: ....

5. (6 points) Let $\mathbb{F}(x,y,z) = ((x-y+z)e^{x^2+y^2+z^2},\ (x+y+z)e^{x^2+y^2+z^2},\ (-x+y-z)e^{x^2+y^2+z^2})$ be a vector field on $\mathbb{R}^3$, C be the circle

$$C = \{(x,y,z) : x^2+y^2 = 1, z = 0\}.$$

Compute the line integral

$$\int_C \mathbb{F} \cdot d\boldsymbol{\alpha},$$

where $\boldsymbol{\alpha}$ is a parametrization of C going counterclockwise.

Solution.

Note: it is not difficult in this case to compute the line integral directly without Stokes' theorem.

Let S be a surface which has C as the boundary, and $\mathbb{r}(u,v) = (X(u,v), Y(u,v), Z(u,v))$ be its parametrization such that the inverse-image of $\boldsymbol{\alpha}$ is also going counterclockwise in the uv-plane. Stokes' theorem says, if we let $\mathbb{n} = \frac{\mathbb{N}}{\|\mathbb{N}\|}$ where $\mathbb{N} = \frac{\partial \mathbb{r}}{\partial u} \times \frac{\partial \mathbb{r}}{\partial v}$, then it holds that $\int_C \mathbb{F} \cdot d\boldsymbol{\alpha} = \int_S \operatorname{curl} \mathbb{F} \cdot \mathbb{n}\, dS$.

For the C above, we can take $S = \{(x,y,z) : x^2+y^2 \le 1, z = 0\}$, and a parametrization $X(u,v) = u, Y(u,v) = v, Z(u,v) = 0$. The corresponding region in the uv-plane is $T = \{(u,v) : u^2+v^2 \le 1\}$. It follows that $\frac{\partial \mathbb{r}}{\partial u} = (1,0,0)$, $\frac{\partial \mathbb{r}}{\partial v} = (0,1,0)$ and hence $\frac{\partial \mathbb{r}}{\partial u} \times \frac{\partial \mathbb{r}}{\partial v} = (0,0,1)$. This last vector is already a unit vector.

After some straightforward computations, we obtain

$$\operatorname{curl} \mathbb{F} = \begin{pmatrix} (1+2y(-x+y-z) - 1 - 2z(x+y+z))e^{x^2+y^2+z^2}, \\ (1+2z(x-y+z) - (-1) - 2x(-x+y-z)e^{x^2+y^2+z^2}, \\ (1+2x(x+y+z) - (-1) - 2y(x-y+z)e^{x^2+y^2+z^2} \end{pmatrix}$$

Therefore, on the uv-plane, namely $x = u, y = v, z = 0$,

$$\begin{aligned} &\operatorname{curl} \mathbb{F}(\mathbb{r}(u,v)) \cdot \mathbb{n}(u,v) \\ &= (2v(-u+v)e^{u^2+v^2}, (2-2u(-u+v))e^{u^2+v^2}, (2+2(u^2+v^2))e^{u^2+v^2}) \cdot (0,0,1) \\ &= 2(1+(u^2+v^2))e^{u^2+v^2}. \end{aligned}$$

Altogether, through Stokes' theorem, the given line integral becomes

$$\begin{aligned} \int_C \mathbb{F} \cdot d\boldsymbol{\alpha} &= \int_S \operatorname{curl} \mathbb{F} \cdot \mathbb{n}\, dS \\ &= \iint_T 2(1+(u^2+v^2))e^{u^2+v^2}\, dudv \\ &= \int_0^{2\pi} \int_0^1 2(1+r^2)e^{r^2} \cdot r\, drd\theta \\ &= \int_0^{2\pi} \int_0^1 (1+t)e^t \cdot dtd\theta \\ &= 2\pi \left[e^t + (te^t - e^t)\right]_0^1 = 2\pi \left[te^t\right]_0^1 = 2\pi e. \end{aligned}$$

where we transformed the uv-integral to the polar coordinate, then made substitution $t = r^2$.