BSc Engineering Sciences – A. Y. 2017/18
Written exam (call II) of the course Mathematical Analysis 2
February 21, 2018

1. (6 points) Find a power series expression for the solution $y(x)$ of the differential equation

$$xy'' + (1+x)y' + 2y = 0$$

such that $y(0) = 1$, $y'(0) = -2$, and determine its radius of convergence.

Solution.

Let us put $y(x) = \sum_{n=0}^{\infty} a_n x^n$. If this is convergent, we have $y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$. If $y(x)$ satisfies the differential equation above, it must hold

$$x\left(\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}\right) + (1+x)\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0.$$

By putting all the terms in the form of power series, we get

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0.$$

By changing the label n in the summations ($n \mapsto n+1$ in the first and second summations), it is equivalent to

$$\sum_{n=1}^{\infty}(n+1)n a_{n+1} x^n + \sum_{n=0}^{\infty}(n+1)a_{n+1} x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0.$$

In the first and third summations, if we put $n = 0$, there is a factor n, hence the summation does not change even if it starts with $n = 0$. Namely,

$$\begin{aligned}
&\sum_{n=0}^{\infty}(n+1)n a_{n+1} x^n + \sum_{n=0}^{\infty}(n+1)a_{n+1} x^n + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n \\
&= \sum_{n=0}^{\infty} \left((n+1)n a_{n+1} + (n+1)a_{n+1} + n a_n + 2a_n\right) x^n = 0.
\end{aligned}$$

Therefore, if the power series is convergent, for each n it must hold

$$(n+1)n a_{n+1} + (n+1)a_{n+1} + n a_n + 2a_n = 0,$$

or equivalently, $a_{n+1} = -\frac{n+2}{(n+1)^2} a_n$ and it follows that $a_n = (-1)^n \frac{n+1}{n!} a_0$. From the conditions $y(0) = 1$, $y'(0) = -2$, it follows that $a_0 = 1$ and $a_1 = -2$, hence in general $a_n = (-1)^n \frac{n+1}{n!}$.

By ratio test, $a_{n+1}x^{n+1}/a_nx^n = -\frac{n+2}{(n+1)^2}x \to 0$ for any $x \in \mathbb{R}$, or the radius of convergence is ∞. Finally,

$$
\begin{aligned}
y(x) &= \sum_{n=0}^{\infty}(-1)^n \frac{n+1}{n!}x^n = \sum_{n=0}^{\infty}(-1)^n \left(\frac{1}{(n-1)!} + \frac{1}{n!}\right) x^n \\
&= -x \cdot \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{n-1}}{(n-1)!} + \sum_{n=0}^{\infty}(-1)^n \frac{x^n}{n!} \\
&= (1-x)e^{-x}.
\end{aligned}
$$

Note: one can obtain 6 points if a_n and the radius of convergence are correctly done.

2. (6 points) Let C be the curve in $\mathbb{R}^2$ defined by

$$C = \{(x, y) : 2(x + y - 1)^2 + (x - y)^2 = 8\}.$$

Find the points on C which are the nearest and the farthest from the origin $(0, 0)$.

Solution.

We need to find all the stationary points of the function $f(x, y) = x^2 + y^2$ (one could take $\sqrt{x^2 + y^2}$, but they share the same points of minima and maxima) under the condition that $g(x, y) = 2(x + y - 1)^2 + (x - y)^2 - 8 = 0$.

By Lagrange's multiplier method, there is $\lambda \in \mathbb{R}$ such that $\lambda \nabla f(x, y) = \nabla g(x, y)$ at stationary points (x, y). Let us compute these gradients:

$$\begin{aligned}
&\nabla f(x, y) = (2x,\ 2y), \\
&\nabla g(x, y) = (4(x + y - 1) + 2(x - y),\ 4(x + y - 1) - (x - y)).
\end{aligned}$$

From the equation of the multiplier method, for a stationary point (x, y), we have

$$\begin{cases} 2\lambda x &= 4(x + y - 1) + 2(x - y), \\ 2\lambda y &= 4(x + y - 1) - 2(x - y), \end{cases}$$

By summing and subtracting the sides of these equations, we obtain

$$\begin{cases} 2\lambda(x + y) &= 8(x + y - 1), \\ 2\lambda(x - y) &= 4(x - y), \end{cases}$$

or equivalently,

$$\begin{cases} (4 - \lambda)(x + y) &= 4, \\ (\lambda - 2)(x - y) &= 0. \end{cases}$$

Let us take the second equation. There are two cases where it can be fulfilled:

(1) $x - y = 0$. Now, (x, y) must also be on C, hence $2(2x - 1)^2 = 8$. From this, it follows that $(x, y) = (\frac{3}{2}, \frac{3}{2}), (-\frac{1}{2}, -\frac{1}{2})$. Correspondingly, $f(\frac{3}{2}, \frac{3}{2}) = \frac{9}{2}, f(-\frac{1}{2}, -\frac{1}{2}) = \frac{1}{2}$.

(2) $\lambda - 2 = 0$. In this case, by combining it with the first equation, we get $x + y = 2$. Again, from the fact that (x, y) is on C, it follows that $2 + (x - y)^2 = 8$, or equivalently, $x - y = \pm\sqrt{6}$. Combined with $x + y = 2$, it yields $(x, y) = \left(1 + \sqrt{\frac{3}{2}}, 1 - \sqrt{\frac{3}{2}}\right), \left(1 - \sqrt{\frac{3}{2}}, 1 + \sqrt{\frac{3}{2}}\right)$ and correspondingly, $f\left(1 \pm \sqrt{\frac{3}{2}}, 1 \mp \sqrt{\frac{3}{2}}\right) = 5$.

As $\frac{1}{2} < \frac{9}{2} < 5$, the nearest point from the origin is $(-\frac{1}{2}, -\frac{1}{2})$ (with the distance $\frac{1}{\sqrt{2}}$) and the farthest points are $\left(1 \pm \sqrt{\frac{3}{2}}, 1 \mp \sqrt{\frac{3}{2}}\right)$ (with the distance $\sqrt{5}$).

3.

(1) (3 points) Let $c > 0$. Find the solution $f(x,t)$ of the partial differential equation

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}$$

with the initial condition $f(x,0) = e^{-x^2}$, $\frac{\partial f}{\partial t}(x,0) = \frac{x}{(x^2+1)^2}$.

(2) (3 points) Let $h(s)$ be a twice continuously differentiable function of $s \in \mathbb{R}$, $\boldsymbol{\alpha} = (\alpha_x, \alpha_y, \alpha_z) \in \mathbb{R}^3$ such that $\|\boldsymbol{\alpha}\|^2 = 1$. Let us define a function g of $(x,y,z,t) \in \mathbb{R}^4$ by

$$g(x,y,z,t) = h(x\alpha_x + y\alpha_y + z\alpha_z - ct).$$

Prove that g satisfies the following partial differential equation (3d wave equation):

$$\frac{\partial^2 g}{\partial t^2} = c^2 \left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2} \right)$$

Solution.

(1) As we learned during the lecture, a general solution of this equation (1d wave equation) is given by $f(x,t) = \frac{1}{2}\left(F(x-ct) + F(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)ds$, where $F(s)$ is a twice continuously differentiable function, $G(s)$ is a once continuously differentiable function. Furthermore, it holds that $f(x,0) = F(x)$ and $\frac{\partial f}{\partial t}(x,0) = G(x)$.

We are given the initial conditions $f(x,0) = e^{-x^2}$, $\frac{\partial f}{\partial t}(x,0) = \frac{x}{(x^2+1)^2}$, hence we can take $F(s) = e^{-s^2}, G(s) = \frac{s}{(s^2+1)^2}$. Note that $\int G(s)ds = -\frac{1}{2(s^2+1)} + \text{Const}$. Altogether, we have

$$f(x,t) = \frac{1}{2}\left(e^{-(x-ct)^2} + e^{-(x+ct)^2}\right) + \frac{1}{4c}\left(\frac{1}{(x-ct)^2+1} - \frac{1}{(x+ct)^2+1}\right).$$

(2) By chain rule, we have

$$\frac{\partial g}{\partial t} = -ch'(x\alpha_x + y\alpha_y + z\alpha_z - ct), \qquad \frac{\partial g}{\partial x} = \alpha_x h'(x\alpha_x + y\alpha_y + z\alpha_z - ct),$$
$$\frac{\partial g}{\partial y} = \alpha_y h'(x\alpha_x + y\alpha_y + z\alpha_z - ct), \qquad \frac{\partial g}{\partial z} = \alpha_z h'(x\alpha_x + y\alpha_y + z\alpha_z - ct).$$

Continuing to the second derivative,

$$\frac{\partial^2 g}{\partial t^2} = c^2 h''(x\alpha_x + y\alpha_y + z\alpha_z - ct), \qquad \frac{\partial^2 g}{\partial x^2} = \alpha_x^2 h''(x\alpha_x + y\alpha_y + z\alpha_z - ct),$$
$$\frac{\partial^2 g}{\partial y^2} = \alpha_y^2 h''(x\alpha_x + y\alpha_y + z\alpha_z - ct), \qquad \frac{\partial^2 g}{\partial z^2} = \alpha_z^2 h''(x\alpha_x + y\alpha_y + z\alpha_z - ct).$$

As $\alpha_x^2 + \alpha_y^2 + \alpha_z^2 = \|\boldsymbol{\alpha}\|^2 = 1$, we have $c^2\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right) = c^2 h''(x\alpha_x + y\alpha_y + z\alpha_z - ct) = \frac{\partial^2 g}{\partial t^2}$.

4. (6 points) Compute the integral

$$\iiint_T dxdydz\, z\sqrt{x^2+y^2}$$

where T is the region bounded by the cylinder $x^2-2x+y^2=0$, the sphere $x^2+y^2+z^2=4$ and the plane $z=0$.

Solution.

Note that $x^2-2x+y^2=(x-1)^2+y^2-1$, hence $x^2-2x+y^2=0$ defines a cylinder based on the circle centered at $(1,0,0)$ with radius 1. If we project the sphere $x^2+y^2+z^2=4$ to the xy-plane, its image is the disk $\{(x,y)\in\mathbb{R}^2 : x^2+y^2\le 2\}$, which includes the disk $\{(x,y)\in\mathbb{R}^2 : (x-1)^2+y^2\le 1\}$.

Let us put $T_0 := \{(x,y)\in\mathbb{R}^2 : (x-1)^2+y^2<1\}$. Then we can write

$$T=\left\{(x,y,z)\in\mathbb{R}^3 : (x,y)\in T_0,\ 0\le z\le\sqrt{4-x^2-y^2}\right\}.$$

Now we can make the given integral into the following iterated integral, and the integration with respect to z can be performed immediately:

$$\begin{aligned}\iint\limits_{T_0} dxdy\int_0^{\sqrt{4-x^2-y^2}} z\sqrt{x^2+y^2}\,dz &= \iint\limits_{T_0}\left[\frac{z^2}{2}\right]_0^{\sqrt{4-x^2-y^2}}\sqrt{x^2+y^2}\,dxdy\\ &=\frac{1}{2}\iint\limits_{T_0}(4-x^2-y^2)\sqrt{x^2+y^2}\,dxdy.\end{aligned}$$

In polar coordinates, T_0 corresponds to $\left\{(r,\theta) : 0\le r\le 2\cos\theta, -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\right\}$, hence

$$\begin{aligned}\frac{1}{2}\iint\limits_{T_0}(4-x^2-y^2)\sqrt{x^2+y^2}\,dxdy &= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta}(4-r^2)r\cdot r\,drd\theta\\ &=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\frac{4}{3}r^3-\frac{1}{5}r^5\right]_0^{2\cos\theta}d\theta\\ &=16\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{\cos^3\theta}{3}-\frac{\cos^5\theta}{5}\right)d\theta.\end{aligned}$$

With $t=\sin\theta$, $\frac{dt}{d\theta}=\cos\theta$, the last expression becomes

$$\begin{aligned}16\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{1-\sin^2\theta}{3}-\frac{(1-\sin^2\theta)^2}{5}\right)\cos\theta d\theta &= 16\int_{-1}^{1}\left(\frac{1-t^2}{3}-\frac{(1-t^2)^2}{5}\right)dt\\ &=16\left[\frac{t}{3}-\frac{t^3}{9}-\frac{t}{5}+\frac{2t^3}{15}-\frac{t^5}{25}\right]_{-1}^{1}\\ &=\frac{832}{225}.\end{aligned}$$

5. (6 points) Let $\mathbb{F}(x,y,z) = (-y^3e^z + y^2e^{xy^2},\ x^3\cos z + 2xye^{xy^2},\ xyz)$ be a vector field on $\mathbb{R}^3$, C be the circle

$$C = \{(x,y,z) : x^2 + y^2 = a^2, z = 0\},$$

where $a > 0$.

Compute the line integral

$$\int_C \mathbb{F} \cdot d\boldsymbol{\alpha},$$

where $\boldsymbol{\alpha}$ is a parametrization of C going counterclockwise.

Solution.

Let S be a surface which has C as the boundary, and $\mathbb{r}(u,v) = (X(u,v), Y(u,v), Z(u,v))$ be its parametrization such that the inverse-image of $\boldsymbol{\alpha}$ is also going counterclockwise in the uv-plane. Stokes' theorem says, if we let $\mathbb{n} = \frac{\mathbb{N}}{\|\mathbb{N}\|}$ where $\mathbb{N} = \frac{\partial \mathbb{r}}{\partial u} \times \frac{\partial \mathbb{r}}{\partial v}$, then it holds that $\int_C \mathbb{F} \cdot d\boldsymbol{\alpha} = \int_S \operatorname{curl} \mathbb{F} \cdot \mathbb{n}\, dS$.

For the C above, we can take $S = \{(x,y,z) : x^2 + y^2 < a^2, z = 0\}$, and a parametrization $X(u,v) = u, Y(u,v) = v, Z(u,v) = 0$. The corresponding region in the uv-plane is $T = \{(u,v) : u^2+v^2 < a^2\}$. It follows that $\frac{\partial \mathbb{r}}{\partial u} = (1,0,0)$, $\frac{\partial \mathbb{r}}{\partial v} = (0,1,0)$ and hence $\frac{\partial \mathbb{r}}{\partial u} \times \frac{\partial \mathbb{r}}{\partial v} = (0,0,1)$. This last vector is already a unit vector.

After some straightforward computations, we obtain

$$\operatorname{curl} \mathbb{F} = (xz + x^3 \sin z,\ -y^3e^z - yz,\ 3x^2\cos z + 3y^2e^z).$$

Therefore, on the uv-plane,

$$\operatorname{curl} \mathbb{F}(\mathbb{r}(u,v)) \cdot \mathbb{n}(u,v) = (-2u^{uv^2} - 4u^2v^2e^{uv^2}, -v^4e^{uv^2}, 3(u^2+v^2)) \cdot (0,0,1) = 3(u^2+v^2).$$

Altogether, through Stokes' theorem, the given line integral becomes

$$\begin{aligned}
\int_C \mathbb{F} \cdot d\boldsymbol{\alpha} &= \int_S \operatorname{curl} \mathbb{F} \cdot \mathbb{n}\, dS \\
&= \iint_T 3(u^2+v^2)\, du dv \\
&= \int_0^{2\pi} \int_0^a 3r^2 \cdot r\, dr d\theta \\
&= 2\pi \cdot \frac{3a^4}{4} = \frac{3\pi a^4}{2},
\end{aligned}$$

where we transformed the uv-integral to the polar coordinate.