BSc Engineering Sciences – A. Y. 2017/18

Written exam of the course Mathematical Analysis 2

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Last name: .... First name: ....

Matriculation: ....

Solve the following problems, motivating in detail the answers.

1. (6 points) Determine the values of $\alpha \in \mathbb{R}$ for which the following improper integral is convergent, and compute it for $\alpha = \frac{1}{2}$:

$$\int_0^{\pi/4} \frac{\cos x}{(\cos^2 x - \sin^2 x)^\alpha} dx.$$

Solution.

The integrand can be rewriten as

$$\frac{\cos x}{(1 - 2\sin^2 x)^\alpha}$$

By substituting $\sin x = t$, hence $\frac{dt}{dx} = \cos x$, the integral can be transformed into

$$\int_0^{\frac{1}{\sqrt{2}}} \frac{1}{(1-2t^2)^\alpha} dt.$$

As $t \to 0$, the integrand remains bounded. As for $t \to \frac{1}{\sqrt{2}}$, we have

$$\frac{1}{(1-2t^2)^\alpha} = \frac{1}{(-2(\frac{1}{\sqrt{2}} - t)^2 + 2\sqrt{2}(\frac{1}{\sqrt{2}} - t))^\alpha}.$$

Now clearly $-2(\frac{1}{\sqrt{2}} - t)^2 + 2\sqrt{2}(\frac{1}{\sqrt{2}} - t) \le 2\sqrt{2}(\frac{1}{\sqrt{2}} - t)$, and for $t \ge 1/\sqrt{2} - 1/2$, $-2(\frac{1}{\sqrt{2}} - t)^2 + 2\sqrt{2}(\frac{1}{\sqrt{2}} - t) \ge (2\sqrt{2} - 1)(\frac{1}{\sqrt{2}} - t)$, and then

$$\frac{1}{(2\sqrt{2}(\frac{1}{\sqrt{2}} - t))^\alpha} \le \frac{1}{(-2(\frac{1}{\sqrt{2}} - t)^2 + 2\sqrt{2}(\frac{1}{\sqrt{2}} - t))^\alpha} \le \frac{1}{((2\sqrt{2}-1)(\frac{1}{\sqrt{2}} - t))^\alpha},$$

so that the integral around $\frac{1}{\sqrt{2}}$ is finite if and only if $\alpha < 1$, by comparing with the function $\frac{1}{(\frac{1}{\sqrt{2}} - t)^\alpha}$. An alternative way leading to the same conclusion is to observe that, for $t \to \frac{1}{\sqrt{2}}$,

$$\frac{1}{(1-2t^2)^\alpha} = \frac{1}{2^\alpha(\frac{1}{\sqrt{2}} - t)^\alpha(\frac{1}{\sqrt{2}} + t)^\alpha} \sim \frac{1}{(2\sqrt{2})^\alpha} \cdot \frac{1}{(\frac{1}{\sqrt{2}} - t)^\alpha},$$

and to apply the asymptotic comparison test.

As for $\alpha = \frac{1}{2}$, substitute further $t = \frac{1}{\sqrt{2}}\sin\theta$, $\frac{dt}{d\theta} = \frac{1}{\sqrt{2}}\cos\theta$ and the integral becomes

$$\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sqrt{2}\cdot\sqrt{1-\sin^2\theta}} d\theta = \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{2\sqrt{2}}.$$

2.

(1) (4 points) Find all the stationary points of the following scalar field, defined on $\mathbb{R}^2$,

$$f(x,y) = e^{x+y}(x^2+xy)$$

and classify them into relative minima, maxima and saddle points.

(2) (2 points) Compute the derivative of the following function on $\mathbb{R}$:

$$f(t) = (1+\cosh t)^{1+\cosh t}.$$

Solution.
(1) For the f given above, it holds that

$$\nabla f(x,y) = (e^{x+y}(x^2+xy+2x+y), e^{x+y}(x^2+xy+x)).$$

At stationary points, $\nabla f(x,y) =$ holds. Namely,

$$e^{x+y}(x^2+xy+2x+y) = 0, e^{x+y}(x^2+xy+x) = 0.$$

As e^{x+y} takes never 0, this is equivalent to

$$x^2+xy+2x+y = 0, x^2+xy+x = 0,$$

and by subtracting the both sides, one obtains $x+y=0$. Substituting $y=-x$ in one of these equations, one obtains $x^2-x^2+x = x = 0$. Therefore, the only stationary point is $(0,0)$.

To classify this point, let us compute the Hessian matrix:

$$\begin{pmatrix} e^{x+y}(x^2+xy+2x+y+2x+2) & e^{x+y}(x^2+xy+2x+y+x+1) \\ e^{x+y}(x^2+xy+2x+y+x+y+1) & e^{x+y}(x^2+xy+x+x) \end{pmatrix}$$

and at the point $(x,y)=(0,0)$, this becomes

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}.$$

Its determinant is $2\cdot 0 - 1\cdot 1 = -1$, therefore, it has both negative and positive eigenvalues, and the point $(0,0)$ is a saddle point.
(2) Take the function $g(x,y) = x^y$ (for $x,y>1$) and define $\boldsymbol{\alpha}(t) = (\alpha_1(t),\alpha_2(t)) = (1+\cosh t, 1+\cosh t)$. Note that $f(t) = g(\boldsymbol{\alpha}(t))$, therefore, by the chain rule, $f'(t) = \frac{\partial g}{\partial x}(\boldsymbol{\alpha}(t))\frac{d\alpha_1}{dt}(t) + \frac{\partial g}{\partial y}(\boldsymbol{\alpha}(t))\frac{d\alpha_2}{dt}(t)$.

We have $\frac{\partial g}{\partial x} = yx^{y-1}$, $\frac{\partial g}{\partial y} = \log x\cdot x^y$ and $\frac{d\alpha_1}{dt}(t) = \frac{d\alpha_2}{dt}(t) = \sinh t$, hence,

$$\begin{aligned} f'(t) &= (1+\cosh t)\cdot(1+\cosh t)^{\cosh t}\cdot\sinh t + \log(1+\cosh t)\cdot(1+\cosh t)^{(1+\cosh t)}\cdot\sinh t \\ &= \sinh t(1+\log(1+\cosh t))(1+\cosh t)^{(1+\cosh t)}. \end{aligned}$$

Note: any other method is of course OK if correct, e.g. writing $f(t) = e^{(1+\cosh t)\log(1+\cosh t)}$.

3. (6 points) Determine whether the following vector field on $\mathbb{R}^2$

$$\mathbb{f}(x,y) = (e^y \cos(xe^y), xe^y \cos(xe^y))$$

is a gradient of some scalar field. If so, find one of these scalar fields φ such that $\mathbb{f}(x,y) = \nabla\varphi(x,y)$.

Solution.

Let us call $\mathbb{f}(x,y) = (f_1(x,y), f_2(x,y))$, where $f_1(x,y) = e^y \cos(xe^y)$, $f_2(x,y) = xe^y \cos(xe^y)$. We compute:

$$\frac{\partial f_1}{\partial y}(x,y) = e^y \cos(xe^y) - xe^{2y} \sin(xe^y),$$
$$\frac{\partial f_2}{\partial x}(x,y) = e^y \cos(xe^y) - xe^{2y} \sin(xe^y),$$

and we see that they coincide. As $\mathbb{R}^2$ is convex, this implies that $\mathbb{f}$ is a gradient.

To find a concrete potential φ, we can take

$$\begin{aligned}\varphi(x,y) &= \int_0^y f_2(0,t)dt + \int_0^x f_1(t,y)dt \\ &= 0 + [\sin(te^y)]_0^x \\ &= \sin(xe^y).\end{aligned}$$

4. (6 points) Compute

$$\iiint_S (x^2 + y^2 - \arctan z)\, dxdydz,$$

where

$$S = \{(x, y, z) \in \mathbb{R}^3 \,:\, x^2 + y^2 \leq 1,\, 0 \leq z \leq 1\}.$$

Solution.
By additivity of integral, we can split it into the following

$$\iiint_S (x^2 + y^2)\, dxdydz - \iiint_S (\arctan z)\, dxdydz.$$

The region S is xy-projectable: $S = \{(x, y, z) \in \mathbb{R}^3 \,:\, (x, y) \in S_0,\, 0 \leq z \leq 1\}$, where $S_0 = \{(x, y) \in \mathbb{R}^2 \,:\, x^2 + y^2 \leq 1\}$, so we can compute the integral by first integrating with respect to z and then xy.

The first integrand does not depend on z, therefore,

$$\begin{aligned}
\iiint_S (x^2 + y^2)\, dxdydz &= \iint_{S_0} \int_0^1 (x^2 + y^2)\, dz\, dxdy \\
&= \iint_{S_0} (x^2 + y^2)\, dxdy \\
&= \int_0^{2\pi} \int_0^1 r^2 r dr d\theta \\
&= \frac{\pi}{2}.
\end{aligned}$$

As for the second integral, note that $\arctan z = (z \arctan z - \frac{1}{2} \log(1 + z^2))'$. Therefore,

$$\begin{aligned}
\iiint_S \arctan z\, dz\, dxdy &= \iint_{S_0} \int_0^1 \arctan z\, dz\, dxdy \\
&= \iint_{S_0} [z \arctan z - \frac{1}{2} \log(1 + z^2))]_0^1\, dxdy \\
&= \iint_{S_0} (\frac{\pi}{4} - \frac{1}{2} \log 2)\, dxdy \\
&= \pi \left(\frac{\pi}{4} - \frac{1}{2} \log 2 \right).
\end{aligned}$$

Altogether,

$$\iiint_S (x^2 + y^2 - \arctan z)\, dxdydz = \pi \left(\frac{1}{2} - \frac{\pi}{4} + \frac{1}{2} \log 2 \right).$$

5. (6 points) Let $\mathbb{F}(x,y,z) = (x^3, y^3, z^3)$ be a vector field on $\mathbb{R}^3$, S be the sphere $\{(x,y,z) : x^2+y^2+z^2 = a^2\}$, where $a > 0$ and $\mathbb{n}$ the outgoing normal unit vector on S at each point of S.

Compute the surface integral

$$\iint\limits_S \mathbb{F} \cdot \mathbb{n}\, dS.$$

Solution.

Thanks to Gauss' theorem (divergence theorem), this integral is equal to the following volume integral

$$\iiint\limits_V \mathrm{div}\mathbb{F}\, dxdydz$$

where $V = \{(x,y,z) : x^2+y^2+z^2 \le a^2\}$.

Let us compute:

$$\mathrm{div}\mathbb{F} = 3x^2 + 3y^2 + 3z^2 = 3(x^2+y^2+z^2).$$

To perform the volume integral, we use the spherical coordinate $x = r\cos\theta\sin\varphi, y = r\sin\theta\sin\varphi, z = r\cos\varphi$. The region Q correponding to V in this change of coordinate is $Q = \{(r,\theta,\varphi) : 0 \le r \le a,\, 0 \le \theta \le 2\pi,\, 0 \le \varphi \le \pi\}$. Recall that the Jacobian determinant is $J(r,\theta,\varphi) = -r^2\sin\varphi$, and note that $x^2+y^2+z^2 = r^2$. Theorefore,

$$\begin{aligned}
\iint\limits_S \mathbb{F} \cdot \mathbb{n}\, dS &= \iiint\limits_V \mathrm{div}\mathbb{F}\, dxdydz \\
&= 3\iiint\limits_Q r^2 \cdot r^2 \sin\varphi\, drd\theta d\varphi \\
&= 3\int_0^{\pi}\int_0^{2\pi}\int_0^{a} r^4 \sin\varphi\, drd\theta d\varphi \\
&= 3 \cdot 2 \cdot 2\pi \cdot \frac{a^5}{5}. \\
&= \frac{12\pi a^5}{5}.
\end{aligned}$$

Note: some people tried to do this without using Gauss' theorem. At best they transformed the surface integral into a correct double integral, in which case, they obtain 3 points.