Sequence and Series

- $a_0, a_1, \cdots a_n, \cdots$: sequence of real / complex numbers.
- Partial sum $\sum_{n=0}^{N} a_n$. If this converges to L, then we write $L = \sum_{n=0}^{\infty} a_n$

<u>Ex. 1</u> Geometric series. $a > 1$. $\displaystyle\sum_{n=0}^{N} \frac{1}{a^n} = \frac{a^{N+1}-1}{a^N(a-1)} = \frac{a}{a-1} - \frac{1}{a^N(a-1)} \to \frac{a}{a-1}$

<u>Ex. 2</u> $\displaystyle\sum_{n=1}^{N} \frac{1}{n(n+1)} = \sum_{n=1}^{N} \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{N+1} \to 1.$

<u>Ex. 3</u> $\sum \frac{1}{n}$ is not convergent. $\sum \frac{1}{n^2}$ is convergent.

Criteria of convergence

If $\sum_{n=0}^{\infty} a_n$ is convergent, then $a_n \to 0$.

If a_n is not convergent to 0, then $\sum_{n=0}^{\infty} a_n$ is not convergent.

[Root test] Let $a_n \geq 0$, and $a_n^{\frac{1}{n}} \to R$.
(a) if $R < 1$, $\sum_{n=0}^{\infty} a_n$ converges
(b) if $R > 1$, $\sum_{n=0}^{\infty} a_n$ diverges

[Ratio test] Let $a_n > 0$, and $\frac{a_{n+1}}{a_n} \to L$.
(a) if $L < 1$, $\sum a_n$ converges
(b) if $L > 1$, $\sum a_n$ diverges.

[Comparison test] Let $a_n, b_n \geq 0$, $c > 0$ such that $a_n \leq c b_n$.
If $\sum b_n$ converges, so does $\sum a_n$.

<u>Exercises.</u>

1. Compute $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}$.
2. $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = 1.$

Determine the convergence.

10.14 $\cancel{3.\ \sum \frac{n}{(4n-3)(4n-1)}}$. 4. $\sum \frac{n^2}{2^n}$ 5. $\sum \frac{|\sin n|}{n^2}$

<u>We know</u> $\sum \frac{1}{n^s}$ is convergent if $s > 1$, divergent if $s \leq 1$.

<u>Compute.</u> $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^{n-1}}$ $\qquad \displaystyle\sum_{n=1}^{\infty} \frac{2^n + n^2 + n}{2^{n+1} n(n+1)}$

Improper integrals

First kind. $\int_a^b f(x)\,dx$ is first defined for a bounded function f.

$I(b) = \int_a^b f(x)\,dx$. If the limit $\lim_{b\to\infty} I(b)$ exists, we write it by $\int_a^\infty f(x)\,dx$.

Similarly, $\int_{-\infty}^{b} f(x)\,dx$, $\int_{-\infty}^{\infty} f(x)\,dx$.

Ex. $\int_0^b e^{-cx}\,dx = -\frac{1}{c}\left[e^{-cx}\right]_0^b = -\frac{1}{c}\left(e^{-cb}-1\right) \to \frac{1}{c}$.

Second kind

Let $f(x)$ be defined on $(a,b]$ (half-interval).

$I(x) = \int_x^b f(t)\,dt$. If $\lim_{x\to a+} I(x)$ exists, we denote it by $\int_a^b f(x)\,dx$

Ex. $f(t) = t^{-s}$, $t \in (0,1]$. $\int_x^b t^{-s}\,ds = \begin{cases} \frac{1}{1-s}\left(b^{1-s} - x^{1-s}\right) & s \neq 1 \\ \log b - \log x & s = 1 \end{cases}$

This is convergent if $s<1$.

Exercises 1. $\int_1^\infty x^s\,ds$ $\int_1^b x^s\,ds = \begin{cases} \left[\frac{b^{1-s}-1}{1-s}\right] & s \neq 0 \\ \log b & s = 1 \end{cases}$

convergent if $s<1$.

2. $\int_2^\infty \frac{x}{x^2+1} - \frac{2}{2x+1}\,dx$ $\int_2^b \frac{x}{x^2+1} - \frac{2}{2x+1}\,dx = \left[\frac{1}{2}\log(x^2+1) - \log(2x+1)\right]_2^b$

$= \left[\frac{1}{2}\log\frac{x^2+1}{(2x+1)^2}\right]_2^b$

Exercises Determine the convergence.

1. $\int_0^\infty \frac{x}{\sqrt{x^4+1}}\,dx$ 2. $\int_{-\infty}^\infty e^{-x^2}\,dx$. 3. $\int_0^\infty \frac{1}{\sqrt{x^3+1}}\,dx$.

4. $\int_0^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$. 7. $\int_0^1 \frac{\log x}{1-x}\,dx$.

Compute the improper integral.

1. $\int_1^\infty \left(\frac{x}{2x^2+1} - \frac{1}{2x+2}\right)dx$.

Criterion $f(x) \geq g(x) \geq 0$. If $\int_a^b f(x)\,dx$ is convergent, then so is $\int_a^b g(x)\,dx$

If $\int_a^b g(x)\,dx$ is divergent, then so is $\int_a^b f(x)\,dx$.

Absolute/conditional convergence

$\{x_n\}$: sequence of numbers. If $\sum |x_n|$ is convergent, we say $\sum x_n$ is absolutely convergent. If $\sum x_n$ is convergent but $\sum |x_n|$ not, then it is conditionally convergent.

Exercise 1. Prove that $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is conditionally convergent.

2. $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$.

Sequence and series of functions

$f_1(x), f_2(x), \cdots f_n(x), \cdots$ functions on $S \subset \mathbb{R}$.

If, for each $x \in S$, $\{f_n(x)\}$ (sequence of numbers) is convergent, we say $\{f_n\}$ is pointwise convergent.

Let $\{f_n\}$ be pointwise convergent, $f(x)$ be the limit. If, for any $\varepsilon > 0$, there is N s.t. $|f_n(x) - f(x)| < \varepsilon$ for $n \geq N$, $\{f_n\}$ is uniformly convergent.

Thm. If $\{f_n\}$ is a sequence of continuous functions, uniformly convergent to f, then f is continuous.

Thm. $\{f_n\}$, f as above, then $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$.

Series of functions $g_N(x) = \sum_{n=0}^{N} f_n(x)$

Power series

Let $\{a_n\}$ be a sequence of numbers. Consider a series of functions $\sum a_n(x-a)^n$

Thm. For a power series $\sum a_n(x-a)^n$, there is $R \geq 0$, or $R = \infty$ (the radius of convergence) s.t. for $r < R$, $\sum a_n(x-a)^n$ is uniformly and absolutely convergent for $|x-a| < r$ and divergent for $|x-a| > R$.

Exercise. Determine the radius of convergence.

1. $\sum \frac{x^n}{2^n}$ 2. $\sum \frac{n!}{n^n} x^n$ 3. $\sum \frac{(-1)^n}{n^2+1} x^n$

2 e 1.

Thm. Let $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ for $x \in (a-r, a+r)$. r = rad of convergence.

Then $\int_a^x f(t)dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-a)^{n+1}$.

$f'(x) = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1}$ (Same rad of convergence).

Especially, $f^{(k)}(a) = k!\, a_k$

If $\sum a_n (x-a)^n = \sum b_n (x-a)^n$, then $a_n = b_n$.

Exercises Determine the rad of convergence and compute the sum.

1. $\sum_{n=0}^{\infty} \frac{x^n}{3^{n+1}}$ 2. $\sum_{n=0}^{\infty} n x^n$ 3. $\sum_{n=0}^{\infty} \frac{x^n}{(n+3)!}$

$|x|<3$, $\frac{1}{3-x}$ $|x|<1$, $\frac{x}{(1-x)^2}$ $R=\infty$. $\frac{1}{x^3}\left(e^x - 1 - x - \frac{x^2}{2}\right)$

Let $f(x)$ be infinitely differentiable.

Taylor's series $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$.

Thm. If $|f^{(n)}(x)| \leq A^n$ for some A, $x \in (-a, a)$, then $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$.

Examples $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

$\exp x = 1 + x + \frac{x^2}{2!} + \cdots$

Exercise Prove the expansion.

1. $\log(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)} x^{n+1}$ $|x|<1$.

2. $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$

3. $\log\sqrt{\frac{1+x}{1-x}} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$ $|x|<1$.

Differential equations

Solve $f'(x) = f(x) + x$ by assuming $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{\infty} n a_n x^{n-1} = x + \sum_{n=0}^{\infty} a_n x^n \Rightarrow a_0 = a_1,\ 2a_2 = 1 + a_1,\ (n+1) a_{n+1} = a_n$.

$f(x) = a_0 + a_0 x + \sum_{n=2}^{\infty} \frac{1+a_0}{n!} x^n$.

Exercise Solve $f'(x) = 2x f(x)$.

Convergence of $\log(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$ $\quad \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \Rightarrow$ Term by Term integration

Thm 11.11* Assume f is infinitely differentiable in $x \in (a-r, a+r)$ and assume that $|f^{(n)}(x)| \le n! \frac{C}{r^n}$ for some $C > 0$.
Then $f(x) = \sum \frac{f^{(n)}(0)}{n!} x^n$.

proof) Let $E_n(x)$ be the error term in the Taylor formula.
We have $|E_n(x)| \le \frac{|x-a|^{n+1}}{n!} \cdot (n+1)! \frac{C}{r^n} \cdot \frac{1}{n+1} \le \frac{C|x-a|^{n+1}}{r^n} \to 0$.

$f(x) = \log(1+x)$ $\quad f^{(n)}(x) = \frac{(n-1)!(-1)^{n-1}}{(1+x)^n}$. $\quad |f^{(n)}(x)| \le n! \left(\frac{1}{2}\right)^n$ for $x \in [-\frac{1}{2}, \frac{1}{2}]$

Differential equations

Solve $f'(x) = f(x) + x$ by assuming $f(x) = \sum a_n x^n$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = x + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \Rightarrow a_0 = a_1,\ 2a_2 = 1 + a_1,\ (n+1)a_{n+1} = a_n$$

$$f(x) = a_0 + a_0 x + \sum_{n=2}^{\infty} \frac{1+a_0}{n!} x^n.$$

Exercise 11.16.4 Solve $x f''(x) + f'(x) - f(x) = 0$.

Scalar fields

$f(x,y)$, $f(x,y,z)$ functions of n-variables.

Partial derivatives $\frac{\partial f}{\partial x}(x,y,z) = \lim_{h \to 0} \frac{f(x+h,y,z) - f(x,y,z)}{h}$ $\left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\ \text{similar}\right)$

"Think other variables as constants".

Exercises 1. $f(x,y) = \frac{1}{y}\cos^2 x$ $\quad \frac{\partial f}{\partial x} = \frac{1}{y} 2\cos x \cdot (-\sin x)$. $\quad \frac{\partial f}{\partial y} = -\frac{1}{y^2}\cos^2 x$.

Compute partial derivatives.

2. $f(x,y) = x^{e^y}$ $\quad \frac{\partial f}{\partial x} = e^y x^{e^y - 1}$. $\quad \frac{\partial f}{\partial y} = \log x \cdot x^{e^y} e^y$

Gradient $\nabla f(x,y) = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right)$ (a vector field).

Consider also $\frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial x^2}, \dots$

Thm. If $\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$ are both continuous, then $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

Chain rule $f(x,y,z)$: scalar field. $\alpha(t) = (X(t), Y(t), Z(t))$: curve
$f(\alpha(t))$ is a function of t. $\frac{d}{dt} f(\alpha(t)) = \nabla f(\alpha(t)) \cdot \alpha'(t)$

Example $f(x,y) = x^2 + y^2$, $\alpha(t) = (t, t^2)$. $\quad \frac{d}{dt} f(\alpha(t)) = 2t + 4t^3$.
$\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 2y$, $\alpha'(t) = (1, 2t)$

Application $f(t) = t^t$ $\quad g(x,y) = x^y$, $\alpha(t) = (t,t)$. $g(\alpha(t)) = f(t)$. $\frac{\partial g}{\partial x} = y x^{y-1}$, $\frac{\partial g}{\partial y} = \log x\, x^y$.
$f'(t) = t\, t^{t-1} + \log t\, t^t$.

Exercise $f(t) = t^{t^t}$. compute $f'(t)$.

$f(x)$: scalar field. <u>Stationary points</u>

<u>Def</u> a is a relative minimum / maximum if $f(a) \leq f(x)$ / $f(a) \geq f(x)$ for $x \in B(a;r)$ for some $r>0$.

Let f be differentiable. If $\nabla f(x) = (0, \dots, 0)$, then x is a stationary point.

Minima and Maxima are stationary. Other stationary points are saddle points.

Hessian matrix $H(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ & & \\ \frac{\partial^2 f}{\partial z \partial x} & \cdot & \frac{\partial^2 f}{\partial z^2} \end{pmatrix}$

<u>Thm</u> Let f be a scalar field with continuous second derivatives.

Let a be a stationary point.

(1) If all eigenvalues of $H(a)$ are positive, then f takes a relative min at a.

(2) negative, max.

(3) If there are both positive and negative eigenvalues, a is a saddle.

<u>Exercises</u> 9.13. Find all stationary points and classify them.

5. $f(x,y) = 2x^2 - xy - 3y^2 - 3x + 7y$.

$\nabla f(x,y) = (4x - y - 3, -x - 6y + 7)$. $\nabla f(x,y) = (0,0) \iff \begin{cases} 4x - y - 3 = 0 \\ -x - 6y + 7 = 0 \end{cases}$ $y = 1, x = 1$.

$H(x,y) = \begin{pmatrix} 4 & -1 \\ -1 & -6 \end{pmatrix}$ $\det H < 0 \Rightarrow (1,1)$ is a saddle.

9. $f(x,y) = x^3 + y^3 - 3xy$.

$\nabla f(x,y) = (3x^2 - 3y, 3y^2 - 3x)$. $\nabla f(x,y) = (0,0) \iff \begin{cases} 3x^2 - 3y = 0 \\ 3y^2 - 3x = 0 \end{cases} \iff (x,y) = (0,0), (1,1)$.

$H(x,y) = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$ $H(1,1) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$ $\det H(1,1) > 0$. $\mathrm{tr}\, H(1,1) > 0$ $(1,1)$ is a min.

$H(0,0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$ $\det H(0,0) < 0$ $(0,0)$ is a saddle.

11. $f(x,y) = e^{2x+3y}(8x^2 - 6xy + 3y^2)$. 12. $f(x,y) = (5x + 7y - 25)e^{-(x^2+xy+y^2)}$

<u>Partial differential equations</u>

Equations about functions.

Let $f(x,y)$ have continuous partial derivatives.

$a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y} = 0$. Solution. With a cont. differentiable $g(t)$, $f(x,y) = g(bx - ay)$.

<u>Wave Equation</u>

$f(x,t)$ has continuous second derivatives.

$\frac{\partial^2 f}{\partial t^2} = c\frac{\partial^2 f}{\partial x^2} = 0$, $c > 0$. Solution: F: twice differentiable. G: differentiable

$$f(x,t) = \frac{F(x+ct) + F(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\,ds.$$

<u>Exercise.</u> Solve $5\frac{\partial f}{\partial x} - 2\frac{\partial f}{\partial y} = 0$ with $f(0,0) = 0$, $\frac{\partial f}{\partial x}(x,0) = e^x$.

Solution. $f(x,y) = g(2x + 5y)$. $g(0) = 0$, $2g'(2x) = e^x \Rightarrow g(t) = e^{\frac{t}{2}}$ $f(x,y) = e^{\frac{2x+5y}{2}}$.

Solve $\frac{\partial^2 f}{\partial t^2} = c\frac{\partial^2 f}{\partial x^2}$ with $f(x,0) = 0$, $\frac{\partial f}{\partial t}(x,0) = x^3 e^{-x^2}$

Lagrange's multiplier method

Finding stationary points under constraints.

If a scalar field $f(x)$ has a relative extremum at a, with constraints $g_1(x)=0, \cdots g_m(x)=0$, $m<n$ and $\nabla g_k(a)$ are linearly independent, then there are $\lambda_1, \cdots \lambda_m \in \mathbb{R}$ such that $\nabla f(a) = \lambda_1 \nabla g_1(a) + \cdots + \lambda_m \nabla g_m(a)$.

$(m+n)$-equations for $(m+n)$-variables.

Exercises 9.15.5 Find extremal values of $f(x,y,z) = x-2y+2z$ on the sphere $x^2+y^2+z^2=1$.

Solution. $\nabla f(x,y,z) = (1,-2,2)$. Define $g(x,y,z) = 1-x^2-y^2-z^2$.

$\nabla g(x,y,z) = (-2x,-2y,-2z)$. By Lagrange, $\lambda(-2x,-2y,-2z) = (1,-2,2)$.

We may assume $\lambda \neq 0$. $(x,y,z) = \left(-\frac{1}{2\lambda}, \frac{1}{\lambda}, -\frac{1}{\lambda}\right)$. $x^2+y^2+z^2=1$.

$\Rightarrow \frac{1}{4\lambda^2} + \frac{1}{\lambda^2} + \frac{1}{\lambda^2} = \frac{9}{4\lambda^2} = 1 \Rightarrow \lambda = \pm\frac{3}{2}$.

$(x,y,z) = \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)$. $f(x,y,z) = \frac{1}{3}$ (max).

$(x,y,z) = \left(-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right)$. $f(x,y,z) = -\frac{1}{3}$ (min).

4. Find the extremal values of $f(x,y) = \cos^2 x + \cos^2 y$, under $x-y=\frac{\pi}{4}$.

6. Find the nearest points from the origin of the surface $z^2-xy=1$.

Line integrals

$f(x)$: vector field on $S \subset \mathbb{R}^n$. $\alpha(t) = (X_1(t), \cdots X_n(t))$: (piecewise smooth) curve in S. C

($X_k(t)$'s are functions on $[a,b] \subset \mathbb{R}$). $\alpha'(t) = (X_1'(t), \cdots X_n'(t))$.

$\int_C f \cdot d\alpha := \int_a^b f(\alpha(t)) \cdot \alpha'(t)\,dt$.

Sometimes this is written as $\int_C f_1 dx_1 + \cdots + \int_C f_n dx_n$.

If there is u s.t. $\alpha(u(t)) = \beta(t)$ with $u'(t)>0$ (same direction) $\int_C f\,d\alpha = \int_C f \cdot d\beta$.

$u'(t)<0$ (opposite direction) $\int_C f\,d\alpha = -\int_C f \cdot d\beta$.

Exercises 10.5.6. Compute the line integral

$f(x,y,z) = (2xy, x^2+z, y)$ on the segment $(1,0,2)$ to $(3,4,1)$.

Solution. We take $\alpha(t) = (2t+1, 4t, 2-t)$, $t \in [0,1]$. $\alpha'(t) = (2,4,-1)$.

$\int_C f \cdot d\alpha = \int_0^1 \{2(2t+1)4t, (2t+1)^2+2-t, 4t\} \cdot (2,4,-1)\,dt = \int_0^1 (48t^2+24t+12)\,dt = 40$.

8. $f(x,y,z) = (x,y,xz-y)$. $\alpha(t) = (t^2, 2t, 4t^3)$. $t \in [0,1]$

10. $f(x,y) = \left(\frac{x+y}{x^2+y^2}, \frac{-(x-y)}{x^2+y^2}\right)$. C is the circle $x^2+y^2=a^2$, counterclockwise.

Length of a curve $\quad s = \int \|\alpha'(t)\|\,dt$.

Thm. Let $\mathbf{f}=\nabla\varphi$, then, for a curve α s.t. $\alpha(a)=\mathbf{a}$, $\alpha(b)=\mathbf{b}$.

$$\int_C \mathbf{f}\cdot d\alpha = \varphi(\mathbf{b})-\varphi(\mathbf{a}).$$

Thm. Let $\mathbf{f}$ be a vector field on S. If, for any curve α s.t. $\alpha(a)=\mathbf{a}$, $\alpha(b)=\mathbf{b}$, $\int_C \mathbf{f}\cdot d\alpha$ gives the same value, then $\mathbf{f}(x)=\nabla\varphi(x)$ where $\int_C \mathbf{f}\cdot d\alpha$, $\alpha(a)=\mathbf{a}$, $\alpha(b)=x$.

Thm. For a vector field $\mathbf{f}$ on S the following are equivalent.

(1) $\mathbf{f}=\nabla\varphi$ for some φ

(2) $\int_C \mathbf{f}\cdot d\alpha$ depends only on $\alpha(a)$ & $\alpha(b)$.

(3) $\int_C \mathbf{f}\cdot d\alpha=0$ for α s.t. $\alpha(a)=\alpha(b)$ (closed path).

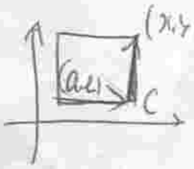
Thm. Let $\mathbf{f}$ be a vector field on S, continuously differentiable.

If $\mathbf{f}=\nabla\varphi$, then $\frac{\partial f_j}{\partial x_k}=\frac{\partial f_k}{\partial x_j}$ for all k,j.

Let S be convex. If $\frac{\partial f_j}{\partial x_k}=\frac{\partial f_k}{\partial x_j}$ for all k,j, then $\mathbf{f}=\nabla\varphi$ for some φ.

Exercise. Show that $\mathbf{f}$ is not gradient. $\mathbf{f}(x,y,z)=(2xy^3, x^2z^2, 3x^2yz^2)$

Constructing the potential.



If $\mathbf{f}=\nabla\varphi$ on $\mathbb{R}^2$, we can take $\varphi(x,y)=\int_C \mathbf{f}\cdot d\alpha = \int_a^x f_1(t,b)\,dt + \int_b^y f_2(x,t)\,dt$.

on $\mathbb{R}^3$, $\varphi(x,y,z)=\int_a^x f_1(t,b,c)\,dt+\int_b^y f_2(x,t,c)\,dt+\int_c^z f_3(x,y,t)\,dt$.

Exercises 10.18 Determine whether $\mathbf{f}$ is a gradient. if so, find φ s.t. $\mathbf{f}=\nabla\varphi$.

1. $\mathbf{f}(x,y)=(x,y)$. Yes. $\varphi=\frac{x^2+y^2}{2}$.
3. $\mathbf{f}(x,y)=(2xe^y+y, x^2e^y+x-2y)$. yes. $\varphi=x^2e^y+xy-y^2$.
8. $\mathbf{f}(x,y,z)=(3y^4z^2, 4x^3z^2, -3x^2y^2)$. no.
11. $\mathbf{f}(x,y,z)=(y^2\cos x+z^3, -4+2y\sin x, 3xz^2+2)$. yes. $\varphi=y^2\sin x+xz^3-4y+2z$.

Exact differential equations

$y=Y(x)$. $P(x,y)$, $Q(x,y)$: known functions. $P(x,y)+Q(x,y)y'=0$ is said to be exact if there is $\varphi(x,y)$ s.t. $(P(x,y),Q(x,y))=\nabla\varphi(x,y)$. Written also $P(x,y)dx+Q(x,y)dy=0$.

A solution $Y(x)$ satisfies $\varphi(x,Y(x))=C$ for some C.

Exercises 10.20 1. $(x+2y)+(2x+y)y'=0$. $\varphi(x,y)=\frac{x^2}{2}+2xy+\frac{y^2}{2}=\left(2x+\frac{y}{2}\right)^2-\frac{7}{2}x^2=C$

$$\frac{y}{2}=-2x\pm\sqrt{C+\frac{7}{2}x^2}.$$

2. $2xy+x^2y'=0$. $\varphi(x,y)=x^2y$.

Even if $P(x,y)+Q(x,y)y'$ is not exact, one might find $\mu(x,y)$ s.t. $\mu(x,y)P(x,y)+\mu(x,y)Q(x,y)y'=0$ is exact.

Example. $y-(2x+y)y'=0$ is not exact. $\mu(x,y)=\frac{1}{y^3}$. $\frac{1}{y^2}+\frac{-2x-y}{y^3}y'=0$ is exact.

Multiple integrals

f: scalar bounded field defined on a bounded rectangle $[a_1, b_1]\times\cdots\times[a_n, b_n]=Q$

Take step functions $s \le f \le t$. (constant on small rectangles)

If $\sup \iint_Q s(x_1\cdots x_n)dx_1\cdots dx_n = \inf \iint_Q t(x_1\cdots x_n)dx_1\cdots dx_n$. f is integrable. $\int\cdots\int_Q f(x_1\cdots x_n)dx_1\cdots dx_n$

S: region in $Q\subset\mathbb{R}^2$, $S=\{(x,y): a\le x\le b,\ \varphi_1(x)\le y\le\varphi_2(x)\}$ (type I).

f: function on S, extended to Q by setting $f(x,y)=0$ if $(x,y)\notin S$.

$$\iint_S f(x,y)\,dxdy := \iint_Q f(x,y)\,dxdy = \int_a^b\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\,dx.$$

Similary, if $V\subset Q\subset\mathbb{R}^3$ is xy-projectable. $V=\{(x,y,z): (x,y)\in S,\ \varphi_1(x,y)\le z\le\varphi_2(x,y)\}$.

$$\iiint_V f(x,y,z)\,dxdydz = \iint_S\int_{\varphi_1(x,y)}^{\varphi_2(x,y)} f(x,y,z)\,dz\,dxdy.$$

Exercises 11.15 8. $S=\{(x,y): 4x^2+9y^2\le 36,\ x\ge 0,\ y\ge 0\}=\{(x,y): 0\le x\le 3,\ 0\le y\le \frac{2}{3}\sqrt{9-x^2}\}$

$$\iint_S (3x+y)\,dxdy = \int_0^3\int_0^{\frac{2}{3}\sqrt{9-x^2}}(3x+y)\,dy\,dx = \int_0^3\left[3xy+\frac{y^2}{2}\right]_0^{\frac{2}{3}\sqrt{9-x^2}}dx$$

$$=\int_0^3 2x\sqrt{9-x^2}+\frac{2}{9}(9-x^2)\,dx = \left[-\frac{2}{3}(9-x^2)^{\frac{3}{2}}+2x-\frac{2}{27}x^3\right]_0^3 = 18+6-2=22.$$

11.34.2 $V=\{(x,y,z): x\ge 0, y\ge 0, z\ge 0,\ x+y+z\le 1\}=\{(x,y,z): 0\le x\le 1,\ 0\le y\le 1-x,\ 0\le 1-x-y\}$.

$$\iiint_V \frac{1}{(1+x+y+z)^3}\,dxdydz = \int_0^1\int_0^{1-x}\int_0^{1-x-y}\frac{1}{(1+x+y+z)^3}\,dz\,dydx = \int_0^1\int_0^{1-x}\left[-\frac{1}{2(1+x+y+z)^2}\right]_0^{1-x-y}dydx$$

$$=\int_0^1\int_0^{1-x}\left(\frac{1}{2(1+x+y)^2}-\frac{1}{8}\right)dydx = \int_0^1\left[-\frac{1}{2(1+x+y)}-\frac{y}{8}\right]_0^{1-x}dx = \int_0^1\left(+\frac{1}{2(1+x)}-\frac{1}{4}-\frac{1-x}{8}\right)dx$$

$$=\left[+\frac{1}{2}\log(1+x)-\frac{3}{8}x+\frac{x^2}{16}\right]_0^1 = +\frac{1}{2}\log 2-\frac{5}{16}.$$

11.15 8(c). $\iint_S (y+2x+20)\,dxdy$. $S=\{(x,y): x^2+y^2\le 16\}$

11.34 3. $\iiint_V xyz\,dxdydz$. $V=\{(x,y,z): x^2+y^2+z^2\le 1,\ x\ge 0, y\ge 0, z\ge 0\}$.

If we integrate $f(x,y)=1$, it gives the area. $f(x,y,z)=1$ volume.

11.15. 7. Compute the volume of the solid bounded by $z=0$, $z=x^2-y^2$, $x=1$, $x=3$.

Green's theorem on $\mathbb{R}^2$

Let P, Q be continuously differentiable scalar fields, S be a connected region in $\mathbb{R}^2$ with a piecewise smooth boundary C parametrized by a single curve $\alpha(t)$.

Then. $$\iint_S\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy = \int_C f\cdot d\alpha,\ \text{where } f(x,y)=(P(x,y), Q(x,y)).$$
$$\left(=\int_C P\,dx+Q\,dy\right).$$

Exercises 11.22. 1(d) Compute $\int_C y^2dx + xdy$ where C is the circle $x^2+y^2=4$, counterclockwise.

$P(x,y) = y^2$, $Q(x,y) = x$. By Green's theorem, with $S = \{(x,y) = x^2+y^2 \le 4\}$

$$\iint_S (1-2y)\,dxdy = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (1-2y)\,dy\,dx = \int_{-2}^{2} [y-y^2]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\,dx$$

$$= 2\int_{-2}^{2}\sqrt{4-x^2}\,dx = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cdot\cos t\cdot 2\cos t\,dt = 8\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos 2t+1}{2}\,dt = 4\pi \qquad 2\sin t = x.$$

1(e). Compute $\int_C y^2dx + xdy$, C is given by $\alpha(t) = (2\cos^3 t, 2\sin^3 t)$, $t \in [0, 2\pi]$

Change of variables

Let S be a region in xy-plane. Assume that there is another region T in uv-plane and the map $\mathbb{r}(u,v) = (X(u,v), Y(u,v))$ is one-to-one from T to S.

$$J(u,v) = \begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial Y}{\partial u} \\ \frac{\partial X}{\partial v} & \frac{\partial Y}{\partial v} \end{vmatrix} \quad \text{Jacobian determinant.}$$

$$\iint_S f(x,y)\,dxdy = \iint_T f(X(u,v), Y(u,v))\,|J(u,v)|\,dudv.$$

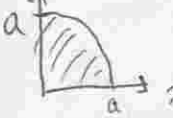
Polar coordinate

$x = X(r,\theta) = r\cos\theta$, $y = Y(r,\theta) = r\sin\theta$. $J(u,v) = r$.

Linear transform

$x = Au+Bv$, $y = Cu+Dv$. $J(u,v) = AD-BC$.

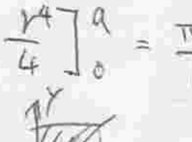
Exercises 11.28.9. Compute $\int_0^a\int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dxdy$



$T = \{(r,\theta) : 0 \le r \le a,\ 0 \le \theta \le \frac{\pi}{2}\}$.

$$= \int_0^{\frac{\pi}{2}}\int_0^a r\cdot r^2\,drd\theta = \int_0^{\frac{\pi}{2}}\left[\frac{r^4}{4}\right]_0^a = \frac{\pi a^4}{8}.$$

11.28.6. Compute $\int_0^a\int_0^x \sqrt{x^2+y^2}\,dy\,dx$.



Compute $\iint_S \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)dxdy$, $S = \{(x,y) : \frac{x^2}{a^2}+\frac{y^2}{b^2} \le 1\}$

Change of variables in $\mathbb{R}^3$

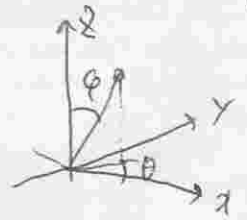
V in xyz-space, Q in uvw-space, $x = X(u,v,w)$, $y = Y(u,v,w)$, $z = Z(u,v,w)$.

$$\iiint_V f(x,y,z)\,dxdydz = \iiint_Q f(X,Y,Z)\,J(u,v,w)\,dudvdw, \quad J(u,v,w) = \begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial Y}{\partial u} & \frac{\partial Z}{\partial u} \\ \frac{\partial X}{\partial v} & \frac{\partial Y}{\partial v} & \frac{\partial Z}{\partial v} \\ \frac{\partial X}{\partial w} & \frac{\partial Y}{\partial w} & \frac{\partial Z}{\partial w} \end{vmatrix}.$$

Cylindrical coordinate

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$. $J(r,\theta,z) = r$.

Spherical coordinate



$x = r\cos\theta\sin\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\varphi$

$$J(r,\theta,\varphi) = \begin{vmatrix} \cos\theta\sin\varphi & \sin\theta\sin\varphi & \cos\varphi \\ -r\sin\theta\sin\varphi & r\cos\theta\sin\varphi & 0 \\ r\cos\theta\cos\varphi & r\sin\theta\cos\varphi & -r\sin\varphi \end{vmatrix} = -r^2\sin\varphi.$$

Example Volume of sphere. $V = \{(x,y,z) : x^2+y^2+z^2 \le a^2\} = \{(r,\theta,\varphi) : 0 \le r \le a,\ 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi\}$

$$\iiint r^2\sin\varphi\,drd\theta d\varphi = \int_0^{2\pi}\int_0^{\pi}\int_0^{r} r^2\sin\varphi\,drd\varphi d\theta = \frac{4}{3}\pi a^3.$$

11.34.5

Exercise Compute $\iiint_V \sqrt{x^2+y^2}\,dxdydz$. $V = \{(x,y,z) : x^2+y^2 \le z^2 \le 1\}$

Surface integrals

A surface S in $\mathbb{R}^3$ can be parametrized by $\mathbb{r}(u,v) = (X(u,v), Y(u,v), Z(u,v))$.
$\mathbb{r}$ is a map from a region in uv-plane to S.

Examples. The upper hemisphere $x^2+y^2+z^2=a^2$, $z \ge 0$. $T_1 = \{(\theta,\varphi) : 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \frac{\pi}{2}\}$

(1) $x = X(\theta,\varphi) = a\cos\theta\sin\varphi$, $y = Y(\theta,\varphi) = a\sin\theta\sin\varphi$, $z = Z(\theta,\varphi) = a\cos\varphi$.

(2) $x = X(u,v) = u$, $y = Y(u,v) = v$, $Z(u,v) = \sqrt{a^2-u^2-v^2}$ $\quad T_2 = \{(u,v) : u^2+v^2 \le a^2\}$

Fundamental vector product: $\frac{\partial \mathbb{r}}{\partial u} = \left(\frac{\partial X}{\partial u}, \frac{\partial Y}{\partial u}, \frac{\partial Z}{\partial u}\right)$, $\frac{\partial \mathbb{r}}{\partial v} = \left(\frac{\partial X}{\partial v}, \frac{\partial Y}{\partial v}, \frac{\partial Z}{\partial v}\right)$. $\frac{\partial \mathbb{r}}{\partial u} \times \frac{\partial \mathbb{r}}{\partial v}$

In Ex (1), $\frac{\partial \mathbb{r}}{\partial \theta} = (-a\sin\theta\sin\varphi, a\cos\theta\sin\varphi, 0)$ $\frac{\partial \mathbb{r}}{\partial \varphi} = (a\cos\theta\cos\varphi, a\sin\theta\cos\varphi, -a\sin\varphi)$.

$$\frac{\partial \mathbb{r}}{\partial \theta} \times \frac{\partial \mathbb{r}}{\partial \varphi} = (-a^2\cos\theta\sin^2\varphi, -a^2\sin\theta\sin^2\varphi, -a^2\sin\varphi\cos\varphi) = -a\sin\varphi \cdot \mathbb{r}$$

(directed towards the center).

Surface integral of a function f on S: $\iint_S f\,dS = \iint_T f(\mathbb{r}(u,v)) \left\| \frac{\partial \mathbb{r}}{\partial u} \times \frac{\partial \mathbb{r}}{\partial v}(u,v) \right\| du\,dv$

Area of S: $\iint_S \left\| \frac{\partial \mathbb{r}}{\partial u} \times \frac{\partial \mathbb{r}}{\partial v} \right\| du\,dv$. $(f=1)$.

Thm. The surface integral does not depend on parametrization.

Surface integral of a vector field: choose a unit normal vector $\mathbb{n} = \pm \frac{\mathbb{N}}{\|\mathbb{N}\|}$, $\mathbb{N} = \frac{\partial \mathbb{r}}{\partial u} \times \frac{\partial \mathbb{r}}{\partial v}$

$$\iint_S \mathbb{F}\cdot\mathbb{n}\,dS = \iint_T \mathbb{F}(\mathbb{r}(u,v)) \cdot \left(\pm \frac{\partial \mathbb{r}}{\partial u} \times \frac{\partial \mathbb{r}}{\partial v}\right) du\,dv$$

Exercise 12.6.5. $\mathbb{r}(u,v) = (u\cos v, u\sin v, u^2)$, $T = \{(u,v) : 0 \le u \le 4,\ 0 \le v \le 2\pi\}$.

Compute the Area.

$\frac{\partial \mathbb{r}}{\partial u} = (\cos v, \sin v, 2u)$, $\frac{\partial \mathbb{r}}{\partial v} = (-u\sin v, u\cos v, 0)$, $\frac{\partial \mathbb{r}}{\partial u} \times \frac{\partial \mathbb{r}}{\partial v} = (-2u^2\cos v, -2u^2\sin v, u)$.

$\left\|\frac{\partial \mathbb{r}}{\partial u} \times \frac{\partial \mathbb{r}}{\partial v}\right\| = \sqrt{4u^4+u^2} = u\sqrt{4u^2+1}$. $\int_0^{2\pi}\int_0^4 u\sqrt{4u^2+1}\,du\,dv = 2\pi \cdot \left[\frac{1}{12}(4u^2+1)^{\frac{3}{2}}\right]_0^4 = \frac{\pi}{6}(65\sqrt{65}-1)$.

12.10.7*. $S = \{(x,y,z) : x^2+y^2+z^2 = a^2,\ z \ge 0\}$.

Compute $\iint_S \mathbb{F}\cdot\mathbb{n}\,dS$, where $\mathbb{F}(x,y,z) = (xz, yz, 0)$, $\mathbb{n}$ has positive z-component.

Take $\mathbb{r}(u,v) = (u, v, \sqrt{a^2-u^2-v^2})$. $\frac{\partial \mathbb{r}}{\partial u} = \left(1, 0, \frac{-u}{\sqrt{a^2-u^2-v^2}}\right)$, $\frac{\partial \mathbb{r}}{\partial v} = \left(0, 1, \frac{-v}{\sqrt{a^2-u^2-v^2}}\right)$

$\frac{\partial \mathbb{r}}{\partial u} \times \frac{\partial \mathbb{r}}{\partial v} = \left(\frac{u}{\sqrt{a^2-u^2-v^2}}, \frac{v}{\sqrt{a^2-u^2-v^2}}, 1\right)$. $\mathbb{F}(\mathbb{r}(u,v)) \cdot \frac{\partial \mathbb{r}}{\partial u} \times \frac{\partial \mathbb{r}}{\partial v}(u,v) = u^2+v^2$.

$\iint_S \mathbb{F}\cdot\mathbb{n}\,dS = \iint_T (u^2+v^2)\,du\,dv = \int_0^{2\pi}\int_0^a r^3\,dr\,d\theta = \frac{\pi}{2}a^4$. $T = \{(u,v) : u^2+v^2 \le a^2\}$

12.10.4. Let S be the triangle with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.
Compute $\iint_S \mathbb{F}\cdot\mathbb{n}\,dS$ where $\mathbb{F}(x,y,z) = (x,y,z)$. $\mathbb{n}$ has positive z-component.

Curl and divergence

$\mathbb{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$.

$\operatorname{div}\mathbb{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$. $\operatorname{curl}\mathbb{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$.

Exercises 12.15 1.(b) Compute $\operatorname{div}\mathbb{F}$ and $\operatorname{curl}\mathbb{F}$.

$\mathbb{F}(x,y,z) = (2z-3y, 3x-z, y-2x)$. $\operatorname{div}\mathbb{F} = 0+0+0 = 0$, $\operatorname{curl}\mathbb{F} = (1+1, 2+2, 3+3) = (2,4,6)$.

1(c) $\mathbb{F} = (x^2\sin y, y^2\sin xz, xy\sin(\cos z))$.

(d) $\mathbb{F} = (e^{xy}, \cos xy, \cos xz^2)$.

Stokes' theorem

Assume that S is a piecewise smooth surface, with a piecewise smooth boundary C, which is a single curve, parametrized by α. Let $\mathbb{r}$ be a parametrization of S from a simply connected region, with the corresponding boundary parametrized counterclockwise. Let $\mathbb{n}$ parallel to $\frac{\partial \mathbb{r}}{\partial u}\times\frac{\partial \mathbb{r}}{\partial v}$. Then, for a continuously differentiable vector field $\mathbb{F}$,
$$\iint_S \operatorname{curl}\mathbb{F}\cdot\mathbb{n}\,dS = \int_C \mathbb{F}\cdot d\alpha.$$

Exercise 12.13.1 $\mathbb{F}(x,y,z)=(y^2, xy, xz)$, $S=\{(x,y,z): x^2+y^2+z^2=1, z\geq 0\}$.
Compute $\iint_S \operatorname{curl}\mathbb{F}\cdot\mathbb{n}\,dS$, where $\mathbb{n}$ has positive z-component.

(1) ~~...~~
$\mathbb{r}(u,v)=(u,v,\sqrt{1-u^2-v^2})$. $\frac{\partial \mathbb{r}}{\partial u}\times\frac{\partial \mathbb{r}}{\partial v}=\left(\frac{u}{\sqrt{1-u^2-v^2}}, \frac{v}{\sqrt{1-u^2-v^2}}, 1\right)$. $T=\{(u,v): u^2+v^2\leq 1\}$
$\operatorname{curl}\mathbb{F}=(0,-z,-y)$. $\operatorname{curl}\mathbb{F}(\mathbb{r}(u,v))=(0,-\sqrt{1-u^2-v^2}, -v)$.
$\iint_S \operatorname{curl}\mathbb{F}\cdot\mathbb{n}\,dS = \iint_T \operatorname{curl}\mathbb{F}(\mathbb{r}(u,v))\cdot\frac{\partial \mathbb{r}}{\partial u}\times\frac{\partial \mathbb{r}}{\partial v}\,du\,dv = \iint_T -2v\,du\,dv=0.$

(2) By Stokes, $\iint_S \operatorname{curl}\mathbb{F}\cdot\mathbb{n}\,dS=\int_C \mathbb{F}\cdot d\alpha$, where $\alpha(t)=(\cos t, \sin t, 0)$, $t\in[-\pi,\pi]$
$=\int \mathbb{F}(\alpha(t))\cdot d\alpha$. $\mathbb{F}(\alpha(t))=(\sin^2 t, \cos t\sin t, 0)$, $\alpha'(t)=(-\sin t, \cos t, 0)$
$=\int_{-\pi}^{\pi}(-\sin^3 t+\cos^2 t\sin t)\,dt=\int_{-\pi}^{\pi}\sin t\,(2\cos^2 t-1)\,dt=0.$

12.13.6. Compute $\int_C \mathbb{F}\cdot d\alpha$, where $\mathbb{F}(x,y,z)=(y+z, z+x, x+y)$,
and C is the intersection of $x^2+y^2=2y$ and $y=z$.
Transform it to a surface integral by Stokes' Theorem.

Gauss' theorem

Let V be a solid in $\mathbb{R}^3$, xy-, yz, zx-projectable, bounded by the surface S, $\mathbb{n}$ be the outgoing normal unit vector on S, $\mathbb{F}$ a continuously differentiable vector field. Then $\iiint_V \operatorname{div}\mathbb{F}\,dx\,dy\,dz=\iint_S \mathbb{F}\cdot\mathbb{n}\,dS$.

Exercise $\mathbb{F}(x,y,z)=(x(x^2+y^2)z, y(x^2+y^2)z, z)$. $V=\{(x,y,z): x^2+y^2\leq 1, 0\leq z\leq 1\}$
$S=\{(x,y,z): x^2+y^2\leq 1, z=0\}\cup\{(x,y,z): x^2+y^2\leq 1, z=1\}\cup\{(x,y,z): x^2+y^2=1, 0\leq z\leq 1\}$
$\operatorname{div}\mathbb{F}=(3x^2+y^2)z+(x^2+3y^2)z+1=4(x^2+y^2)z+1$. $T=\{(x,y): x^2+y^2\leq 1\}$
$\iiint_V (4(x^2+y^2)z+1)\,dx\,dy\,dz=\iint_T\int_0^1 (4(x^2+y^2)z+1)\,dz\,dx\,dy=\iint_T \left[2(x^2+y^2)z^2+z\right]_0^1 dx\,dy$
$=\iint_T (2(x^2+y^2)+1)\,dx\,dy=2\pi.$
On $S=S_0\cup S_1\cup S_s$. On S_0, $\mathbb{F}=0$. On S_1, $\mathbb{n}=(0,0,1)$. $\mathbb{F}\cdot\mathbb{n}=1$. $\iint_{S_1}\mathbb{F}\cdot\mathbb{n}\,dS=\pi$.
On S_s. $\mathbb{r}(\theta,z)=(\cos\theta, \sin\theta, z)$. $\frac{\partial \mathbb{r}}{\partial\theta}=(-\sin\theta,\cos\theta,0)$, $\frac{\partial \mathbb{r}}{\partial z}=(0,0,1)$. $\frac{\partial \mathbb{r}}{\partial\theta}\times\frac{\partial \mathbb{r}}{\partial z}=(\cos\theta,\sin\theta,0)$
$\mathbb{F}(\mathbb{r}(\theta,z))=(\cos\theta\cdot z, \sin\theta\cdot z, z)$
$\iint_{S_s}\mathbb{F}\cdot\mathbb{n}\,dS=\iint_{S_s} z\,d\theta\,dz=\int_0^{2\pi}\int_0^1 z\,d\theta\,dz=\pi$
$\pi+\pi=2\pi$.

Exercise. $\mathbb{F}(x,y,z)=(x,y,z)$. $V=\{(x,y,z): x^2+y^2+z^2\leq a^2\}$ Check that $\iiint \operatorname{div}\mathbb{F}\,dx\,dy\,dz=\iint_S \mathbb{F}\cdot\mathbb{n}\,dS$