ratio test (Thm 10.13)

Let Σa_n be a series of positive terms such that

$\frac{a_{n+1}}{a_n} \to L$ (as $n \to \infty$)

(a) If $L < 1$, the series converges.

(b) If $L > 1$, the series diverges.

(If $L = 1$, more considerations are needed).

11.7 Exercises.

Determine the radius of convergence.

1. $\Sigma \frac{z^n}{2^n}$ 2. $\Sigma \frac{z^n}{(n+1)2^n}$, 3. $\Sigma \frac{(z+3)^n}{(n+1)2^n}$ 4. $\Sigma \frac{(-1)^n 2^{2n} z^n}{2n}$, 5. $\Sigma[1-(-2)^n]z^n$

17. If $f_n(x) = nx e^{-nx^2}$, for $n = 1, 2, \dots$ and $x \in \mathbb{R}$.

show that $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 \lim_{n\to 0} f_n(x)\,dx$.

18. If $f_n(x) = \frac{\sin nx}{n}$, $f(x) = \lim_{n\to\infty} f_n(x)$,

show that $\lim_{n\to\infty} f_n'(x) \neq f'(x)$

11.13 Exercises

Find the radius of convergence and compute the sum.

1. $\sum_{n=0}^{\infty} (-1)\, x^{2n}$ 2. $\sum_{n=0}^{\infty} \frac{x^n}{3^{n+1}}$

Prove the expansion.

11. $a^x = \sum_{n=0}^{\infty} \frac{(\log a)^n}{n!} x^n$, $a > 0$. 12. $\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$.

(Thm 11.11. If $|f^{(n)}(x)| \leq A^n$, then the Taylor series converges to f)

24. Let $f(x) = e^{-\frac{1}{x^2}}$ for $x \neq 0$ and $f(0) = 0$.

Show that $f^{(n)}(0) = 0$, hence Taylor's series does not converge to f.

11.16 1. Solve $(1-x^2)y'' - 2xy' + 6y = 0$.

with $y(0) = 1$, $y'(0) = 0$.

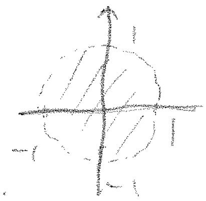
4. $y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$. Prove that $xy'' + y' - y = 0$

Exercises 8.3

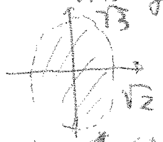
2\. Are the following sets open? $(x, y) \in \mathbb{R}^2$

(a) $x^2 + y^2 < 1$.

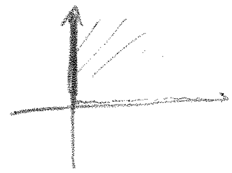
<u>Yes</u> Take an arbitrary (x_1, y_1) s.t. $x_1^2 + y_1^2 < 1$.
Put $r_1 = \sqrt{x_1^2 + y_1^2}$. Then $B((x_1, y_1), 1 - r_1)$ is contained in $x^2 + y^2 < 1$ by the triangle inequality.



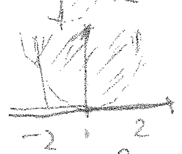
(b) $3x^2 + 2y^2 < 6$. <u>Yes</u>



(d) $x \geq 0$, $y > 0$. <u>No.</u>



(e) $y > x^2$, $|x| < 2$. <u>Yes.</u>



8.5 Determine where the following functions are defined and are continuous

1(a) $f(x, y) = x^4 + y^4 - 4x^2y^2$ <u>$(x, y) \in \mathbb{R}^2$</u>

By putting $g(x, y) = x$, $h(x, y) = y$, $f(x, y) = g(x, y)^4 + h(x, y)^4 - 4g(x, y)^2 h(x, y)^2$
and one can apply Theorem 8.1. and Example 1.

(b) $f(x, y) = \log(x^2 + y^2)$ $x^2 + y^2 > 0$.

By putting $g_1(x, y) = x$, $g_2(x, y) = y$, $h(z) = \log z$, $f(x, y) = h(g_1(x, y)^2 + g_2(x, y)^2)$
and one can apply Theorem 8.1, Example 1 and Theorem 8.2.

3 $f(x, y) = \dfrac{x - y}{x + y}$ if $x + y \neq 1$. f is discontinuous at $(0, 0)$.

$\lim_{x \to 0} f(x, y) = -1$, hence $\lim_{y \to 0} \left[\lim_{x \to 0} f(x, y)\right] = -1$, while

$\lim_{y \to 0} f(x, y) = 1$, hence $\lim_{x \to 0} \left[\lim_{y \to 0} f(x, y)\right] = 1$.

8.14 Find the gradient $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$, or $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$.

1\. (a) $f(x, y) = x^2 + y^2 \sin(xy)$. $\frac{\partial f}{\partial x} = 2x + y^3 \cos(xy)$, $\frac{\partial f}{\partial y} = 2y \sin(xy) + xy^2 \cos(xy)$.

(b) $f(x, y) = e^x \cos y$ $\frac{\partial f}{\partial x} = e^x \cos y$. $\frac{\partial f}{\partial y} = -e^x \sin y$

(c) $f(x, y, z) = x^2 y^3 z^4$ $\frac{\partial f}{\partial x} = 2xy^3z^4$, $\frac{\partial f}{\partial y} = 3x^2y^2z^4$, $\frac{\partial f}{\partial z} = 4x^2y^3z^3$

2\. Evaluate the directional derivative.

(a) $f(x, y, z) = x^2 + 2y^2 + 3z^2$ at $(1, 1, 0)$ in the direction $(1, -1, 2)$

$\nabla f = (2x, 4y, 6z)$, $\nabla f(1, 1, 0) \cdot (1, -1, 2) = (2, 4, 0) \cdot (1, -1, 2) = -2$.

8.9 Check that $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ for the following function.

10 $f(x, y) = x^4 + y^4 - 4x^2y^2$

$\frac{\partial f}{\partial y} = 4y^3 - 8x^2y$. $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -16xy$

$\frac{\partial f}{\partial x} = 4x^3 - 8xy^2$. $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -16xy$.

9.3 Exercises

1. Determine the solution of $4\frac{\partial f}{\partial x}+3\frac{\partial f}{\partial y}=0$ with $f(x,0)=\sin x$.

$(4e_1+3e_2)\cdot\nabla f=0 \Rightarrow f$ is constant on $3x-4y=c$.

$\Rightarrow f(x,y)=g(3x-4y)$ for some g. $f(x,0)=g(3x)=\sin x$

$\Rightarrow g(x)=\sin\frac{x}{3}$, $f(x,y)=g(3x-4y)=\sin\frac{3x-4y}{3}$

9.5 1. Let $k>0$, $g(x,t)=\frac{1}{2}\frac{x}{\sqrt{kt}}$, $f(x,t)=\int_0^{g(x,t)}e^{-u^2}du$

(a) Show that $\frac{\partial f}{\partial x}=e^{-g^2}\frac{\partial g}{\partial x}$, $\frac{\partial f}{\partial t}=e^{-g^2}\frac{\partial g}{\partial t}$

$f(x,t)=F(g(x,t))-F(0)$ where F is a primitive function of e^{-u^2}.

By the derivative of a composed function,

$\frac{\partial f}{\partial x}=\frac{\partial g}{\partial x}\cdot e^{-g^2}$, $\frac{\partial f}{\partial t}=\frac{\partial g}{\partial t}\cdot e^{-g^2}$

(b). Show that $k\frac{\partial^2 f}{\partial x^2}=\frac{\partial f}{\partial t}$.

$\frac{\partial g}{\partial x}=\frac{1}{2\sqrt{kt}}$, $\frac{\partial^2 g}{\partial x^2}=0$, $\frac{\partial g}{\partial t}=-\frac{x}{4\sqrt{kt^3}}$

$k\frac{\partial^2 f}{\partial x^2}=k\left(\frac{\partial g}{\partial x}\right)^2\cdot(-2g)\cdot e^{-g^2}=k\frac{1}{4kt}\cdot\left(-\frac{x}{\sqrt{kt}}\right)\cdot e^{-g^2}=-\frac{x}{4\sqrt{kt^3}}\cdot e^{-g^2}$

$\frac{\partial f}{\partial t}=-\frac{x}{4\sqrt{kt^3}}\cdot e^{-g^2}$, therefore, $\frac{\partial f}{\partial t}=k\frac{\partial^2 f}{\partial x^2}$.

2. Consider a function $f(x,y)=g(\sqrt{x^2+y^2})$

(a) Prove that, for $(x,y)\neq(0,0)$, $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=\frac{1}{r}g'(r)+g''(r)$, $r=\sqrt{x^2+y^2}$

$\frac{\partial f}{\partial x}=\frac{2x}{2\sqrt{x^2+y^2}}g'(r)$. $\frac{\partial^2 f}{\partial x^2}=\frac{1}{r}g'(r)-\frac{2x^2}{2(x^2+y^2)^{\frac{3}{2}}}g'(r)+\frac{x^2}{r^2}g''(r)$

$\frac{\partial^2 f}{\partial y^2}=\frac{1}{r}g'(r)-\frac{y^2}{r^{\frac{3}{2}}}g'(r)+\frac{y^2}{r^2}g''(r)$.

$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=\frac{2}{r}g'(r)-\frac{x^2+y^2}{r^3}g'(r)+\frac{x^2+y^2}{r^2}g''(r)=\frac{1}{r}g'(r)+g''(r)$.

(b) Assume that $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$ and prove that $f(x,y)=a\log(x^2+y^2)+b$

We have $\frac{1}{r}g'(r)+g''(r)=0$. Put $h(r)=g'(r)$, then $\frac{1}{r}h(r)+h'(r)=0$

$\Rightarrow h(r)=\frac{a'}{r}$, $a'\in\mathbb{R}$ (Thm 8.2). $\Rightarrow g(r)=a'\log r+b$

$f(x,y)=a\log\sqrt{x^2+y^2}+b=\frac{a'}{2}\log(x^2+y^2)+b$

Put $\frac{a'}{2}=a$.

9.8 Exercises.

4. Consider two surfaces $2x^2+3y^2-z^2-25=0$, $x^2+y^2-z^2=0$.
The intersection C can be parametrized as $(X(z), Y(z), z)$ and passes the point $P=(\sqrt{7}, 3, 4)$.

(a) Find a unit tangent vector of C at P by implicit computations.

Put $F(x,y,z)=2x^2+3y^2-z^2-25$, $G(x,y,z)=x^2+y^2-z^2$.

We use the formula $$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{pmatrix}^{-1} \begin{pmatrix} -\frac{\partial F}{\partial z} \\ -\frac{\partial G}{\partial z} \end{pmatrix}.$$

$\frac{\partial F}{\partial x}=4x$, $\frac{\partial F}{\partial y}=6y$, $\frac{\partial F}{\partial z}=-2z$, $\frac{\partial G}{\partial x}=2x$, $\frac{\partial G}{\partial y}=2y$, $\frac{\partial G}{\partial z}=-2z$.

$$\begin{pmatrix} X'(\sqrt{7},3,4) \\ Y'(\sqrt{7},3,4) \end{pmatrix} = \begin{pmatrix} 4\sqrt{7} & 18 \\ 2\sqrt{7} & 6 \end{pmatrix}^{-1} \begin{pmatrix} -(-8) \\ -(-8) \end{pmatrix} = \frac{1}{-12\sqrt{7}} \begin{pmatrix} 6 & -18 \\ -2\sqrt{7} & 4\sqrt{7} \end{pmatrix} \begin{pmatrix} 8 \\ 8 \end{pmatrix} = \begin{pmatrix} \frac{8}{\sqrt{7}} \\ -\frac{4}{3} \end{pmatrix}$$

(b) We have, from $3G-F=0$, $x^2+25=2z^2$, or, $x=\sqrt{2z^2-25}$

$F-2G=0$, $y^2=25-z^2$, or, $y=\sqrt{25-z^2}$.

$$X'(z)=\frac{4z}{2\sqrt{2z^2-25}}=\frac{2z}{\sqrt{2z^2-25}}, \quad Y'(z)=\frac{-2z}{2\sqrt{25-z^2}}=-\frac{z}{\sqrt{25-z^2}}$$

At P, $$\begin{pmatrix} X' \\ Y' \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{8}{\sqrt{7}} \\ -\frac{4}{3} \\ 1 \end{pmatrix}$$

9. The equation $x+z+(y+z)^2=6$ defines $z=f(x,y)$ implicitly.
Compute $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x \partial y}$ in terms of x, y, z.

Put $F(x,y,z)=x+z+(y+z)^2-6$. $\frac{\partial F}{\partial x}=1$, $\frac{\partial F}{\partial y}=2(y+z)$, $\frac{\partial F}{\partial z}=1+2(y+z)$.

$$\frac{\partial f}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}=-\frac{1}{1+2(y+z)}, \quad \frac{\partial f}{\partial y}=-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}=-\frac{2(y+z)}{1+2(y+z)}$$

$$\frac{\partial f}{\partial y}=-\frac{2(y+f(x,y))}{1+2(y+f(x,y))}, \quad \frac{\partial^2 f}{\partial x \partial y}=\frac{2\cdot\left(-\frac{1}{1+2(y+z)}\cdot(1+2(y+z))\right)-2(y+z)\cdot 2\,\frac{-1}{1+2(y+z)}}{(1+2(y+f(x,y)))^2}$$

$$=\frac{2}{(1+2(y+z))^3}$$

9.13 Exercises

Locate and classify the stationary points.

1. $f(x,y)=x^2+(y-1)^2$

5. $f(x,y)=2x^2-xy-3y^2-3x+7y$

10. $f(x,y)=\sin x\cosh y$.

21. Let $x_1,\dots,x_n$ distinct numbers. $y_1,\dots,y_n\in\mathbb{R}$

Let $a,b\in\mathbb{R}$. $f(x)=ax+b$.

$E(a,b)=\sum_{j=1}^{n}|f(x_j)-y_j|^2$. Find a,b which minimize $E(a,b)$.

9.15 2. Find the maximum and minimum distances from the origin to the curve $5x^2+6xy+5y^2=8$.

3. Assume $a,b\in\mathbb{R}$, $a,b>0$.

(a) Find the extreme values of $f(x,y)=\frac{x}{a}+\frac{y}{b}$ on $x^2+y^2=1$.

(b) Find the extreme values of $f(x,y)=x^2+y^2$ on $\frac{x}{a}+\frac{y}{b}=1$.

8.14 1(f) $f(x,y,z)=x^{y^z}$. Compute ∇f.

Chain rule. $f(x)=x^x$. compute $f'(x)$.

Exercises 10.5 Compute the line integral.

1. 2. $f(x,y)=(2a-y,\ x)$. $\alpha(t)=(a(t-\sin t),\ a(1-\cos t))$, $t\in[0,2\pi]$

3. $f(x,y,z)=(y^2-z^2,\ 2yz,\ -x^2)$ $\alpha(t)=(t,\ t^2,\ t^3)$. $t\in[0,1]$.

12. A wire has a shape $x^2+y^2=a^2$. density $\varphi(x,y)=|x|+|y|$.
Compute the mass.

Exercises 10.13

5. Show that the following are not a gradient. find a closed path α s.t. $\int f\cdot d\alpha\neq 0$.

(a) $f(x,y,z)=(y,\ x,\ x)$. (b) $f(x,y,z)=(xy,\ x^2+1,\ z^2)$

9. For a continuous function f, $f(x,y)=\left(f(\sqrt{x^2+y^2})\,x,\ f(\sqrt{x^2+y^2})\,y\right)$ is a gradient.

Example 2 $S=\{(x,y)\in\mathbb{R}^2,\ (x,y)\neq(0,0)\}$.

$$f(x,y)=\left(\frac{-y}{x^2+y^2},\ \frac{x}{x^2+y^2}\right).$$

Show that $D_1f_2=D_2f_1$.

Compute, for $\alpha(t)=(\cos t,\ \sin t)$. $t\in[0,2\pi]$,

$\int f\cdot d\alpha=2\pi$. hence f is not a gradient.

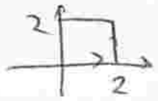
Exercises 11.15

Evaluate the double integrals.

1. $\iint_S x\cos(x+y)\,dx\,dy$, $S=$ 

5. $\iint_S (x^2-y^2)\,dx\,dy$, $S=\{(x,y): 0\le x\le \pi,\ 0\le y\le \sin x\}$.

11.22. Use the Green's theorem to evaluate $\int_C (y^2\,dx + x\,dy)$

for 

11.28 Compute in the polar coordinate

6. $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\,dy\,dx$

$(x-a)^2+y^2\le a^2$

$-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, $0\le r\le a\cos\theta$.

12.4. Compute $\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}$.

7. $\mathbf{r}(u,v)=(a\sin u\cosh v,\ b\cos u\cosh v,\ c\sinh v)$.

8. $\mathbf{r}(u,v)=(u+v,\ u-v,\ 4v^2)$

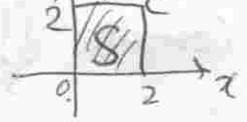
12.6. Compute the area.

2. $x+y+z=a$, $x^2+y^2\le a^2$.

Exercises 11.22.

Use Green's theorem to evaluate $\int_C (y^2 dx + x dy)$, where

1(a) C is the boundary of the square (counter-clockwise)



Solution). Define the vector field $f(x,y) = (y^2, x)$. Or, $P(x,y) = y^2$, $Q(x,y) = x$.

With α parametrizing C, the given integral is $\int_C f \cdot d\alpha$,

and by Green's theorem, $\int_C f \cdot d\alpha = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx dy = \iint_S (1-2y) dx dy$

As the square S is of type I. $S = \{(x,y) : 0 \le x \le 2, 0 \le y \le 2\}$,

this can be evaluated by the iterated integral

$$\int_0^2 \int_0^2 (1-2y) dy\, dx = \int_0^2 [y-y^2]_0^2 dx = \int_0^2 (-2) dx = -4.$$

12.4. Compute $\frac{\partial \mathbb{r}}{\partial u} \times \frac{\partial \mathbb{r}}{\partial v}$. (Recall that for $\mathbb{W}_1 = (x_1, y_1, z_1)$, $\mathbb{W}_2 = (x_2, y_2, z_2)$, $\mathbb{W}_1 \times \mathbb{W}_2 = (y_1 z_2 - z_1 y_2, z_1 x_2 - x_1 z_2, x_1 y_2 - y_1 x_2)$)

7. $\mathbb{r}(u,v) = (a \sin u \cosh v, b \cos u \cosh v, c \sinh v)$.

Solution $\frac{\partial \mathbb{r}}{\partial u} = (a \cos u \cosh v, -b \sin u \cosh v, 0)$.

$\frac{\partial \mathbb{r}}{\partial v} = (a \sin u \sinh v, b \cos u \sinh v, c \cosh v)$.

$\frac{\partial \mathbb{r}}{\partial u} \times \frac{\partial \mathbb{r}}{\partial v} = (-bc \sin u \cosh^2 v, -ac \cos u \cosh^2 v, ab(\cos^2 u + \sin^2 u) \sinh v \cosh v)$

$= (-bc \sin u \cosh^2 v, -ac \cos u \cosh^2 v, ab \sinh v \cosh v)$.

8. $\mathbb{r}(u,v) = (u+v, u-v, 4v^2)$.

Solution $\frac{\partial \mathbb{r}}{\partial u} = (1, 1, 0)$. $\frac{\partial \mathbb{r}}{\partial v} = (1, -1, 8v)$.

$\frac{\partial \mathbb{r}}{\partial u} \times \frac{\partial \mathbb{r}}{\partial v} = (8v, -8v, -2)$.

12-6. Compute the area. 2. the intersection of $x+y+z=a$, $x^2+y^2 \le a^2$.

Solution. The surface can be written, with $z = a-x-y$, as

$S = \{(x,y,z) : x^2+y^2 \le a^2, z = a-x-y\}$. Put $S_0 = \{(x,y) : x^2+y^2 \le a^2\}$

In other words, S is parametrized by $\mathbb{r}(u,v) = (u, v, a-u-v)$, $u^2+v^2 \le a^2$.

$\frac{\partial \mathbb{r}}{\partial u} = (1, 0, -1)$, $\frac{\partial \mathbb{r}}{\partial u} = (0, 1, -1)$. $\frac{\partial \mathbb{r}}{\partial u} \times \frac{\partial \mathbb{r}}{\partial v} = (1,1,1)$, $\left\|\frac{\partial \mathbb{r}}{\partial u} \times \frac{\partial \mathbb{r}}{\partial v}\right\| = \sqrt{3}$.

The area is $\iint_{S_0} \left\|\frac{\partial \mathbb{r}}{\partial u} \times \frac{\partial \mathbb{r}}{\partial v}\right\| dx dy = \sqrt{3} \iint_{S_0} dx dy = \pi a^2 \sqrt{3}$ (S_0 is a disk with radius a)

Exercise 12.10. $\underline{1(a)}$. Let $S: x^2+y^2+z^2=1, z\geq 0$ (hemisphere), and $\mathbb{F}(x,y,z)=(x,y,0)$ be a vector field.

Compute $\iint_S \mathbb{F}\cdot \mathbb{n}_1 dS$ with the parametrization $z=\sqrt{1-x^2-y^2}$.

$\underline{\text{Solution}}$. With $S_0 := \{(x,y): x^2+y^2\leq 1\}$, S can be parametrized by

$\mathbb{r}(u,v)=(u,v,\sqrt{1-u^2-v^2})$, $(u,v)\in S_0$.

$\frac{\partial \mathbb{r}}{\partial u}=\left(1,0,\frac{-u}{\sqrt{1-u^2-v^2}}\right)$, $\frac{\partial \mathbb{r}}{\partial v}=\left(0,1,\frac{-v}{\sqrt{1-u^2-v^2}}\right)$. $\frac{\partial \mathbb{r}}{\partial u}\times\frac{\partial \mathbb{r}}{\partial v}=\left(\frac{u}{\sqrt{1-u^2-v^2}},\frac{v}{\sqrt{1-u^2-v^2}},1\right)$

As we know, $\iint_S \mathbb{F}\cdot\mathbb{n}_1 dS=\iint_{S_0}\mathbb{F}\cdot\left(\frac{\partial \mathbb{r}}{\partial u}\times\frac{\partial \mathbb{r}}{\partial v}\right)du\,dv$,

and $\mathbb{F}(\mathbb{r}(u,v))=(u,v,0)$. $\mathbb{F}\cdot\left(\frac{\partial \mathbb{r}}{\partial u}\times\frac{\partial \mathbb{r}}{\partial v}\right)=\frac{u^2+v^2}{\sqrt{1-u^2-v^2}}$. Using the polar coordinate,

$$\iint_{S_0}\frac{u^2+v^2}{\sqrt{1-u^2-v^2}}du\,dv=\int_0^{2\pi}\int_0^1\frac{r^2}{\sqrt{1-r^2}}\cdot r\,dr\,d\theta=\int_0^{2\pi}\int_0^1\frac{r^3}{\sqrt{1-r^2}}dr\,d\theta.$$

With $t=r^2$, $\frac{dt}{dr}=2r$. we get

$$\int_0^1\frac{r^3}{\sqrt{1-r^2}}dr=\frac{1}{2}\int_0^1\frac{t}{\sqrt{1-t}}dt=\frac{1}{2}\int_0^1\frac{(t-1)+1}{\sqrt{1-t}}dt$$

$$=\frac{1}{2}\int_0^1\left(-\sqrt{1-t}+\frac{1}{\sqrt{1-t}}\right)dt=\frac{1}{2}\left[-\frac{2}{3}(1-t)^{\frac{3}{2}}+2(1-t)^{\frac{1}{2}}\right]_0^1=\frac{1}{2}\cdot\frac{4}{3}=\frac{2}{3}.$$

Altogether, $\int_0^{2\pi}\int_0^1\frac{r^3}{\sqrt{1-r^2}}dr\,d\theta=\int_0^{2\pi}\frac{2}{3}d\theta=\frac{4}{3}\pi$.

12.13 $\underline{2.}$ $S: z=1-x^2-y^2, z\geq 0$, $\mathbb{F}(x,y,z)=(y,z,x)$.

Compute $\iint \text{curl}\,\mathbb{F}\cdot\mathbb{n}_1 dS$, using Stokes' theorem.

$\underline{\text{Solution}}$. The surface S can be parametrized by $\mathbb{r}(u,v)=(u,v,1-u^2-v^2)$, $(u,v)\in S_0=\{(u,v): u^2+v^2\leq 1\}$. This has the boundary

$C:=\{(x,y,z): x^2+y^2=1, z=0\}$. Parametrize it by $\alpha(t)=(\cos t,\sin t,0)$.

By Stokes' theorem, $\iint_S \text{curl}\,\mathbb{F}\cdot\mathbb{n}_1 dS=\int_C \mathbb{F}(\alpha(t))\cdot d\alpha$.

By noting that $\mathbb{F}(\alpha(t))=(\sin t,0,\cos t)$, $\alpha'(t)=(-\sin t,\cos t,0)$.

we have $\int_C \mathbb{F}(\alpha(t))\cdot d\alpha=\int_0^{2\pi}\mathbb{F}(\alpha(t))\cdot\alpha'(t)dt=\int_0^{2\pi}-\sin^2 t\,dt$.

Using $\sin^2 t=\frac{1}{2}(1-\cos 2t)$, $\int_0^{2\pi}-\sin^2 t\,dt=-\frac{1}{2}\int_0^{2\pi}(1-\cos 2t)dt=-\pi$.

12.13 5. Let C be the curve of the intersection $x^2+y^2+z^2=a$, $x+y+z=0$.

Prove that $\int_C \mathbb{F}\cdot d\alpha = \pi a^2\sqrt{3}$, where $\mathbb{F}(x,y,z)=(y,z,x)$.

Solution. From $z=-x-y$, $x^2+y^2+(-x-y)^2=a \Leftrightarrow 2x^2+2y^2+2xy=a^2$, $z=-x-y$

$\Leftrightarrow \frac{3}{2}(x+y)^2+\frac{1}{2}(x-y)^2=a$, $z=-x-y$, we see that C is an ellipse, parametrized by $\mathbb{r}(u,v)=(u,v,-u-v)$, $\frac{3}{2}(u+v)^2+\frac{1}{2}(u-v)^2=a^2$.

C is the boundary of the surface $S: \mathbb{r}(u,v)=(u,v,-u-v)$, $(u,v)\in S_0$,

where $S_0:=\{(u,v): \frac{3}{2}(u+v)^2+\frac{1}{2}(u-v)^2\le a^2\}$.

$\frac{\partial \mathbb{r}}{\partial u}=(1,0,-1)$, $\frac{\partial \mathbb{r}}{\partial v}=(0,1,-1)$. $\frac{\partial \mathbb{r}}{\partial u}\times\frac{\partial \mathbb{r}}{\partial v}=(1,1,1)$.

$\mathrm{curl}\,\mathbb{F}=(1,1,1)$. By Stokes' theorem,

$$\int_C \mathbb{F}d\alpha = \iint_S \mathbb{F}\cdot\mathbb{n}\,dS = \iint_{S_0} \mathbb{F}(\mathbb{r}(u,v))\cdot\frac{\partial \mathbb{r}}{\partial u}\times\frac{\partial \mathbb{r}}{\partial v}\,du\,dv = \iint_{S_0} 3\,du\,dv.$$

To perform the last integral, introduce $\begin{cases} s=\sqrt{\frac{3}{2}}(u+v) \\ t=\sqrt{\frac{1}{2}}(u-v) \end{cases} \Leftrightarrow \begin{cases} u=\frac{1}{\sqrt6}s+\frac{1}{\sqrt2}t \\ v=\frac{1}{\sqrt6}s-\frac{1}{\sqrt2}t \end{cases}$

$$J=\begin{vmatrix} \frac{\partial u}{\partial s} & \frac{\partial v}{\partial s} \\ \frac{\partial u}{\partial t} & \frac{\partial v}{\partial t}\end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt6} & \frac{1}{\sqrt6} \\ \frac{1}{\sqrt2} & -\frac{1}{\sqrt2}\end{vmatrix} = -\frac{1}{\sqrt3}.$$ and $(u,v)\in S_0 \Leftrightarrow s^2+t^2\le a^2$.

$S_1=\{(s,t): s^2+t^2\le a^2\}$

$$\iint_{S_0} 3\,du\,dv = 3\cdot\iint_{S_1}\left|-\frac{1}{\sqrt3}\right|ds\,dt=\sqrt3\cdot\pi a^2,$$ because S_1 is a disk with radius a.

12.21 1. Let S be the surface of the unit cube $V:=\{(x,y,z); 0\le x\le 1, 0\le y\le 1, 0\le z\le 1\}$

$\mathbb{n}$ be the unit outer normal vector of S.

$\mathbb{F}(x,y,z)=(x^2,y^2,z^2)$ be a vector field.

Compute $\iint_S \mathbb{F}\cdot\mathbb{n}\,dS$ and $\iiint_V \mathrm{div}\,\mathbb{F}\,dx\,dy\,dz$.

Solution — S consists of 6 squares. Let $S_{x,0}=\{(0,y,z): 0\le y\le 1, 0\le z\le 1\}$, $S_{x,1}=\{(1,y,z): \begin{smallmatrix}0\le y\le 1\\ 0\le z\le 1\end{smallmatrix}\}$

$\mathbb{n}$ on $S_{x,0}$ is $(-1,0,0)$. $\mathbb{F}=(0,y^2,z^2)$. so $\mathbb{F}\cdot\mathbb{n}=0$.

$\mathbb{n}$ on $S_{x,1}$ is $(1,0,0)$. $\mathbb{F}=(1,y^2,z^2)$. so $\mathbb{F}\cdot\mathbb{n}=1$. $\iint_{S_{x,1}} dS=\int_0^1\int_0^1 dy\,dz=1$.

Similarly, $\iint_{S_{y,0}}\mathbb{F}\cdot\mathbb{n}\,dS=0$, $\iint_{S_{y,1}}\mathbb{F}\cdot\mathbb{n}\,dS=1$, $\iint_{S_{z,0}}\mathbb{F}\cdot\mathbb{n}\,dS=0$. $\iint_{S_{z,1}}\mathbb{F}\cdot\mathbb{n}\,dS=1 \Rightarrow \iint_S\mathbb{F}\cdot\mathbb{n}\,dS=3$.

— $\mathrm{div}\,\mathbb{F}=2x+2y+2z$.

$$\iiint_V(2x+2y+2z)\,dx\,dy\,dz=\int_0^1\int_0^1\int_0^1(2x+2y+2z)\,dx\,dy\,dz=1+1+1=3$$