



Solitons and Nonsmooth Diffeomorphisms in Conformal Nets

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Received: 12 November 2018 / Accepted: 25 January 2019

Published online: 3 May 2019 – © The Author(s) 2019

Abstract: We show that any solitonic representation of a conformal (diffeomorphism covariant) net on S^1 has positive energy and construct an uncountable family of mutually inequivalent solitonic representations of any conformal net, using nonsmooth diffeomorphisms. On the loop group nets, we show that these representations induce representations of the subgroup of loops compactly supported in $S^1 \setminus \{-1\}$ that do not extend to the whole loop group. In the case of the $U(1)$ -current net, we extend the diffeomorphism covariance to the Sobolev diffeomorphisms $\mathcal{D}^s(S^1)$, $s > 2$, and show that the positive-energy vacuum representations of $\text{Diff}_+(S^1)$ with integer central charges extend to $\mathcal{D}^s(S^1)$. The solitonic representations constructed above for the $U(1)$ -current net and for Virasoro nets with integral central charge are continuously covariant with respect to the stabilizer subgroup of $\text{Diff}_+(S^1)$ of -1 of the circle.

1. Introduction

In two-dimensional quantum field theory, solitons appear in the presence of inequivalent vacuum sectors. In [[Fro76](#)], Frölich proposed an operator-algebraic formulation of solitons as superselection sectors localized in a half-space. Existence of such solitons has been obtained for a wide class of models [[Sch96](#),[Sch98](#),[Mug99](#)], and general structural results have been obtained [[Fre93](#),[Reh98](#)]. In two-dimensional conformal field theory, the vacuum is unique due to dilation invariance [[Rob74](#)] and translation-invariant states are not always localized in half-space [[Tan18](#)], yet solitons appear through α -induction [[LR95](#),[BE98](#),[BE99a](#)], and the interrelationship between solitons has led to the operator-algebraic formulation of modular invariant [[BE99b](#)]. In this way, solitons play a crucial role in the study of conformal field theories.

S. Del Vecchio: Supported by ERC advanced Grant 669240 QUEST “Quantum Algebraic Structures and Models” and GNAMPA-INDAM.

Y. Tanimoto: Supported by Programma per giovani ricercatori, anno 2014 “Rita Levi Montalcini” of the Italian Ministry of Education, University and Research.

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In the operator-algebraic framework, a conformal field theory is realized as a family of von Neumann algebras satisfying certain axioms (conformal net), and the superselection sectors, including solitons, are the equivalence classes of its representations. When a conformal net has a subnet, α -induction yields solitons for the larger net from a sector of the smaller net. Note that the Virasoro nets, the conformal nets generated by the stress–energy tensor alone, do not have any Möbius covariant subnet [Car98]. Therefore, it is a natural question whether the Virasoro nets admit any nontrivial soliton. Recently Henriques in [Hen17a] proved that the category of solitons $\text{Sol}(\mathcal{A})$ of a completely rational conformal net \mathcal{A} is a bicommutant category whose Drinfel'd center corresponds to the category of DHR sectors of \mathcal{A} . This fact implies the existence of non-trivial solitons for all the conformal nets with central charge $c < 1$ and μ -index > 1 (hence including the Virasoro nets with $c < 1$), yet the existence of such solitons is only implicit. In this paper, we construct for any conformal net a family of concrete, proper, automorphic solitons (with index 1) parametrized by \mathbb{R}_+ using nonsmooth diffeomorphisms.

The (smooth) diffeomorphism covariance is the defining property of conformal nets. The representations of this large spacetime symmetry group can be further extended to certain nonsmooth diffeomorphisms depending on their regularity [CW05, Wei06]. Here, we show that these representations cannot be extended to some less-smooth diffeomorphisms, and exploit this to construct proper irreducible solitons (a similar construction has been implicitly given in [LX04, KLX05] which yielded reducible, type III solitons). We also show that any soliton has positive energy, this time by exploiting unitarily representable nonsmooth diffeomorphisms.¹

This distinction between implementable and non-implementable diffeomorphisms is central in this work. We say that a nonsmooth diffeomorphism γ is unitarily implemented in a conformal net if there is a unitary operator whose adjoint action realizes γ on the quantum observables. When a diffeomorphism is not implementable, its action may give rise to a new sector. In any conformal net, the Sobolev-class diffeomorphisms $\mathcal{D}^s(S^1)$ with $s > 3$ are implementable [CDIT]. Here we show, using the Tomita–Takesaki modular theory, that nonsmooth diffeomorphisms that have discontinuous derivatives are not implementable, although they are implementable when restricted to local algebras. In this way, we obtain an uncountable family of inequivalent solitons for any conformal net. On the other hand, the implementability of many nonsmooth diffeomorphisms is inherited by sectors, and indeed positivity of energy in any soliton is proved in this way.

As an application, we construct irreducible unitary projective positive-energy representations of the subgroup ΛG of the loop group LG , consisting of loops with support not containing the point -1 that do not extend to LG . The existence of such representations was marked as an open problem in [PS86, P.174, Remark]. Similar representations are constructed for the group B_0 of diffeomorphisms of S^1 preserving the point -1 . These results can be seen as an application of the modular theory of von Neumann algebras to the representation theory of infinite-dimensional groups.

We also pursue the question of which nonsmooth diffeomorphisms are implementable. We take the $U(1)$ -current net (the derivative of the massless free field, or the Heisenberg algebra) which has the central charge $c = 1$, and show that it is covariant with respect to Sobolev-class diffeomorphisms $\mathcal{D}^s(S^1)$, $s > 2$. This is done by the Shale(–Stinespring) criterion of unitary implementation [Sha62], and improves the implementation with $s > 3$ for general conformal nets [CDIT]. This implies that some unitary represen-

¹ Positivity of energy has been proved for finite index representations without using conformal covariance [BCL98]. Our proof depends on conformal covariance and normality on half lines, but makes no assumption on the index.

tations of $\text{Diff}_+(S^1)$ with integer c can be extended to $\mathcal{D}^s(S^1)$ -diffeomorphisms with $s > 2$. This is to our knowledge the largest group that is implementable for some c , and no other implementable diffeomorphism is known. As a consequence, the solitons we construct in Sect. 3.2 are continuously B_0 -covariant in this case.

This paper is organized as follows: in Sect. 2 we recall the notions of conformal net and its representation theory together with some facts about the diffeomorphisms groups and loop groups. In Sect. 3.1 we prove that every soliton is translation covariant with positive energy. The covariance can be further extended, but we do not know the continuity. In Sect. 3.2 we construct a family of proper solitons arising from nonsmooth diffeomorphisms. It is shown in Sect. 3.3 that a soliton that is locally Möb-covariant (not just covariant with respect to the universal covering $\widetilde{\text{Möb}}$) extends to a DHR representation. Sect. 4 is dedicated to concrete examples: we use the results in Sect. 3 to prove that there exist irreducible positive energy representations of B_0 and ΛG that do not extend to $\text{Diff}_+(S^1)$ and LG , respectively. Furthermore, in Sect. 5 we show that the $U(1)$ -current net and the Virasoro nets with positive integer central charge are $\mathcal{D}^s(S^1)$ -covariant, $s > 2$. In Sect. 6 we summarize open problems. In “Appendix”, we prove that any conformal net is covariant with respect to piecewise smooth C^1 -diffeomorphisms.

2. Preliminaries

2.1. Conformal nets. Let \mathcal{I} be the set of nonempty, non-dense, connected open intervals of the unit circle S^1 . For $I \in \mathcal{I}$, I^c denotes the interior of the complement of the interval $I \in \mathcal{I}$, namely $I^c = (S^1 \setminus I)^\circ$. The Möbius group $\text{Möb} = \text{PSL}(2, \mathbb{R})$ acts on S^1 by linear fractional transformations (see Sect. 2.3.1).

A **Möbius covariant net on S^1** (\mathcal{A}, U, Ω) consists of a family $\{\mathcal{A}(I), I \in \mathcal{I}\}$ of von Neumann algebras acting on a separable complex Hilbert space \mathcal{H} , a strongly continuous unitary representation U of Möb and a “vacuum” vector $\Omega \in \mathcal{H}$ satisfying the following properties:

- (CN1) **Isotony**: if $I_1 \subset I_2$, $I_1, I_2 \in \mathcal{I}$, then $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)$.
- (CN2) **Locality**: if $I_1 \cap I_2 = \emptyset$, $I_1, I_2 \in \mathcal{I}$, then $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)'$.
- (CN3) **Möbius covariance**: for $g \in \text{Möb}$, $I \in \mathcal{I}$, $\text{Ad } U(g)(\mathcal{A}(I)) = \mathcal{A}(gI)$.
- (CN4) **Positivity of energy**: the representation U has positive energy, i.e. the conformal Hamiltonian L_0 (the generator of rotations) has non-negative spectrum.
- (CN5) **Vacuum**: Ω is the unique (up to a scalar) vector such that $U(g)\Omega = \Omega$ for $g \in \text{Möb}$, and Ω is cyclic for $\bigvee_{I \in \mathcal{I}} \mathcal{A}(I)$.

Positivity of energy is actually equivalent to positivity of the generator of translations [Kos02, Proposition 1], see Sect. 2.3.1. With these assumptions, the following automatically hold, see [GF93, Lemma 2.9, Theorem 2.19(ii)] [FJ96, Section 3] and the arguments of [Bau95, Theorem 1.2.6]:

- (CN6) **Haag duality**: for every $I \in \mathcal{I}$, $\mathcal{A}(I') = \mathcal{A}(I)'$ where $\mathcal{A}(I)'$ is the commutant of $\mathcal{A}(I)$.
- (CN7) **Additivity**: if $I, I_\alpha \in \mathcal{I}$ and $I \subset \bigcup_\alpha I_\alpha$, then $\mathcal{A}(I) \subset \bigvee_\alpha \mathcal{A}(I_\alpha)$.
- (CN8) **Bisognano–Wichmann property**: if $I_{(0,\pi)} = \{z \in S^1 : \text{Im}(z) > 0\} \in \mathcal{I}$ and $\Delta_{I_{(0,\pi)}}$ is the modular operator associated to $\mathcal{A}(I_{(0,\pi)})$ and Ω then

$$\Delta_{I_{(0,\pi)}}^{it} = U(\delta(-2\pi t)),$$

where δ is the one parameter group of dilations (as a subgroup of Möb through the Cayley transform, see Section 2.3.1).

(CN9) **Irreducibility**: each $\mathcal{A}(I)$ is a type III₁ factor and $\bigvee_{I \in \bar{\mathcal{I}}_{\mathbb{R}}} \mathcal{A}(I) = \mathcal{B}(\mathcal{H})$, where $\bar{\mathcal{I}}_{\mathbb{R}}$ is the set of intervals not containing the point -1 , see Sect. 2.2.

By a **conformal net** (or diffeomorphism covariant net) we shall mean a Möbius covariant net that satisfies the following:

(CN10) U extends to a projective unitary representation of $\text{Diff}_+(S^1)$ on \mathcal{H} such that for all $I \in \mathcal{I}$ we have

$$\text{Ad } U(\gamma)(\mathcal{A}(I)) = \mathcal{A}(\gamma I), \quad \gamma \in \text{Diff}_+(S^1),$$

and

$$\text{Ad } U(\gamma)(x) = x, \quad x \in \mathcal{A}(I), \quad \text{supp } \gamma \in I', \quad (2.1)$$

where $\text{supp } \gamma$ is the closure of the complement of the set of $z \in S^1$ such that $\gamma(z) = z$.

In a conformal net, the following is automatic [MTW18]:

(CN11) **The split property**: if $\bar{I} \subset \tilde{I}$ where \bar{I} is the closure of I , then there is a type I factor $\mathcal{R}_{I, \tilde{I}}$ such that $\mathcal{A}(I) \subset \mathcal{R}_{I, \tilde{I}} \subset \mathcal{A}(\tilde{I})$.

2.2. Representations of conformal nets. DHR representations. A **DHR (Doplicher–Haag–Roberts) representation** ρ of a conformal net \mathcal{A} is a family of maps $\{\rho_I\}_{I \in \mathcal{I}}$ where ρ_I is a normal (σ -weakly continuous) representation of the von Neumann algebra $\mathcal{A}(I)$ on a fixed Hilbert space \mathcal{H}_ρ with the compatibility property $\rho_{I_2}|_{\mathcal{A}(I_1)} = \rho_{I_1}$, $I_1 \subset I_2$.

We say that two representations ρ_1, ρ_2 are unitarily equivalent if there exists an intertwining unitary operator U from \mathcal{H}_{ρ_1} and \mathcal{H}_{ρ_2} , i.e. $U\rho_{1,I}(x) = \rho_{2,I}(x)U$ for every $x \in \mathcal{A}(I)$ and $I \in \mathcal{I}$. A representation ρ is said to be **irreducible** if $\bigvee_{I \in \mathcal{I}} \rho_I(\mathcal{A}(I)) = \mathcal{B}(\mathcal{H}_\rho)$. The collection of identity maps $\rho_0 = \{\rho_{0,I}\}$ where $\rho_{0,I}(x) = x$, $x \in \mathcal{A}(I)$ is called the **vacuum representation**.

A group \mathbf{G} may act on S^1 . In this paper, \mathbf{G} will be either Möb, $\text{Diff}_+(S^1)$ or some groups that contain $\text{Diff}_+(S^1)$ such that U can be extended to them (see Sections 3.2 and 5). A DHR representation ρ is said to be **G -covariant** if there exists a unitary projective representation U_ρ of \mathbf{G} such that

$$\text{Ad } U_\rho(\gamma)(\rho_I(x)) = \rho_{\gamma I}(\text{Ad } U(\gamma)(x)),$$

for all $\gamma \in \mathbf{G}$. If \mathbf{G} is a topological group and acts on S^1 continuously, we often require that U is continuous in the strong operator topology.

Solitons. Let $\bar{\mathcal{I}}_{\mathbb{R}}$ be the set of open, non-empty, connected subsets of the real line \mathbb{R} , identified with $S^1 \setminus \{-1\} \subset \mathbb{C}$ via Cayley transform (see Sect. 2.3.1). Namely, $\bar{\mathcal{I}}_{\mathbb{R}}$ is the family of bounded open intervals and open half-lines of \mathbb{R} .

A **soliton** (or solitonic representation to be precise) σ of a conformal net \mathcal{A} is a family of maps $\{\sigma_I\}_{I \in \bar{\mathcal{I}}_{\mathbb{R}}}$ where σ_I is a normal representation of the von Neumann algebra $\mathcal{A}(I)$ on a fixed Hilbert space \mathcal{H}_σ with the compatibility property $\sigma_{I_2}|_{\mathcal{A}(I_1)} = \sigma_{I_1}$, $I_1 \subset I_2$. We say that the soliton σ is **proper** if there is no DHR representation of the conformal net \mathcal{A} that agrees with σ when restricted to the family of intervals $\bar{\mathcal{I}}_{\mathbb{R}}$.

Let \mathbf{G} again be a group acting on S^1 , and $\mathbf{G}_0 \subset \mathbf{G}$ be a subgroup whose elements preserve the point $-1 \in S^1$. A soliton σ of \mathcal{A} is **\mathbf{G}_0 -covariant** if there is a unitary projective representation U_σ of \mathbf{G}_0 such that $\text{Ad } U_\sigma(\gamma)(\sigma_I(x)) = \sigma_{\gamma I}(\text{Ad } U(\gamma)(x))$ with $x \in \mathcal{A}(I)$. Similarly, a soliton σ is **locally \mathbf{G} -covariant**² if there is a unitary projective representation U_σ of \mathbf{G} such that for a neighborhood \mathcal{U} of the unit element of \mathbf{G} and $I \in \overline{\mathcal{I}}_{\mathbb{R}}$ such that $\gamma I \subset \mathbb{R}$ for $\gamma \in \mathcal{U}$, it holds that $\text{Ad } U_\sigma(\gamma)(\sigma_I(x)) = \sigma_{\gamma I}(\text{Ad } U(\gamma)(x))$, $x \in \mathcal{A}(I)$, $\gamma \in \mathcal{U}$.

Consider the case where \mathbf{G}_0 includes the translation group of \mathbb{R} . We say that a soliton σ has **positive energy** if the unitary representation U_σ above can be chosen in such a way that the restriction to the one-parameter subgroup of translations is continuous in the strong operator topology and has a positive generator. Note that, if σ is not irreducible, U_σ that implements covariance is not unique, and other implementations may fail to have positive generator.

With \mathbb{R}_\pm considered as intervals in $\overline{\mathcal{I}}_{\mathbb{R}}$, we define the **index** of σ as the Jones index of the inclusion $\sigma(\mathcal{A}(\mathbb{R}_+)) \subset \sigma(\mathcal{A}(\mathbb{R}_-))'$. This is a natural generalization of the index of DHR representations.

2.3. The spacetime symmetry groups.

2.3.1. The Möbius group The group $\text{SL}(2, \mathbb{R})$ of 2×2 real matrices with determinant one acts on the compactified real line $\mathbb{R} \cup \{\infty\}$ by linear fractional transformations:

$$g : t \rightarrow gt := \frac{at + b}{ct + d} \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R}).$$

The kernel of this action is $\{\pm 1\}$. By identifying the compactified real line $\mathbb{R} \cup \{\infty\}$ with the circle S^1 via Cayley transform

$$C : S^1 \setminus \{-1\} \rightarrow \mathbb{R}, \quad z \mapsto i \frac{1 - z}{1 + z}, \quad (2.2)$$

with inverse

$$C^{-1} : \mathbb{R} \rightarrow S^1 \setminus \{-1\}, \quad t \mapsto \frac{1 + it}{1 - it}, \quad (2.3)$$

the group $\text{PSL}(2, \mathbb{R}) := \text{SL}(2, \mathbb{R}) / \{\pm 1\}$ can be identified with a subgroup of diffeomorphisms of the circle S^1 , the Möbius group Möb .

The following are important subgroups of $\text{PSL}(2, \mathbb{R})$:

$$R(\theta) = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \quad \delta(t) = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, \quad \tau(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

and they are called the rotation, dilation and translation subgroup, respectively, and act in the following way:

$$\begin{aligned} R(\theta)z &= e^{i\theta}z && \text{on } S^1, \\ \delta(t)s &= e^t s && \text{on } \mathbb{R}, \\ \tau(t)s &= s + t && \text{on } \mathbb{R}. \end{aligned}$$

The expressions of $\delta(t)$ and $\tau(t)$ in the circle picture are available in [Wei05, (A.6)].

² In [CHK+15, Definition 2.2], the notion “ \mathbf{G} -covariance” is defined through a unitary projective representation of the universal covering group $\tilde{\mathbf{G}}$. Differently from this, we distinguish explicitly projective representations of \mathbf{G} and of $\tilde{\mathbf{G}}$ and add “locally”.

2.3.2. The (smooth) diffeomorphism groups. The Lie group. Let us denote by $\text{Diff}_+(S^1)$ the group of orientation preserving, smooth diffeomorphisms of the circle $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ and $\text{Vect}(S^1)$ denote the set of smooth vector fields on S^1 . The group $\text{Diff}_+(S^1)$ is an infinite-dimensional Lie group whose Lie algebra is identified with the real topological vector space $\text{Vect}(S^1)$ of smooth vector fields on S^1 with C^∞ -topology [Mil84]. The exponential map $\text{Exp} : \text{Vect}(S^1) \rightarrow \text{Diff}_+(S^1)$ maps $tf \in \text{Vect}(S^1)$ to the one-parameter group $\text{Exp}(tf) \in \text{Diff}_+(S^1)$ of diffeomorphisms of S^1 satisfying the ordinary differential equation

$$\frac{d\text{Exp}(tf)(z)}{dt} = f(\text{Exp}(tf)(z)), \quad \text{Exp}(0)(z) = z.$$

We identify $\text{Vect}(S^1)$ with $C^\infty(S^1, \mathbb{R})$ and for $f \in C^\infty(S^1, \mathbb{R})$ we denote by f' the derivative of f with respect to the angle θ :

$$f'(z) = \left. \frac{d}{d\theta} f(e^{i\theta}) \right|_{e^{i\theta}=z}.$$

We consider a diffeomorphism $\gamma \in \text{Diff}_+(S^1)$ as a map from $S^1 \subset \mathbb{C}$ in S^1 . With this convention, its action on $f \in \text{Vect}(S^1)$ is

$$(\gamma_* f)(e^{i\theta}) = -ie^{-i\theta} \left. \frac{d}{d\varphi} \gamma(e^{i\varphi}) \right|_{e^{i\varphi}=\gamma^{-1}(e^{i\theta})} f(\gamma^{-1}(e^{i\theta})).$$

The Lie algebra. The linear space $\text{Vect}(S^1)$ is endowed with the Lie bracket given by

$$[f, g] = f'g - fg'.$$

As a Lie algebra, $\text{Vect}(S^1)$ admits the Gelfand–Fuchs 2-cocycle

$$\omega(f, g) = \frac{1}{48\pi} \int_{S^1} (f(e^{i\theta})g'''(e^{i\theta}) - f'''(e^{i\theta})g(e^{i\theta}))d\theta.$$

The Virasoro algebra Vir is the central extension of the complexification of the algebra generated by the trigonometric polynomials in $\text{Vect}(S^1)$ defined by the 2-cocycle ω . It can be explicitly described as the complex Lie algebra generated by $L_n, n \in \mathbb{Z}$, and the central element κ , with brackets

$$[L_n, L_m] = (n - m)L_{n+m} + \delta_{n+m,0} \frac{n^3 - n}{12} \kappa.$$

Positive-energy representations. Consider a representation $\pi_V : \text{Vir} \rightarrow \text{End}(V)$ of Vir on a complex vector space V endowed with a non-degenerate, positive-definite scalar product $\langle \cdot, \cdot \rangle$. We call π_V a **unitary positive energy representation** if the following hold:

- Unitarity: $\langle v, \pi_V(L_n)w \rangle = \langle \pi_V(L_{-n})v, w \rangle$ for every $v, w \in V$ and $n \in \mathbb{Z}$;
- Positivity of the energy: $V = \bigoplus_{\lambda \in \mathbb{R}_+ \cup \{0\}} V_\lambda$, where $V_\lambda := \ker(\pi_V(L_0) - \lambda \mathbb{1}_V)$. The lowest eigenvalue of $\pi_V(L_0)$ is called lowest weight;
- Central charge: $\pi_V(\kappa) = c \mathbb{1}_V$;

There exists an irreducible unitary positive energy representation $\pi_{c,h}$ with central charge c and lowest weight h if and only if $c \geq 1$ and $h \geq 0$ (continuous series representation) or $(c, h) = (c(m), h_{p,q}(m))$, where $c(m) = 1 - \frac{6}{(m+2)(m+3)}$, $h_{p,q}(m) = \frac{(p(m+1)-qm)^2-1}{4m(m+1)}$, $m = 3, 4, \dots$, $p = 1, 2, \dots, m-1$, $q = 1, 2, \dots, p$, (discrete series representation) [KR87, DMS97]. In this case the representation space V is denoted by $\mathcal{H}^{\text{fin}}(c, h)$. We denote by $\mathcal{H}(c, h)$ the Hilbert space completion of the vector space $\mathcal{H}^{\text{fin}}(c, h)$ associated with the unique irreducible unitary positive energy representation of Vir with central charge c and lowest weight h .

The stress–energy tensor. In a (possibly infinite) direct sum representation $\pi_{\mathcal{H}}$ of π_{c,h_j} with the same central charge c , the conformal Hamiltonian L_0 is diagonalized, and on the linear span of its eigenvectors \mathcal{H}^{fin} (the space of finite energy vectors), the Virasoro algebra acts algebraically as unbounded operators. With a slight abuse of notation, we denote by L_n the elements of Vir represented in \mathcal{H} .

For a smooth complex-valued function f on S^1 with finitely many non-zero Fourier coefficients, the (chiral) stress–energy tensor associated with f is the operator

$$T(f) = \sum_{n \in \mathbb{Z}} L_n \hat{f}_n$$

acting on \mathcal{H} , where

$$\hat{f}_n = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-in\theta} f(e^{i\theta}).$$

The stress–energy tensor T can be extended to certain nonsmooth real functions $f \in \mathcal{S}_{\frac{3}{2}}(S^1, \mathbb{R})$ by the linear energy bounds, yielding a self-adjoint unbounded operator $T(f)$. We will review these results in detail in Sect. 2.3.3.

Such a representation integrates to a projective unitary representation of the universal covering group $\widetilde{\text{Diff}}_+(S^1)$ of $\text{Diff}_+(S^1)$, namely, there is a projective unitary representation U of $\widetilde{\text{Diff}}_+(S^1)$ such that $U(\text{Exp}(f)) = e^{iT(f)}$ up to a scalar.³ If the h_j 's appearing in the direct summand differ from each other by an integer, then U reduces to a projective representation of $\text{Diff}_+(S^1)$ (2π -rotation is a scalar).

The stress–energy tensor T satisfies the following covariance [FH05, Proposition 5.1, Proposition 3.1]:

Proposition 2.1. *The stress–energy tensor T on \mathcal{H} and its integration U as above satisfy*

$$\begin{aligned} \text{Ad } U(\gamma)(T(f)) &= T(\gamma_*(f)) + \beta(\dot{\gamma}, f) \\ \beta(\dot{\gamma}, f) &:= \frac{c}{24\pi} \int_0^{2\pi} \{\dot{\gamma}, z\}|_{z=e^{i\theta}} f(e^{i\theta}) e^{i2\theta} d\theta \end{aligned}$$

on vectors in \mathcal{H}^{fin} , for $f \in \text{Vect}(S^1)$ and $\gamma \in \widetilde{\text{Diff}}_+(S^1)$, where $\dot{\gamma} \in \text{Diff}_+(S^1)$ is the image of γ under the covering map. Furthermore, the commutation relations

$$i[T(g), T(f)] = T(g'f - f'g) + c\omega(g, f),$$

³ This scalar cannot be made trivial [FH05, (5.11)].

hold for arbitrary $f, g \in C^\infty(S^1)$, on vectors $\psi \in \mathcal{H}^{\text{fin}}$, where

$$\{\dot{\gamma}, z\} = \frac{\frac{d^3}{dz^3}\dot{\gamma}(z)}{\frac{d}{dz}\dot{\gamma}(z)} - \frac{3}{2} \left(\frac{\frac{d^2}{dz^2}\dot{\gamma}(z)}{\frac{d}{dz}\dot{\gamma}(z)} \right)^2$$

is the Schwarzian derivative of $\dot{\gamma}$ and $\frac{d}{dz}\dot{\gamma}(z) = -i\bar{z}\frac{d}{d\theta}\dot{\gamma}(e^{i\theta}) \Big|_{e^{i\theta}=z}$.

If we consider the Cayley transform (2.2), (2.3), a vector field $f \in \text{Vect}(S^1)$ in real line coordinates is given by

$$C_*(f)(t) = \frac{(1+t^2)}{2} f(C^{-1}(t)).$$

With the Schwartz class functions $\mathcal{S}(\mathbb{R})$, the stress energy tensor satisfies the following quantum-energy inequalities [FH05, Theorem 4.1].

Theorem 2.2. *Let $f \in \text{Vect}(S^1)$ with $C_*(f) \in \mathcal{S}(\mathbb{R})$ and $C_*(f)(t) \geq 0 \forall t \in \mathbb{R}$. For $\psi \in \mathcal{D}(L_0)$, it holds that*

$$(\psi, T(f)\psi) \geq -\frac{c}{12\pi} \int_{\mathbb{R}} \left(\frac{d}{dt} \sqrt{C_*(f)(t)} \right)^2 dt,$$

where the derivative is given by

$$\frac{d}{dt} \sqrt{C_*(f)(t)} = \begin{cases} (\frac{d}{dt} C_*(f)(t)) / (2\sqrt{C_*(f)(t)}) & \text{if } C_*(f)(t) \neq 0 \\ 0 & \text{if } C_*(f)(t) = 0. \end{cases}$$

Translations The generator of translations is by definition $\mathfrak{t}(t) := \frac{\partial}{\partial s} (\tau(t)s) \Big|_{s=0} = 1$.

The corresponding field on S^1 is

$$\mathfrak{t}(e^{i\theta}) = 1 + \cos \theta.$$

Subgroups B_n The stabilizer subgroup $B_0 \subset \text{Diff}_+(S^1)$ of -1 is a Lie subgroup with Lie algebra given by those vector fields $f \in \text{Vect}(S^1)$ such that $f(-1) = 0$. The dilation and translation subgroups of $\text{Diff}_+(S^1)$ are in B_0 . Similarly, B_n is the subgroup of B_0 whose elements have vanishing 1st, 2nd, ..., n -th derivatives at -1 . The group B_1 still contains translations, but not dilations. These groups have the natural C^∞ topology, but we often treat them without topology.

2.3.3. Piecewise smooth diffeomorphisms. Let $\text{Diff}_+^{1,\text{ps}}(S^1)$ be the group of piecewise smooth C^1 -diffeomorphisms of S^1 , namely, $\gamma \in \text{Diff}_+^{1,\text{ps}}(S^1)$ is a C^1 -diffeomorphism and S^1 can be decomposed into finitely many closed intervals (with a possibly common end point) on each of which γ is smooth and has finite derivatives in all orders at the end points. It has been known that some elements of $\text{Diff}_+^{1,\text{ps}}(S^1)$ can be implemented in a conformal net [CW05, Wei06]. Let us recall these elements.

For a real-valued continuous function f of the circle, set

$$\|f\|_{\frac{3}{2}} := \sum_{n \in \mathbb{Z}} |\hat{f}_n| (1 + |n|^{\frac{3}{2}}),$$

where $\hat{f}_n := \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(e^{i\theta}) d\theta$ is the n -th Fourier coefficient⁴ of f . We denote with $\mathcal{S}_{\frac{3}{2}}(S^1, \mathbb{R})$ the class of functions $f \in L^1(S^1, \mathbb{R})$ such that $\|f\|_{\frac{3}{2}}$ is finite, endowed with the topology induced by the norm $\|\cdot\|_{\frac{3}{2}}$. By [Wei06, Lemma 2.2], if f is piecewise smooth and C^1 on the whole S^1 , then $f \in \mathcal{S}_{\frac{3}{2}}(S^1, \mathbb{R})$.

Let T be the stress–energy tensor on $\mathcal{H} = \bigoplus_j \mathcal{H}(c, h_j)$. In [CW05, Proposition 4.2, Theorem 4.4, Proposition 4.5] it has been shown that, if $f \in \mathcal{S}_{\frac{3}{2}}(S^1, \mathbb{R})$, then the operator $T(f) = \sum_{n \in \mathbb{Z}} L_n f_n$ on the domain \mathcal{H}^{fin} is convergent and essentially self-adjoint on any core of L_0 . In addition, if $\|f - f_n\|_{\frac{3}{2}} \rightarrow 0$ for $f, f_n \in \mathcal{S}_{\frac{3}{2}}(S^1, \mathbb{R})$, then $T(f_n) \rightarrow T(f)$ in the strong resolvent sense.

From these results, it follows that certain piecewise smooth C^1 -diffeomorphisms of the form $\text{Exp}(f)$ are implemented in a conformal net. Actually, we prove in “Appendix A” that any conformal net is $\text{Diff}_+^{1,\text{ps}}(S^1)$ -covariant. We do not consider topology on $\text{Diff}_+^{1,\text{ps}}(S^1)$. It follows that any soliton is $\text{Diff}_{+,1}^{1,\text{ps}}(S^1)$ -covariant, where $\text{Diff}_{+,1}^{1,\text{ps}}(S^1) := \{\gamma \in \text{Diff}_+^{1,\text{ps}}(S^1) : \gamma(-1) = -1, \gamma'(-1) = 1\}$ (Theorem 3.4). The solitons we construct in Sect. 3.2 are $\text{Diff}_{+,0}^{1,\text{ps}}(S^1)$ -covariant where $\text{Diff}_{+,0}^{1,\text{ps}}(S^1) := \{\gamma \in \text{Diff}_+^{1,\text{ps}}(S^1) : \gamma(-1) = -1\}$.

2.3.4. The Groups of Sobolev-class diffeomorphisms. For s real, the Sobolev spaces $H^s(S^1)$ are defined by

$$H^s(S^1) := \{f \in L^2(S^1) : \|f\|_{H^s} < \infty\}, \text{ where } \|f\|_{H^s} := \left(\sum_{k \in \mathbb{Z}} (1 + k^2)^s |\hat{f}_k|^2 \right)^{\frac{1}{2}}.$$

If $s > \frac{3}{2}$, then the set $\mathcal{D}^s(S^1)$ of Sobolev-class diffeomorphisms

$$\mathcal{D}^s(S^1) := \{\gamma \in \text{Diff}_+^1(S^1) : \tilde{\gamma} - \iota \in H^s(S^1)\},$$

where $\tilde{\gamma}$ is a lift of γ to \mathbb{R} , is a topological group [IKT13, Theorem B.2] (see also [CDIT, Lemma 2.5]). We also have the following continuity of the action on $H^s(S^1)$ [IKT13, Theorem B.2].

Lemma 2.3. *For $s > 3/2$, the map $(f, \gamma) \mapsto f \circ \gamma$ from $H^s(S^1) \times \mathcal{D}^s(S^1)$ into $H^s(S^1)$ is continuous.*

3. General Results on Solitons

Throughout this Section, (\mathcal{A}, U, Ω) is a conformal net in the sense of Section 2.

It has been observed [Hen17b, Section 3.3.1] that, by [Wei06], any soliton can be made translation covariant. In the next Section we show further that it has always positive energy, proving [Hen17b, Conjecture 32]. We suspect that the converse implication [Hen17b, Conjecture 34] could be negative, cf. [Tan18, Tan11]. In addition, we present a general scheme to construct solitons for any conformal net.

⁴ This should be distinguished from a sequence of functions f_n .

3.1. Positivity of energy. Let us first observe that $\text{Exp}(tg)$ makes sense if g is C^1 , because then the existence and uniqueness of solution of the ordinary differential equation are assured [CL55, Chapter 1, Theorem 2.3]. We need some preparatory results on representations of these elements.

Lemma 3.1. *Let $g \in C^\infty(S^1, \mathbb{R})$ and f be a real piecewise smooth and C^1 -function on S^1 . Then it holds that*

$$\text{Ad } e^{iT(g)}(T(f)) = T(\text{Exp}(g)_*(f)) + \beta(\text{Exp}(g), f).$$

Proof. Let us fix s such that $2 < s < \frac{5}{2}$. Note that $f \in H^s(S^1)$. Indeed, f'' is defined except a finite number of points and of bounded variation, and by the proof of [Wei06, Lemma 2.2], we have $|k|^2 \hat{f}_k| \leq \left| \frac{\text{Var}(f'')}{k} \right|$, where $\text{Var}(f'')$ is the variation of f'' , see [Kat04, Theorem 4.5]. From this it is immediate that $|k|^{2s} |\hat{f}_k|^2 \leq \left| \frac{\text{Var}(f'')^2}{k^{6-2s}} \right|$ and the right-hand side is summable in k as $6 - 2s > 1$, hence $f \in H^s(S^1)$.

Next, let us observe that $H^s(S^1) \subset \mathcal{S}_{\frac{3}{2}}(S^1)$. Indeed,

$$\sum_k (1 + |k|)^{\frac{3}{2}} |\hat{f}_k| \leq \sum_k (1 + |k|)^s |\hat{f}_k| \cdot (1 + |k|)^{\frac{3}{2}-s} \leq 2 \sum_k (1 + |k|^2)^{\frac{s}{2}} |\hat{f}_k| \cdot (1 + |k|)^{\frac{3}{2}-s}$$

and the right-hand side can be seen as a scalar product of two $\ell^2(\mathbb{Z})$ sequences (because $s > 2$), hence it holds that $\|f\|_{\frac{3}{2}} \leq \text{Const.} \|f\|_{H^s}$, where the constant depends on s but not on f .

We know that there is a sequence $\{f_n\} \subset C^\infty(S^1, \mathbb{R})$, $\|f - f_n\|_{H^s} \rightarrow 0$. For $f_n \in C^\infty(S^1, \mathbb{R})$, we have by Proposition 2.1

$$\text{Ad } e^{iT(g)}(e^{iT(f_n)}) = e^{i(T(\text{Exp}(g)_*(f_n)) + \beta(\text{Exp}(g), f_n))}. \quad (3.1)$$

By the above observations, we have $f_n \rightarrow f$ in $\mathcal{S}_{\frac{3}{2}}(S^1, \mathbb{R})$. By [CDIT, Lemma 2.5(a)] (see also [IKT13, Lemma B.2]), $f \mapsto \text{Exp}(g)_*(f)$ is continuous in $H^s(S^1)$, hence $\text{Exp}(g)_*(f_n) \rightarrow \text{Exp}(g)_*(f)$ in $\mathcal{S}_{\frac{3}{2}}(S^1)$. By [CW05, Proposition 4.5], $T(\text{Exp}(g)_*(f_n)) \rightarrow T(\text{Exp}(g)_*(f))$ in the strong resolvent sense, and it is also clear that $\beta(\text{Exp}(g), f_n) \rightarrow \beta(\text{Exp}(g), f)$. Therefore, by taking the limit of (3.1), we obtain the claim. \square

Remark 3.2. If $f \in C^1$ and not C^2 , then $f \notin H^s(S^1)$ for $s > \frac{5}{2}$ since with such s it holds that $H^s(S^1) \subset C^2(S^1)$ by the Sobolev–Morrey embedding.

In [Wei06, Proposition 2.3], it is claimed that the same conclusion holds for $f \in \mathcal{S}_{\frac{3}{2}}(S^1)$, but a proof of the convergence $\gamma_*(f_n) \rightarrow \gamma_*(f)$ in $\|\cdot\|_{\frac{3}{2}}$ is missing. Yet, the main results of the paper remain valid because one needs only the conclusion for f that is piecewise smooth and C^1 . It is also possible to prove [Wei06, Proposition 2.3] (this is due to Sebastiano Carpi, we thank him for agreeing that we include it in the present article): if f is smooth, then $\text{Ad } e^{iT(g)}(T(f)) = T(\text{Exp}(g)_*(f)) + \beta(\text{Exp}(g), f)$ holds on \mathcal{H}^{fin} . For a fixed $\xi \in \mathcal{H}^{\text{fin}}$, $T(f)\xi$ is continuous in $f \in H^{\frac{3}{2}}(S^1)$ from the estimates in [CW05]. If we take a sequence f_n of smooth functions convergent to $f \in \mathcal{S}_{\frac{3}{2}}(S^1)$ as above, then it is convergent in $H^{\frac{3}{2}}(S^1)$, and $\text{Exp}(g)_*(f_n)$ is convergent to $\text{Exp}(g)_*(f)$ in $H^{\frac{3}{2}}(S^1)$ to $\text{Exp}(g)_*(f)$. This implies that $T(\text{Exp}(g)_*(f_n))\xi = (\text{Ad } e^{iT(g)}(T(f_n)) - \beta(\text{Exp}(g), f_n))\xi$ is convergent to $T(\text{Exp}(g)_*(f))\xi$. Therefore, the

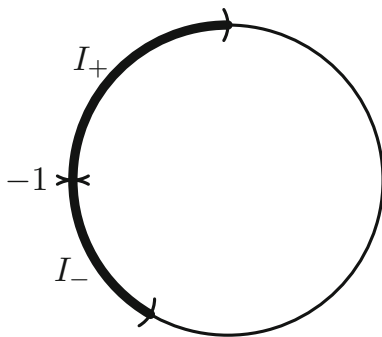


Fig. 1. Intervals I_{\pm}

equality $\text{Ad } e^{iT(g)}(T(f)) = T(\text{Exp}(g)_*(f)) + \beta(\text{Exp}(g), f)$ holds on \mathcal{H}^{fin} , but it is a core of the right-hand side, hence so is it for the left-hand side and the equality as operators follows.

Lemma 3.3. *Let $g, f \in C^\infty(S^1, \mathbb{R})$ and $g(-1) = g'(-1) = f(-1) = f'(-1) = 0$ and compactly supported. Let I_{\pm} be disjoint intervals in S^1 one of whose boundary points is -1 (see Fig. 1. Let $f = f_- + f_+$, $f_{\pm} \in \mathcal{S}_{\frac{3}{2}}(S^1)$ be the decomposition of f into two pieces cut at the point -1 (which is possible by [Wei06, Lemma 2.2]), and similarly introduce $g = g_- + g_+$, $g_{\pm} \in \mathcal{S}_{\frac{3}{2}}(S^1)$, and assume that $\text{supp } f_{\pm}, \text{supp } g_{\pm} \subset I_{\pm}$.*

Then it holds that

$$\text{Ad } e^{iT(g_-)}(T(f_-)) = T(\text{Exp}(g_-)_*(f_-)) + \beta(\text{Exp}(g_-), f_-),$$

where $\beta(\text{Exp}(g_-), f_-)$ is defined by a similar formula as before:

$$\beta(\text{Exp}(g_-), f) := \frac{c}{24\pi} \int_{\text{supp } g_-} \left\{ \text{Exp}(g_-), z \right\} \Big|_{z=e^{i\theta}} f(e^{i\theta}) e^{i2\theta} d\theta, \quad (3.2)$$

where the integral is restricted to $\text{supp } g_-$ on which the Schwarzian derivative is defined.

Proof. Let $t \in \mathbb{R}$. Since f_- is piecewise smooth and C^1 and g is smooth, by Lemma 3.1 we have

$$\text{Ad } e^{iT(g)}(e^{iT(tf_-)}) = e^{iT(\text{Exp}(g)_*(tf_-))} e^{i\beta(\text{Exp}(g), tf_-)}.$$

Furthermore, note that

$$\begin{aligned} \beta(\text{Exp}(g), tf_-) &= \frac{c}{24\pi} \int_0^{2\pi} \left\{ \text{Exp}(g), z \right\} \Big|_{z=e^{i\theta}} tf_-(e^{i\theta}) e^{i2\theta} d\theta \\ &= \frac{c}{24\pi} \int_{\text{supp } g_-} \left\{ \text{Exp}(g_-), z \right\} \Big|_{z=e^{i\theta}} tf_-(e^{i\theta}) e^{i2\theta} d\theta \\ &= \beta(\text{Exp}(g_-), tf_-), \end{aligned}$$

because $\text{Exp}(g_-)$ and f_- has support contained in a common interval and $\text{Exp}(g_-)$ is smooth there.

Note that $g_{\pm} \in \mathcal{S}_{\frac{3}{2}}(S^1, \mathbb{R})$, hence $e^{iT(tf_{\pm})}$ and $e^{iT(g_{\pm})}$ are affiliated with $\mathcal{A}(I_{\pm})$ by [Wei06, Proposition 2.3], and it follows that $e^{iT(g)} = e^{iT(g_-)}e^{iT(g_+)}$. By the assumed support property, we have

$$\begin{aligned} \text{Ad} e^{iT(g)}(e^{iT(tf_-)}) &= \text{Ad}(e^{iT(g_-)} \cdot e^{iT(g_+)}) (e^{iT(tf_-)}) \\ &= e^{iT(\text{Exp}(g_-)_*(tf_-))} \cdot e^{i\beta(\text{Exp}(g_-), tf_-)}. \end{aligned}$$

By taking the derivative with respect to t , we obtain

$$\text{Ad} e^{iT(g)}(T(f_-)) = \text{Ad} e^{iT(g_-)}(T(f_-)) = T(\text{Exp}(g_-)_*(f_-)) + \beta(\text{Exp}(g_-), f_-),$$

on the full domain. \square

Theorem 3.4. *A soliton σ of a conformal net \mathcal{A} is $\text{Diff}_{+,1}^{1,\text{ps}}(S^1)$ -covariant, the translations act continuously and have positive energy.*

Proof. Our strategy is to write the translation as a product of three elements localized in half-lines or an interval, in each of which σ is normal. For this purpose, let $I_{(\theta_1, \theta_2)} = \{e^{i\theta} : \theta_1 < \theta < \theta_2\}$ be an interval on S^1 and we take a C^∞ -function $h_+ : S^1 \setminus \{-1\} \rightarrow \mathbb{R}$ that is equal to 0 on $I_{(-\pi, 0)}$ and equal to 1 on $I_{(\frac{\pi}{2}, \pi)}$. Similarly, let $h_-(x)$ be a C^∞ -function that is equal to 1 on $I_{(-\pi, -\frac{\pi}{2})}$ and equal to 0 on $I_{(0, \pi)}$. They have disjoint supports.

Let us first prove the following relation:

$$\text{Ad} e^{iT(h_-t)}(T(t)) = T(\text{Exp}(th_-t)_*(t)) + \beta(\text{Exp}(th_-t), t), \quad (3.3)$$

where $t(e^{i\theta}) = 1 + \cos \theta$ is the generator of translations (see Sect. 2.3.2). Note that h_-t is supported in a certain interval I_- , one of whose boundary is -1 , hence so is $\text{Exp}(th_-t)$. We decompose t into two pieces $t_+, t_- \in \mathcal{S}_{\frac{3}{2}}(S^1, \mathbb{R})$ such that $t_-(\theta) = t(\theta)$ on I_- and $t_+ = t - t_-$. Note that $\beta(\text{Exp}(th_-t), t_-) = \beta(\text{Exp}(th_-t), t)$, since the supports of $\text{Exp}(th_-t)$ and of t_+ are disjoint, see (3.2). As h_-t coincides with t on an interval, one of whose boundary point is -1 , we can apply Lemma 3.3 to obtain

$$\begin{aligned} \text{Ad} e^{iT(h_-t)}(T(t_-)) &= T(\text{Exp}(th_-t)_*(t_-)) + \beta(\text{Exp}(th_-t), t_-) \\ &= T(\text{Exp}(th_-t)_*(t_-)) + \beta(\text{Exp}(th_-t), t). \end{aligned} \quad (3.4)$$

On the other hand, since h_-t and t_+ have disjoint support (see Fig. 2), we have

$$\text{Ad} e^{iT(h_-t)}(T(t_+)) = T(t_+). \quad (3.5)$$

Note that $\text{Exp}(th_-t)_*t = \text{Exp}(th_-t)_*t_+ + \text{Exp}(th_-t)_*t_- = t_+ + \text{Exp}(th_-t)_*t_-$. By adding the sides of (3.4) and (3.5), we obtain on the intersection of the domains

$$\text{Ad} e^{iT(h_-t)}(T(t)) = T(\text{Exp}(th_-t)_*(t)) + \beta(\text{Exp}(th_-t), t).$$

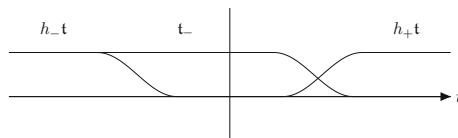


Fig. 2. Vector fields h_-t , t_- , h_+t in the real line picture

The intersection of the operators in (3.4), (3.5) includes $C^\infty(L_0) = \bigcap_n \mathcal{D}(L_0^n)$, hence so does the sum. The right-hand side of the last expression is essentially self-adjoint by [CW05, Theorem 4.4] (cited in Sect. 2.3.3). Hence the left-hand side is a self-adjoint extension of the right-hand side, and therefore, they must coincide on the full domain.

Next, we write $e^{iT(t)}$ as

$$e^{iT(t)} = e^{iT(h_-t)} \cdot e^{-iT(h_-t)} e^{iT(t)} e^{-iT(h_+t)} \cdot e^{iT(h_+t)}.$$

We claim that $e^{-iT(h_-t)} e^{iT(t)} e^{-iT(h_+t)}$ is localized on an interval whose closure does not contain -1 (such an interval depends on t). This follows from (3.3). Indeed, $\text{Exp}(th_-t)_*(t)$ agrees with t in a neighborhood of -1 (depending on t) and

$$\begin{aligned} e^{-iT(h_-t)} e^{iT(t)} e^{-iT(h_+t)} &= e^{iT(\text{Exp}(h_-t)_*(t))} e^{i\beta(\text{Exp}(th_-t), t)} \cdot e^{-iT(h_-t)} e^{-iT(h_+t)} \\ &= e^{iT(\text{Exp}(h_-t)_*(t))} e^{i\beta(\text{Exp}(th_-t), t)} e^{-iT(h_-t+h_+t)}, \end{aligned}$$

where we used the linearity of T on functions of class $\mathcal{S}_2(S^1, \mathbb{R})$, and the last expression is localized in a bounded interval: as $h_-t + h_+t$ equals t in a neighborhood of $-1 \in S^1$, $\text{Ad} e^{-iT(h_-t+h_+t)}$ implements the same action on $\mathcal{A}(I_{t,\epsilon})$ for some neighborhood $I_{t,\epsilon}$ for small t as the action of $\text{Ad} e^{iT(\text{Exp}(h_-t)_*(t))}$. In other words, $\text{Ad} e^{iT(\text{Exp}(h_-t)_*(t))} e^{-iT(h_-t+h_+t)}$ is trivial on $\mathcal{A}(I_{t,\epsilon})$, which implies that $e^{iT(\text{Exp}(h_-t)_*(t))} e^{-iT(h_-t+h_+t)}$ is localized in $I'_{t,\epsilon}$.

We introduce a representation of the translation group by

$$U_\sigma(t) := \sigma(e^{iT(h_-t)}) \sigma(e^{-iT(h_-t)} e^{iT(t)} e^{-iT(h_+t)}) \sigma(e^{iT(h_+t)}).$$

Note first the relation $\beta(\gamma_1 \circ \gamma_2, f) = \beta(\gamma_1, \gamma_{2*}(f)) + \beta(\gamma_2, f)$ for smooth γ , which is equivalent to $0 = \beta(\text{id}, f) = \beta(\gamma^{-1}, \gamma_*f) + \beta(\gamma, f)$. This continues to hold for nonsmooth $\gamma = \text{Exp}(th_\pm t)$ and $f = t$. Indeed, $\text{Exp}(th_\pm t)$ can be continued to a smooth diffeomorphisms with compact support, say I , and let I_\pm be the intervals obtained by removing -1 from I . The function $\beta(\text{Exp}(th_\pm t), f)$ is defined by the integral (3.2), which can be extended to $f = t_\pm$, the restriction of t to I_\pm . Hence the relation $\beta(\text{Exp}(-th_\pm t), \text{Exp}(th_\pm t)_*(t_\pm)) + \beta(\text{Exp}(th_\pm t), t_\pm) = 0$ holds. Again by linearity in the second variable, we have for $t_1, t_2 \in \mathbb{R}$

$$\begin{aligned} 0 &= \beta(\text{id}, t) = \beta(\text{Exp}(t_2 h_- t), \text{Exp}(-t_2 h_- t)_*(t)) + \beta(\text{Exp}(-t_2 h_- t), t), \\ 0 &= \beta(\text{id}, t) = \beta(\text{Exp}(-t_1 h_+ t), \text{Exp}(t_1 h_+ t)_*(t)) + \beta(\text{Exp}(t_1 h_+ t), t). \end{aligned} \quad (3.6)$$

By recalling that h_- and h_+ have disjoint supports, with the help of (3.3) and an analogous relation for h_+ , this yields a one-parameter group in t :

$$\begin{aligned} U_\sigma(t_1) U_\sigma(t_2) &= \sigma(e^{it_1 T(h_-t)}) \sigma(e^{-it_1 T(h_-t)} e^{it_1 T(t)} e^{-it_1 T(h_+t)}) \sigma(e^{it_1 T(h_+t)}) \\ &\quad \cdot \sigma(e^{it_2 T(h_-t)}) \sigma(e^{-it_2 T(h_-t)} e^{it_2 T(t)} e^{-it_2 T(h_+t)}) \sigma(e^{it_2 T(h_+t)}) \\ &= \sigma(e^{it_1 T(h_-t)}) \sigma(e^{it_2 T(h_-t)}) \sigma(e^{-it_1 T(h_-t)} e^{it_1 T(\text{Exp}(-t_2 h_- t)_*(t))} e^{i\beta(\text{Exp}(-t_2 h_- t), t)} e^{-it_1 T(h_+t)}) \\ &\quad \cdot \sigma(e^{it_1 T(h_+t)}) \sigma(e^{-it_2 T(h_-t)} e^{it_2 T(t)} e^{-it_2 T(h_+t)}) \sigma(e^{it_2 T(h_+t)}) \\ &= \sigma(e^{i(t_1+t_2)T(h_-t)}) \sigma(e^{-it_1 T(h_-t)} e^{it_1 T(\text{Exp}(-t_2 h_- t)_*(t))} e^{i\beta(\text{Exp}(-t_2 h_- t), t)} e^{-it_1 T(h_+t)}) \\ &\quad \cdot \sigma(e^{-it_2 T(h_-t)} e^{it_2 T(\text{Exp}(t_1 h_+ t)_*(t))} e^{i\beta(\text{Exp}(t_1 h_+ t), t)} e^{-it_2 T(h_+t)}) \sigma(e^{it_1 T(h_+t)}) \sigma(e^{it_2 T(h_+t)}) \\ &= \sigma(e^{i(t_1+t_2)T(h_-t)}) \end{aligned}$$

$$\begin{aligned}
 & \cdot \sigma(e^{-it_1 T(h-t)}) e^{it_1 T(\text{Exp}(-t_2 h-t)_*(t))} e^{-it_1 T(h+t)} e^{-it_2 T(h-t)} e^{it_2 T(\text{Exp}(t_1 h+t)_*(t))} e^{-it_2 T(h+t)} \\
 & \cdot \sigma(e^{it_1 T(h+t)}) \sigma(e^{it_2 T(h+t)}) \cdot e^{i\beta(\text{Exp}(-t_2 h-t), t)} e^{i\beta(\text{Exp}(t_1 h+t), t)} \\
 & = \sigma(e^{i(t_1+t_2)T(h-t)}) \\
 & \cdot \sigma(e^{-i(t_1+t_2)T(h-t)}) e^{it_1 T(t)} e^{it_2 T(t)} e^{i\beta(\text{Exp}(t_2 h-t), \text{Exp}(-t_2 h-t)_*(t))} e^{-it_1 T(h+t)} e^{it_2 T(\text{Exp}(t_1 h+t)_*(t))} e^{-it_2 T(h+t)} \\
 & \cdot \sigma(e^{it_1 T(h+t)}) \sigma(e^{it_2 T(h+t)}) \cdot e^{i\beta(\text{Exp}(-t_2 h-t), t)} e^{i\beta(\text{Exp}(t_1 h+t), t)} \\
 & = \sigma(e^{i(t_1+t_2)T(h-t)}) \\
 & \cdot \sigma(e^{-i(t_1+t_2)T(h-t)}) e^{it_1 T(t)} e^{it_2 T(t)} e^{i\beta(\text{Exp}(-t_1 h+t), \text{Exp}(t_1 h+t)_*(t))} e^{-i(t_1+t_2)T(h+t)} \\
 & \cdot \sigma(e^{it_1 T(h+t)}) \sigma(e^{it_2 T(h+t)}) \cdot e^{i\beta(\text{Exp}(-t_2 h-t), t)} e^{i\beta(\text{Exp}(t_2 h-t), \text{Exp}(-t_2 h-t)_*(t))} e^{i\beta(\text{Exp}(t_1 h+t), t)} \\
 & = \sigma(e^{i(t_1+t_2)T(h-t)}) \sigma(e^{-(t_1+t_2)T(h-t)}) e^{i(t_1+t_2)T(t)} e^{-i(t_1+t_2)T(h+t)} \\
 & \cdot \sigma(e^{i(t_1+t_2)T(h+t)}) \cdot e^{i\beta(\text{Exp}(-t_2 h-t), t)} e^{i\beta(\text{Exp}(t_2 h-t), \text{Exp}(-t_2 h-t)_*(t))} \\
 & \cdot e^{i\beta(\text{Exp}(t_1 h+t), t)} e^{i\beta(\text{Exp}(-t_1 h+t), \text{Exp}(t_1 h+t)_*(t))} \\
 & = U_\sigma(t_1 + t_2),
 \end{aligned}$$

where the scalars cancel by (3.6). To show continuity of $U^\sigma(t)$, it is enough to see that $U^\sigma(t) \rightarrow \mathbb{1}$ as $t \rightarrow 0$ in the strong operator topology. All the factors in the product

$$U^\sigma(t) = \sigma(e^{itT(h-t)}) \sigma(e^{-itT(h-t)}) e^{itT(t)} e^{-itT(h+t)} \sigma(e^{itT(h+t)})$$

are unitary, and hence uniformly bounded. The first and the third factors tend to $\mathbb{1}$ by normality of σ , as they are supported in a fixed half-line. As for the second factor, the product $e^{-itT(h-t)} e^{itT(t)} e^{-itT(h+t)}$ tends to $\mathbb{1}$ in the vacuum representation, and this is localized in a fixed interval for small t , hence by normality of σ the second factor tends to $\mathbb{1}$ as well.

It remains to prove the positivity of energy. We do this by showing that $U_\sigma(t)$ can be obtained as a limit in the strong resolvent sense of a sequence of one-parameter unitary groups with positive generator. Let t_1 be a C^∞ -vector field on S^1 such that $C_*(t_1)$ is equal to 1 on $(-\infty, 1)$ and equal to 0 on $(2, +\infty)$. From t_1 we construct a sequence of vector fields (see Fig. 3)

$$C_*(t_n)(t) := C_*(t_1) \left(\frac{t}{n} \right), \quad t \in \mathbb{R}, \quad n \in \mathbb{N}.$$

We fix $2 < s < \frac{5}{2}$ (cf. the proof of Lemma 3.1). Let us show that $t_n \rightarrow t$ in the $H^s(S^1)$ -topology. For this it is sufficient to show that $\left\{ \frac{d^3}{d\theta^3} t_n \right\}_{n \in \mathbb{N}}$ is a sequence of functions in $L^1(S^1)$ uniformly bounded in n and that $t_n \rightarrow \tau$ in $L^1(S^1)$: this implies that $|k^3 \hat{t}_n(k)| < \text{Const.}$, where $\hat{t}_n(k)$ is the k -th Fourier coefficient of t_n , or equivalently, $|k^{2s} \hat{t}_n(k)|^2 < \frac{\text{Const.}}{k^{6-2s}}$, and the right-hand side is summable in k since $6 - 2s > 1$. From

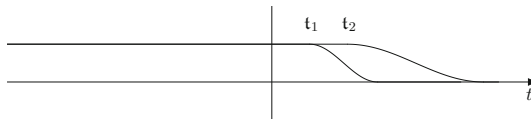


Fig. 3. Vector fields t_n in the real line picture

the convergence $t_n \rightarrow t$ in L^1 we obtain the convergence of each $\hat{t}_n(k)$. Therefore, by the Lebesgue dominated convergence theorem (applied to the measurable set \mathbb{Z} with the counting measure), we obtain the convergence $t_n \rightarrow t$ in $H^s(S^1)$. Let us make these necessary estimates separately.

Explicitly, as $\frac{dC}{d\theta}(e^{i\theta}) = \frac{1}{1+\cos\theta}$, we have

$$t_n(e^{i\theta}) = (1 + \cos \theta) \cdot C_*(t_1) \left(\frac{C(e^{i\theta})}{n} \right)$$

and $C_*(t_1)$ is bounded. The third derivative of t_n is

$$\begin{aligned} \frac{d^3}{d\theta^3} t_n(e^{i\theta}) &= \sin \theta \cdot C_*(t_1) \left(\frac{C(e^{i\theta})}{n} \right) - \frac{2 \cos \theta}{n(1 + \cos \theta)} \frac{d}{dt} (C_*(t_1)) \left(\frac{C(e^{i\theta})}{n} \right) \\ &\quad - \frac{\sin^2 \theta}{n(1 + \cos \theta)^2} \frac{d}{dt} (C_*(t_1)) \left(\frac{C(e^{i\theta})}{n} \right) \\ &\quad + \frac{1}{n^3(1 + \cos \theta)^2} \frac{d^3}{dt^3} (C_*(t_1)) \left(\frac{C(e^{i\theta})}{n} \right). \end{aligned} \quad (3.7)$$

Recall that $C_*(t_1)$ is constant on $(-\infty, 1) \cup (2, \infty)$. The first term of the right-hand side of (3.7) is clearly uniformly bounded in n on S^1 . For the second term of the right-hand side of (3.7), by the change of variable $t = C(e^{i\theta})$, we have:

$$\begin{aligned} &\int_{-\pi}^{\pi} \left| \frac{2 \cos \theta}{n(1 + \cos \theta)} \frac{d}{dt} (C_*(t_1)) \left(\frac{C(e^{i\theta})}{n} \right) \right| d\theta \\ &= \frac{1}{n} \int_n^{2n} \left| 2 \cos(-i \log C^{-1}(t)) \frac{d}{dt} (C_*(t_1)) \left(\frac{t}{n} \right) \right| dt \end{aligned}$$

which is uniformly bounded in n . Next, by calculating directly one has $1 + \cos(-i \log C^{-1}(t)) = \frac{2}{1+t^2}$ and $\sin(-i \log C^{-1}(t)) = \frac{2t}{1+t^2}$, hence the third term is

$$\begin{aligned} &\int_{-\pi}^{\pi} \left| \frac{2 \sin^2 \theta}{n(1 + \cos \theta)^2} \frac{d}{dt} (C_*(t_1)) \left(\frac{C(e^{i\theta})}{n} \right) \right| d\theta \\ &= \int_n^{2n} \left| \frac{2 \sin^2 \theta}{n(1 + \cos \theta)} \frac{d}{dt} (C_*(t_1)) \left(\frac{t}{n} \right) \right| dt \\ &= \frac{1}{n} \int_n^{2n} \left| \frac{4t^2}{1+t^2} \frac{d}{dt} (C_*(t_1)) \left(\frac{t}{n} \right) \right| dt \end{aligned}$$

which is uniformly bounded in n , since the integrand is bounded. The fourth term is also uniformly bounded in n since

$$\begin{aligned} &\int_{-\pi}^{\pi} \left| \frac{1}{n^3(1 + \cos \theta)^2} \frac{d^3}{dt^3} (C_*(t_1)) \left(\frac{C(e^{i\theta})}{n} \right) \right| d\theta \\ &= \int_n^{2n} \left| \frac{1+t^2}{2n^3} \frac{d^3}{dt^3} (C_*(t_1)) \left(\frac{t}{n} \right) \right| dt \\ &\leq \frac{1}{n} \int_n^{2n} \left| \frac{1+4n^2}{2n^2} \frac{d^3}{dt^3} (C_*(t_1)) \left(\frac{t}{n} \right) \right| dt. \end{aligned}$$

Finally, we show that $t_n \rightarrow t$ in $L^1(S^1)$ using the boundedness of $C_*(t_1)$:

$$\begin{aligned} \int_{-\pi}^{\pi} |t(e^{i\theta}) - t_n(e^{i\theta})| d\theta &= \int_{-\pi}^{\pi} \left| (1 + \cos \theta) \left(1 - C_*(t_1) \left(\frac{C(e^{i\theta})}{n} \right) \right) \right| d\theta \\ &= \int_n^{+\infty} \left| (1 + \cos(-i \log C^{-1}(t)))^2 \left(1 - C_*(t_1) \left(\frac{t}{n} \right) \right) \right| dt \\ &= \int_n^{+\infty} \left| \frac{4}{(1+t^2)^2} \left(1 - C_*(t_1) \left(\frac{t}{n} \right) \right) \right| dt \longrightarrow 0 \quad (\text{as } n \rightarrow \infty). \end{aligned}$$

This completes the proof that $t_n \rightarrow t$ in $H^s(S^1)$, $2 < s < \frac{5}{2}$.

For each fixed t , the representation $U_\sigma(t)$ can be obtained as the limit of $\sigma(e^{itT(t_n)})$ in the strong topology. Indeed,

$$\sigma(e^{itT(t_n)}) = \sigma(e^{itT(h_-t_n)})\sigma(e^{-itT(h_-t_n)}e^{itT(t_n)}e^{-itT(h_+t_n)})\sigma(e^{itT(h_+t_n)}).$$

Note that h_- , h_+ , t_n belong to $H^s(S^1)$, and the product is (jointly) continuous [CDIT, Lemma 2.4][IKT13, Lemma B.4] as $\frac{3}{2} < 2 < s$, hence both h_-t_n and h_+t_n are convergent in $H^s(S^1)$, and by the argument of Lemma 3.1, they are convergent in $S_{\frac{3}{2}}(S^1, \mathbb{R})$, hence the corresponding operators are convergent in the strong resolvent sense. Furthermore, each of these sequences are localized in a fixed interval or a half line, hence by the normality of σ on half lines, the convergence follows. In other words, if we write $U^\sigma(t) =: e^{itT^\sigma}$, then $\sigma(T(t_n))$ (the generator of $\sigma(e^{itT(t_n)})$, which is defined by local normality) is convergent to T^σ in the strong resolvent sense.

We have by Theorem 2.2 that $T(t_1) \geq \alpha$ for some $\alpha \in \mathbb{R}$. By the fact that the Schwarzian derivative of a Möbius transformation is 0, it follows that the quantum energy inequalities of Theorem 2.2 are invariant under dilations, and therefore,

$$T(\delta_*^n(t_1)) = T(nt_n) \geq \alpha,$$

which implies

$$T(t_n) \geq \frac{\alpha}{n}.$$

Since $T(t_n)$ is localized on a half-line, by local normality of σ , we have $\sigma(T(t_n)) \geq \frac{\alpha}{n}$. By [RS75, Theorem VIII.23] and the convergence in the strong resolvent sense, T^σ is positive as well.

By Proposition A.4, the net (\mathcal{A}, U, Ω) is locally $\text{Diff}_{+,1}^{1,\text{ps}}(S^1)$ -covariant. Any element $\gamma \in \text{Diff}_{+,1}^{1,\text{ps}}(S^1)$ can be decomposed into a product $\gamma = \gamma_- \circ (\gamma_-^{-1} \circ \gamma \circ \gamma_+^{-1}) \circ \gamma_+$, where $\gamma_\pm \in \text{Diff}_{+,1}^{1,\text{ps}}(S^1)$ have supports as in the proof for $U^\sigma(t)$. It is straightforward to see that the definition

$$U^\sigma(\gamma) = \sigma(U(\gamma_-))\sigma(U(\gamma_-^{-1} \circ \gamma \circ \gamma_+^{-1}))\sigma(U(\gamma_+))$$

does not depend on the decomposition of γ . If I is a left half-line, we can choose γ_- , γ_+ such that $I \cap \text{supp } \gamma_+ = \emptyset$ and $\gamma_-|_I = \gamma$. Now for $x \in \mathcal{A}(I)$ the covariance $\sigma(\text{Ad } U(\gamma)(x)) = \text{Ad } U^\sigma(\gamma)(\sigma(x))$ follows because the both sides are localized in I_- , and by the definition $U^\sigma(\gamma \circ \gamma_+^{-1}) = \sigma(U(\gamma \circ \gamma_+^{-1}))$. The proof of covariance for a right half-line I is similar. \square

3.2. Solitons from nonsmooth diffeomorphisms. Here we construct families of continuously many proper solitons for any conformal net \mathcal{A} , using the diffeomorphism covariance.

Nonsmooth extendable diffeomorphisms. Let $\text{Diff}_+(S^1, -1) \subset \text{Diff}_+^0(S^1)$ be the class of orientation-preserving homeomorphisms ν of S^1 that have the following properties

- $\nu(-1) = -1$,
- ν is a smooth map from $S^1 \setminus \{-1\}$ onto $S^1 \setminus \{-1\}$, and the left and right derivatives at all orders exist at the point -1 , and the first order left and right derivatives are nonzero.

By Borel's theorem [Hor90, Theorem 1.2.6], for any sequence of real numbers $\{\lambda_n\}_{n \geq 1}$, there is a smooth function f such that $f(-1) = -1$ and $\frac{d^n f}{d\theta^n}(-1) = \lambda_n$. Let us take λ_n as the left derivatives of ν at $\theta = -1$. Then we can find a smooth function f such that $f(-1) = -1$ $\frac{d^n f}{d\theta^n}(-1) = \lambda_n$. Let I be an interval containing -1 and call I_- , I_+ the subintervals resulting from I by removing -1 . There is a function \tilde{f} that agrees with ν on I_- and with f on I_+ . This is smooth even at $\theta = -1$, because all the left and right derivatives coincide. Now, since the first derivative at $\theta = -1$ is nonzero, it defines a diffeomorphism of a small interval containing I_- , hence the restriction to this small interval can be continued to a diffeomorphism of S^1 (see [CDIT, Lemma 3.9]). Let us call it ν_{I_-} , then $\nu_{I_-} \in \text{Diff}_+(S^1)$ and agrees with ν on I_- . Similarly, we can find $\nu_{I_+} \in \text{Diff}_+(S^1)$ that agrees with ν on I_+ .

Irreducible solitons. Let \mathcal{A} be a conformal net on S^1 on the Hilbert space \mathcal{H} and U its associated projective representation of $\text{Diff}_+(S^1)$. For $\nu \in \text{Diff}_+(S^1, -1)$ and for every $I \in \overline{\mathcal{I}}_{\mathbb{R}}$ we choose $\nu_I \in \text{Diff}_+(S^1)$ that agrees with ν on I (there is such ν_I even if one of the endpoint of I is -1 (half-lines in the \mathbb{R} picture) by the remark above). We denote by σ_ν the family of maps $\sigma_\nu := \{\sigma_\nu^I\}$ where

$$\begin{aligned} \sigma_\nu^I : \mathcal{A}(I) &\longrightarrow \mathcal{B}(\mathcal{H}) \\ x &\longmapsto \sigma_\nu^I(x) := \text{Ad } U(\nu_I)(x) \end{aligned}$$

and $I \in \overline{\mathcal{I}}_{\mathbb{R}}$, $\nu \in \text{Diff}_+(S^1, -1)$.

Proposition 3.5. *Let $\nu \in \text{Diff}_+(S^1, -1)$. Then $\sigma_\nu = \{\sigma_\nu^I\}$ is an irreducible soliton of the conformal net \mathcal{A} with index 1.*

Proof. Normality on each I follows because each map σ_ν^I is given by the adjoint action $\text{Ad } U(\nu_I)$. We show that the family of maps σ_ν is compatible, namely that, if $I \subset \hat{I}$ for $I, \hat{I} \in \overline{\mathcal{I}}_{\mathbb{R}}$, then $\sigma_\nu^{\hat{I}} \upharpoonright_{\mathcal{A}(I)} = \sigma_\nu^I$. By definition, $\nu_I, \nu_{\hat{I}} \in \text{Diff}_+(S^1)$ agree with ν on I and \hat{I} , respectively, hence they agree on I . Then on $\mathcal{A}(I)$ we have

$$\text{Ad } U(\nu_I) = \text{Ad } U(\nu_{\hat{I}}) \circ \text{Ad } U(\nu_{\hat{I}}^{-1} \circ \nu_I) = \text{Ad } U(\nu_{\hat{I}}),$$

because $\nu_{\hat{I}}^{-1} \circ \nu_I$ is a diffeomorphism of the circle localized in I' and in this case $\text{Ad } U(\nu_{\hat{I}}^{-1} \circ \nu_I)$ acts trivially on I by (2.1).

Irreducibility follows because $\bigvee_{I \in \overline{\mathcal{I}}_{\mathbb{R}}} \sigma(\mathcal{A}(I)) = \bigvee_{I \in \overline{\mathcal{I}}_{\mathbb{R}}} \sigma(\mathcal{A}(\nu I)) = \bigvee_{I \in \overline{\mathcal{I}}_{\mathbb{R}}} \mathcal{A}(I)$ and the weak closure of the right-hand side is $\mathcal{B}(\mathcal{H})$ by (CN9). The index of σ_ν is 1 because

$$\sigma_\nu(\mathcal{A}(\mathbb{R}_+)) = \mathcal{A}(\nu \mathbb{R}_+) = \mathcal{A}((\nu \mathbb{R}_+)'')' = \mathcal{A}(\nu \mathbb{R}_-)' = \sigma_\nu(\mathcal{A}(\mathbb{R}_-))'.$$

□

Now we show that if ν has different left and right derivatives, then σ_ν is a proper soliton. Modular theory is used as a tool to show non-triviality of the constructed soliton. Let us introduce the notation for left and right derivatives (they are in \mathbb{R}_+ because ν is orientation-preserving):

$$\partial_\pm \nu(-1) = -i \lim_{\theta \rightarrow 0^\pm} \frac{\nu(-e^{i\theta}) - \nu(-1)}{\theta}.$$

We also denote their ratio by $r(\nu) := \partial_+ \nu(-1)/\partial_- \nu(-1) \in \mathbb{R}_+$. For this purpose, we need the results from Appendix A. Actually, in order to show that there are proper solitons, we can take ν that coincides with dilations with different parameter in a neighborhood of -1 . In this way, ν_π is a product of ψ_t below for some t and an element in $\text{Diff}_+(S^1)$, and one does not have to invoke Proposition A.4.

Theorem 3.6. *Let $\nu \in \text{Diff}_+(S^1, -1)$ and assume that $r(\nu) \neq 1$. Then σ_ν is a proper, irreducible, $\text{Diff}_{+,0}^{1,\text{ps}}(S^1)$ -covariant soliton of \mathcal{A} . Furthermore, let $\nu_1, \nu_2 \in \text{Diff}_+(S^1, -1)$. The solitons σ_{ν_1} and σ_{ν_2} are unitarily equivalent if and only if $r(\nu_1) = r(\nu_2)$.*

Proof. Let us first consider the function

$$\psi_t(e^{i\theta}) := \begin{cases} e^{i\theta} & \text{if } \theta \in [-\pi, 0) \\ \delta(t)(e^{i\theta}) & \text{if } \theta \in [0, \pi) \end{cases}$$

which is smooth except $-1, 1$. It is constructed just by gluing the identity map on $[-\pi, 0)$ and the dilation on $[0, \pi)$ in the circle picture (they correspond to \mathbb{R}_- and \mathbb{R}_+ in the \mathbb{R} picture, respectively). By passing to the \mathbb{R} picture (which does not affect $r(\nu)$), it is immediate that $r(\psi_t) = e^{-t}$.

We may assume that $\nu(1) = 1$, because if $\nu(1) \neq 1$, then there is a smooth diffeomorphism $\underline{\nu}$ such that $\text{supp } \underline{\nu}$ does not contain -1 and $\underline{\nu}(1) = \nu(1)$. We then have $\underline{\nu}^{-1} \circ \nu(1) = 1$. As such $\underline{\nu}^{-1}$ is represented by $U(\underline{\nu}^{-1})$, the questions of properness, irreducibility and covariance of σ_ν are equivalent to that of $\sigma_{\underline{\nu}^{-1} \circ \nu} = \text{Ad } U(\underline{\nu}^{-1}) \circ \sigma_\nu$.

From $\nu \in \text{Diff}_+(S^1, -1)$ with $\nu(1) = 1$, we construct a homeomorphism ν_π of S^1 that is smooth except two points, the points -1 and 1 :

$$\nu_\pi := \nu \circ R_\pi \circ \nu^{-1} \circ R_\pi,$$

where R_π the rotation by π . If $t = -\log r(\nu)$, then $\psi_t \circ \nu_\pi^{-1} \in \text{Diff}_+^{1,\text{ps}}(S^1)$ because the discrepancy of the derivatives at 1 and -1 cancel exactly.

We show that σ_ν is a proper soliton, i.e. it does not extend to a DHR representation. Let us suppose the contrary, namely that it were a restriction of a DHR representation. We denote the extension by σ_ν . Then σ_ν is rotation covariant [DFK04, Theorem 6], namely there is a unitary representation of the universal covering of S^1 , $\theta \rightarrow U^v(R_\theta)$, such that

$$\text{Ad } U^v(R_\theta) \circ \sigma_\nu = \sigma_\nu \circ \text{Ad } U(R_\theta).$$

Furthermore, σ_ν is invertible, because σ_ν has index 1.

Consider the following composition as a DHR representation:

$$\rho := \text{Ad } U(\psi_t \circ \nu_\pi^{-1}) \circ \sigma_\nu \circ \text{Ad } U(R_\pi) \circ \sigma_{\nu^{-1}} \circ \text{Ad } U(R_\pi).$$

It follows that this is implemented by a unitary $U_\rho := U(\psi_t \circ v_\pi^{-1})U^\nu(R_\pi)U(R_\pi)$ since

$$\begin{aligned}\rho &= \text{Ad } U(\psi_t \circ v_\pi^{-1}) \circ \sigma_\nu \circ \text{Ad } U(R_\pi) \circ \sigma_{\nu^{-1}} \circ \text{Ad } U(R_\pi) \\ &= \text{Ad } U(\psi_t \circ v_\pi^{-1}) \circ \text{Ad } U^\nu(R_\pi) \circ \sigma_\nu \circ \sigma_{\nu^{-1}} \circ \text{Ad } U(R_\pi) \\ &= \text{Ad } U(\psi_t \circ v_\pi^{-1}) \circ \text{Ad } U^\nu(R_\pi) \circ \text{Ad } U(R_\pi).\end{aligned}$$

On the other hand, by construction we have $\rho(x) = x$ for $x \in \mathcal{A}(I_{(-\pi,0)})$ where $I_{(-\pi,0)} = I'_{(0,\pi)}$ and $\rho(x) = \text{Ad } U(\delta(t))$ for $x \in \mathcal{A}(I_{(0,\pi)})$. Therefore, the unitary U_ρ must belong to $\mathcal{A}(I_{(0,\pi)})$ by Haag duality (CN6). At the same time, by the Bisognano-Wichmann property (CN8), the dilation $\text{Ad } U(\delta(t))$ is the modular automorphism $\sigma_{\mathcal{A}(I_{(0,\pi)})}^{t/2\pi}$ of $\mathcal{A}(I_{(0,\pi)})$ with respect to the vacuum vector Ω . This is a contradiction because the modular automorphisms for $t \neq 0$ cannot be inner for $\mathcal{A}(I_{(0,\pi)})$ which is a type III₁ factor [Str81, Section 28.11]. Therefore, we conclude that σ_ν does not extend to a DHR representation.

Next, let $\nu_1, \nu_2 \in \text{Diff}_+(S^1, -1)$. It holds that $r(\nu_1) = r(\nu_2)$, if and only if $r(\nu_1^{-1} \circ \nu_2) = 1$, and by Proposition A.4 and the argument of the previous paragraphs, if and only if $\nu_1^{-1} \circ \nu_2$ is implemented by a unitary, or in other words, if and only if σ_{ν_1} and σ_{ν_2} are unitarily equivalent.

Finally, let us prove $\text{Diff}_{+,0}^{1,\text{ps}}(S^1)$ -covariance. We already know from Theorem 3.4 that any soliton is $\text{Diff}_{+,1}^{1,\text{ps}}(S^1)$ -covariant, hence it is enough to show that σ_ν is dilation covariant. This follows because $r(\delta(t) \circ \nu \circ \delta(-t)) = r(\nu)$, hence σ_ν and $\sigma_{\delta(t) \circ \nu \circ \delta(-t)}$ are unitarily equivalent. This unitary implements the dilation $\delta(t)$. \square

Type III solitons. Instead of considering functions $\nu \in \text{Diff}_+(S^1, -1)$, we can do a similar construction using a function ν with the following properties:

- ν is smooth on $S^1 \setminus \{-1\}$ and the left and right derivatives at all orders exist at the point -1 and the first order left and right derivatives are nonzero.
- ν is injective and orientation preserving.
- $\nu(S^1 \setminus \{-1\})$ is a proper interval I_ν of S^1 .

If we take such a ν , σ_ν still yields a soliton of the conformal net \mathcal{A} , since the arguments of Proposition 3.5 remain valid except for irreducibility and index. This type of construction was implicitly presented in [LX04] and [KLX05], namely, by taking the construction of solitons of the net $\mathcal{A} \otimes \mathcal{A}$ in [LX04] and restrict it to $\mathcal{A} \otimes \mathbb{C}1$. In this case, one obtains solitons σ_ν which are of type III (namely $\left(\bigcup_{I \subset \mathbb{R}} \sigma_\nu(\mathcal{A}(I))\right)''$ is a type III factor). For completeness we show that this type of construction also yields a proper soliton. We further prove that such a soliton is locally $\widetilde{\mathcal{D}^s(S^1)}$ -covariant with $s > 3$, namely, there is a unitary representation U^σ of $\widetilde{\mathcal{D}^s(S^1)}$ such that $\text{Ad } U^\sigma(\gamma)(\mathcal{A}(I)) = \mathcal{A}(\gamma I)$ as long as γ is contained in a neighborhood \mathcal{U} of the unit element of $\mathcal{D}^s(S^1)$ such that $\gamma I \subset \mathbb{R}$ for $\gamma \in \mathcal{U}$.

Theorem 3.7. *For ν as above, σ_ν is a proper soliton of type III, locally $\widetilde{\mathcal{D}^s(S^1)}$ -covariant with $s > 3$. For any pair ν_1, ν_2 , two solitons σ_{ν_1} and σ_{ν_2} are unitarily equivalent.*

Proof. We must show that the representation σ_ν does not extend to a representation of the net \mathcal{A} of the circle. Let I_- and I_+ two disjoint intervals that are made by removing

the point -1 from an interval $I \ni -1$. As $\nu(I_-)$ and $\nu(I_+)$ are separated by nonzero distance, by the split property (CN11),

$$\begin{aligned}\sigma_\nu(\mathcal{A}(I_-) \vee \mathcal{A}(I_+)) &= \sigma_\nu^{I_-}(\mathcal{A}(I_-)) \vee \sigma_\nu^{I_+}(\mathcal{A}(I_+)) \\ &= \mathcal{A}(\nu(I_-)) \vee \mathcal{A}(\nu(I_+)) \\ &\simeq \mathcal{A}(\nu(I_-)) \otimes \mathcal{A}(\nu(I_+)) \\ &= \sigma_\nu^{I_-}(\mathcal{A}(I_-)) \otimes \sigma_\nu^{I_+}(\mathcal{A}(I_+)) \\ &\simeq \mathcal{A}(I_-) \otimes \mathcal{A}(I_+),\end{aligned}$$

where \simeq means unitary equivalence mapping the algebra on the left (respectively right) to the algebra on the left (respectively right). Let us assume that σ extended to a DHR representation. Then it would be normal on $\mathcal{A}(I)$, and hence $\sigma_\nu(\mathcal{A}(I_-) \vee \mathcal{A}(I_+)) \simeq \mathcal{A}(I_-) \vee \mathcal{A}(I_+)$. But we know from [Buc74, page 292, Example b)] that $\mathcal{A}(I_-) \vee \mathcal{A}(I_+)$ is not isomorphic to $\mathcal{A}(I_-) \otimes \mathcal{A}(I_+)$, so this is a contradiction, and we conclude that σ_ν does not extend to a DHR representation.

Let us next show the unitary equivalence between σ_{ν_1} and σ_{ν_2} . This follows because $\nu_2 \circ \nu_1^{-1}$ is a diffeomorphism from I_{ν_1} to I_{ν_2} that can be extended to $\nu_{1,2} \in \text{Diff}_+(S^1)$. Now $U(\nu_{1,2})$ intertwines σ_{ν_1} and σ_{ν_2} because by definition, on $\mathcal{A}(I)$ with $I \in \overline{\mathcal{I}}_{\mathbb{R}}$,

$$\sigma_{\nu_2} = \text{Ad } U(\nu_2^I) = \text{Ad } U(\nu_{1,2})U(\nu_1^I) = \text{Ad } U(\nu_{1,2}) \circ \sigma_{\nu_1},$$

where we used the fact that $\nu_{1,2} \circ \nu_1^I = \nu_2^I$ when restricted to I .

With the result of the previous paragraph, in order to show local $\widetilde{\mathcal{D}^s(S^1)}$ -covariance, we can take a specific ν . We take ν as the square root map, namely, $\nu(e^{i\theta}) = e^{i\theta/2}$, where $\theta \in [-\pi, \pi)$. Note that $\mathcal{D}^s(S^1)$ contains a copy of the 2-cover $\mathcal{D}^s(S^1)^{(2)}$ of $\mathcal{D}^s(S^1)$, and the square root map ν locally intertwines the action of $\mathcal{D}^s(S^1)^{(2)}$ on the half-circle $[-\frac{\pi}{2}, \frac{\pi}{2})$ and the action of $\mathcal{D}^s(S^1)$ on S^1 (see [LX04, Section 2], [Wei05, Section 3.1 §3]). Therefore, for $\gamma \in \widetilde{\mathcal{D}^s(S^1)}$ and $I \in \overline{\mathcal{I}}_{\mathbb{R}}$ as in the definition of local covariance and with the quotient map $q^{(2)} : \widetilde{\mathcal{D}^s(S^1)} \rightarrow \mathcal{D}^s(S^1)^{(2)}$, $\text{Ad } U(q^{(2)}(\gamma)) \circ \sigma_\nu = \sigma_\nu \circ \text{Ad } U(\gamma)$. Namely, σ_ν is locally $\widetilde{\mathcal{D}^s(S^1)}$ -covariant. \square

As σ_ν is not irreducible, the implementation of $\widetilde{\mathcal{D}^s(S^1)}$ is not unique. It is also possible to use the n -th root map in such a way that the covariance is implemented by the restriction of U to $\mathcal{D}^s(S^1)^{(n)}$. The representation $\gamma \mapsto U(q^{(2)}(\gamma))$ used here has positive energy by [Wei06, Corollary 3.6] (applied to the vacuum representation).

3.3. Möbius covariance implies DHR. In Theorem 3.7 we saw that there is a proper, locally $\widetilde{\mathcal{D}^s(S^1)}$ -covariant soliton on any conformal net \mathcal{A} . In particular, it is locally Möb-covariant. In contrast, here we show that there is no locally Möb-covariant proper soliton. The key is that $\text{Ad } U^\sigma(R_{2\pi})$ is trivial in this case. This is essentially contained in [CHK+15, Lemma 2.6] and [CKL08, Proposition 19], but we give a direct proof in our present setting.

Proposition 3.8. *Let σ be a locally Möb-covariant soliton of a conformal net \mathcal{A} with a representation U^σ of Möb. Then σ extends to a DHR representation of \mathcal{A} .*

Proof. To extend σ to $\mathcal{A}(I)$ where I contains -1 , we choose a rotation R_θ such that $R_\theta I$ does not contain -1 . We define $\sigma^I := \text{Ad } U^\sigma(R_{-\theta}) \circ \sigma \circ \text{Ad } U(R_\theta)$, and show that this is well-defined. Indeed, assume that there are two such $0 < \theta_1, \theta_2 < 2\pi$. Then, either the rotation from 0 to $\theta_1 - \theta_2$ or the rotation 0 to $\theta_1 - \theta_2 + 2\pi$ brings $R_{\theta_2} I$ to $R_{\theta_1} I$ inside $S^1 \setminus \{-1\}$. As $R_{2\pi}$ is a scalar, without the loss of generality, we may consider the case of $\theta_1 - \theta_2$. By the assumption of locally Möb-covariance,

$$\begin{aligned} & \text{Ad } U^\sigma(R_{-\theta_1}) \circ \sigma \circ \text{Ad } U(R_{\theta_1}) \\ &= \text{Ad } U^\sigma(R_{-\theta_2}) U^\sigma(R_{-\theta_1+\theta_2}) \circ \sigma \circ \text{Ad } U(R_{\theta_1-\theta_2}) U(R_{\theta_2}) \\ &= \text{Ad } U^\sigma(R_{-\theta_2}) \circ \sigma \circ \text{Ad } U(R_{\theta_2}), \end{aligned}$$

hence the definition does not depend on θ under the condition above. The compatibility condition follows from this, because if θ can be used for a larger interval, it works also for a smaller interval. \square

There are also many irreducible locally $\widetilde{\text{Möb}}$ -covariant solitons: consider an inclusion of conformal nets $\mathcal{A} \subset \mathcal{B}$, take a certain DHR representation ρ of \mathcal{A} and its α -induction to \mathcal{B} . For a generic ρ , the α -induction is not a DHR representation but just a soliton, and often such soliton can be decomposed into irreducible ones. For concrete examples, see e.g. [CHK+15, Section 5] and here one considers ρ 's with index 1 (automorphisms), hence their α -induction to \mathcal{B} are already irreducible (this is due to Sebastiano Carpi, we thank him for his comments).

4. Applications to Infinite-Dimensional Groups

Some conformal nets arise from a particular representation (often called the “vacuum representation”) of infinite-dimensional groups. Our construction in Sect. 3.2 gives rise to a new class of representation of certain subgroups of them.

4.1. The Virasoro net and representations of B_0 . The Virasoro net with central charge c is the conformal net induced by the stress–energy tensor $T_{c,0}$ in the vacuum representation $\mathcal{H}(c, 0)$ of the Virasoro algebra Vir :

$$\text{Vir}_c(I) = \{e^{iT_{c,0}(f)} : f \in C^\infty(S^1), \text{ real-valued, } \text{supp } f \subset I\}''.$$

With the lowest weight vector Ω_c and the unitary projective representation of U_c of $\text{Diff}_+(S^1)$, $(\text{Vir}_c, U_c, \Omega_c)$ is a conformal net, see [Car04, Section 2.4].

The solitons in Sect. 3.2 give rise to positive energy representations of B_0 that do not extend to positive-energy representations of $\text{Diff}_+(S^1)$. Let $\nu \in \text{Diff}_+(S^1, -1)$. For any $\gamma \in B_0$, $\nu \circ \gamma \circ \nu^{-1}$ is C^1 , as the discontinuity of the first derivative of ν at the point of infinity gets cancelled. We set

$$\begin{aligned} \alpha_\nu : B_0 &\rightarrow \text{Diff}_+^{1,\text{ps}}(S^1) \\ \gamma &\mapsto \nu \circ \gamma \circ \nu^{-1}. \end{aligned}$$

Clearly α_ν is a homomorphism of B_0 into $\text{Diff}_+^{1,\text{ps}}(S^1)$, and U_c extends to $\text{Diff}_+^{1,\text{ps}}(S^1)$ by Proposition A.4 (although we do not know continuity). We construct a projective unitary representation U_c^ν of B_0 by

$$U_c^\nu(g) := (U_c \circ \alpha_\nu)(g). \quad (4.1)$$

Proposition 4.1. *Let $v \in \text{Diff}_+(S^1, -1)$ and assume that $r(v) \neq 1$. The representation U_c^v defined in (4.1) is not a restriction of any positive-energy representation of $\text{Diff}_+(S^1)$. In addition, $U_c^{v_1} \simeq U_c^{v_2}$ if and only if $r(v_1) = r(v_2)$.*

Proof. The representation U_c^v is irreducible, since $\bigvee_{I \in \overline{\mathbb{I}}_{\mathbb{R}}} \mathcal{A}(I) = \mathcal{B}(\mathcal{H})$ by (CN9) and the left-hand side is generated by $\{U_c(\gamma) : \text{supp } \gamma \in \mathbb{R}\} = \{U_c^v(\gamma) : \text{supp } \gamma \in \mathbb{R}\}$. Suppose that U_c^v were a restriction of a positive-energy representation of $\text{Diff}_+(S^1)$. By irreducibility, it would have to be a (c', h') -representation of Sect. 2.3.2.

Any (c', h') -representation $U_{c', h'}^v$ is a projective representation of $\text{Diff}_+(S^1)$, in particular, the 2π -rotation is a scalar. Furthermore, it holds⁵ that $\text{Ad } U_{c', h'}^v(R_\theta)(U_c^v(\gamma)) = U_c^v(R_\theta \circ \gamma \circ R_{-\theta})$ as long as θ is small enough so that $R_\theta \text{supp } \gamma \subset S^1 \setminus \{-1\}$, because $U_{c', h'}^v$ is an extension of the projective representation U_c^v . Therefore, the restriction of it to Möb makes the soliton σ_v of Vir_c locally Möb-covariant. By Proposition 3.8, σ_v would extend to a DHR representation, but this contradicts with Theorem 3.6. This shows that U_c^v does not extend to any (c', h') -representation.

The claim about unitary equivalence also follows from Theorem 3.6, by passing to σ_{v_1} and σ_{v_2} . \square

In Sect. 5, we show that the representations U_n , where n is a positive integer, extend to $\mathcal{D}^s(S^1)$, $s > 2$ by continuity. As $\mathcal{D}^s(S^1)$ includes $\text{Diff}_+^{1, \text{ps}}(S^1)$ if $2 < s < 5/2$, the representations U_n^v are strongly continuous when $n \in \mathbb{Z}_+$ and the solitons σ_v are continuously covariant with respect to B_0 , see Proposition 5.8.

4.2. Loop group nets and representations of ΛG . We review the loop group nets, following [Kos03, Section III.2]. These results are based on [GW84, Kac90, GF93, TL99].

Let G be a simple, compact, connected and simply connected Lie group. The group of smooth maps from S^1 to G is denoted by LG . With ΛG we denote the group of smooth maps $\mathbb{R} \rightarrow G$ with compact support. This group ΛG is identified through Cayley transform with the subgroup of LG of elements whose support does not contain -1 .

The Lie algebra $L\mathfrak{g}$ consisting of smooth maps from S^1 to \mathfrak{g} with the pointwise operation is called the loop algebra, and it is the Lie algebra of the loop group LG in the infinite-dimensional sense (see [PS86]). A 2-cocycle of $L\mathfrak{g}$ is a bilinear form $\omega : L\mathfrak{g} \times L\mathfrak{g} \rightarrow \mathbb{R}$ such that

$$\omega([x, y], z) + \omega([y, z], x) + \omega([z, x], y) = 0.$$

With such an ω it is possible to construct a central extension of $\widetilde{L\mathfrak{g}}$ of $L\mathfrak{g}$ by a one-dimensional centre: as a vector space, $\widetilde{L\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{R}$ with bracket

$$[(x, a), (y, b)] := ([x, y], \omega(x, y))$$

with $x, y \in L\mathfrak{g}$ and $a, b \in \mathbb{R}$. If \mathfrak{g} is simple, every continuous G -invariant 2-cocycle ω has the form

$$\omega(x, y) = \frac{1}{2\pi} \int_0^{2\pi} \langle x(\theta), y'(\theta) \rangle d\theta$$

⁵ This equation holds including phase: $\text{Ad } U_{c', h'}^v(R_\theta)$ does not give phase by Proposition 2.1, where the phase $\beta(R_\theta, f)$ vanishes since R_θ is a Möbius transformation, and exponentials $\text{Exp}(f)$ generate the whole group $\text{Diff}_+(S^1)$, since $\text{Diff}_+(S^1)$ is algebraically simple [Mat74].

where $\langle \cdot, \cdot \rangle$ is a symmetric invariant form on \mathfrak{g} , which is unique up to a scalar.

The constant functions in $L\mathfrak{g}$ can be identified with \mathfrak{g} . Let us fix a basis $\{j_0^a\}$ in $i\mathfrak{g}$. The complexification $\widetilde{L\mathfrak{g}}_{\mathbb{C}}$ of $\widetilde{L\mathfrak{g}}$ contains functions $e^{in\theta}$ multiplied with an element j_0^a of the basis in $i\mathfrak{g}$. Let us denote these elements by j_n^a . They satisfy the commutation relation

$$[j_m^a, j_n^b] = if_c^{ab} j_{m+n}^c + \omega(j_0^a, j_0^b) m \delta_{m, -n}.$$

One can construct the “vacuum” representations $\pi_{\ell,0}^G$, namely, a representation that contains a vector Ω such that $\pi_{\ell,0}^G(j_n^a)\Omega = 0$ for $n \geq 0$. The central element $(0, a)$ is represented by a scalar ℓ , which is called the level. One can introduce a scalar product on this representation space in such a way that $\pi_{\ell,0}^G(j_n^a)^\dagger = \pi_{\ell,0}^G(j_{-n}^a)$ and $\|\Omega\| = 1$. Such a scalar product is positive definite if and only if the level ℓ is positive integer. Furthermore, for a smooth element f with $f(\theta) \in i\mathfrak{g}$, $\pi_{\ell,0}^G(f)$ is an essentially self-adjoint operator on the domain generated by Ω and $\{\pi_{\ell,0}^G(j_{-n}^a)\}$.

A projective unitary representation V of a group on a Hilbert space \mathcal{H} is a map from the group into $\mathcal{U}(\mathcal{H})$ such that $V(g)V(h) = c(g, h)V(gh)$ for some scalar $c(g, h)$. The vacuum representation of $L\mathfrak{g}$ at level ℓ integrates to a projective unitary representations of LG : for a smooth element f with $f(\theta) \in i\mathfrak{g}$, it holds that $V_{\ell,0}(\text{Exp} f) = e^{i\pi_{\ell,0}^G(f)}$ up to a scalar. Furthermore, there is a projective unitary representation U of $\text{Diff}_+(S^1)$ such that $U(\gamma)V_{\ell,0}(g)U(\gamma)^* = V_{\ell,0}(g \circ \gamma^{-1})$.

A projective unitary representation V of LG on \mathcal{H} is said to have positive energy if there exists a strongly continuous unitary representation U of the rotation group \mathbb{T} on the same Hilbert space with positive generator such that

$$U(R_\theta)V(g)U(R_\theta)^* = V(g_\theta),$$

where $g_\theta(e^{i\varphi}) := g(e^{i(\varphi-\theta)})$. Correspondingly, we say that a projective unitary representation V of ΛG has positive energy if there exists a strongly continuous unitary representation U of the translation group \mathbb{R} with positive generator such that

$$U(\tau_t)V(g)U(\tau_t)^* = V(g_t)$$

where $g_t(s) = f(s - t)$. We have [PS86, Proposition 9.2.6]:

Proposition 4.2. *The restriction to ΛG of a positive energy representation of LG is a positive energy representation of ΛG .*

Let $V_{\ell,0}^G$ be the vacuum representation of level ℓ , which has positive energy with respect to U . With the family of von Neumann algebras

$$\mathcal{A}_{G,\ell}(I) := \left\{ V_{\ell,0}^G(g) : \text{supp } g \subset I \right\}''$$

$(\mathcal{A}_{G,\ell}, U, \Omega)$ is a conformal net. They are called the **loop group nets** with group G at level ℓ .

Proposition 4.3. *There exist irreducible positive energy representations of ΛG that do not extend to positive energy representations of LG .*

Proof. Fix a level ℓ and consider the conformal net $\mathcal{A}_{G,\ell}$. Then we can construct a representation $V_{\ell,0}^{G,v}$ of ΛG by

$$V_{\ell,0}^{G,v} := \sigma_v \circ V_{\ell,0}^G,$$

where σ_v is a proper soliton of the conformal net $\mathcal{A}_{G,\ell}$ with $v \in \text{Diff}_+(S^1, -1)$ as in Sect. 3.2. By Theorem 3.4, $V_{\ell,0}^{G,v}$ it has positive energy.

Suppose that $V_{\ell,0}^{G,v}$ were the restriction V of a positive energy representation of LG . Then $V_{\ell,0}^{G,v}$ would also be irreducible as a representation of LG as in Proposition 4.1. Such V must have positive energy by [GW84, Theorem 7.4], namely, there is a projective unitary representation U^v of $\text{Diff}_+(S^1)$ whose restriction to the rotations has positive generator and $U^v(\gamma)V(g)U^v(\gamma) = V(g \circ \gamma^{-1})$. As the restriction of V to ΛG is $V_{\ell,0}^{G,v}$, U^v makes $\widehat{V_{\ell,0}^{G,v}}$ locally $\text{Diff}_+(S^1)$ -covariant, especially it is locally Möb-covariant (and not just locally $\widehat{\text{Möb}}$ -covariant, namely the 2π -rotation is trivial). Accordingly, the soliton σ_v is also locally Möb-covariant. By Proposition 3.8, it should extend to a DHR representation of the net. This contradicts with Theorem 3.6. Therefore, $V_{\ell,0}^{G,v}$ does not arise from the restriction of any positive energy representation of LG . \square

The existence of such representation has been marked as an open problem in [PS86, P.174, Remark]. The type III soliton from Sect. 3.2 gives another such representation, since it cannot be locally Möb-covariant as we saw in Theorem 3.7 and Proposition 3.8.

5. Sobolev Diffeomorphism Covariance of the U(1)-Current Net

Here we take the U(1)-current net where there is the criterion of Shale–Stinespring to determine whether an automorphism of the algebra can be unitarily implemented. Following the strategy of [Vro13], we show that $\mathcal{D}^s(S^1)$ -diffeomorphisms are implemented with $s > 2$, and this group includes $\text{Diff}_+^{1,\text{ps}}(S^1)$ and $\text{Diff}_+^3(S^1)$ and also the C^1 -piecewise Möbius group [Wei05].

Let \mathcal{K} be a complex Hilbert space with the scalar product $\langle \cdot, \cdot \rangle$. The C^* -algebra generated by the operators $W(f)$, $f \in \mathcal{K}$, satisfying the relations $W(f)W(g) = e^{-i\text{Im} \langle f, g \rangle / 2} W(f+g)$ and $W(0) = \mathbb{1}$, is called the CCR algebra. There is a representation of the CCR algebra on the symmetric Fock space $\Gamma_+(\mathcal{K}) = \bigoplus \text{Sym} \mathcal{K}^{\otimes n}$ with the Fock vacuum Ω and $W(f)\Omega = \sum_n \frac{1}{n!} f^{\otimes n}$. We denote the Weyl operators in this representation with the same $W(f)$. If $f \in \mathcal{K}$ and A is a *real linear*, invertible operator on \mathcal{K} that preserves the symplectic bilinear form $\text{Im} \langle \cdot, \cdot \rangle$, then the map $W(f) \mapsto W(Af)$ is a $*$ -automorphism of the CCR algebra. Such a $*$ -automorphism is unitarily implemented in the Fock representation if and only if $\frac{1}{2}J[A, J]$ is a Hilbert-Schmidt operator, where J is the multiplication by the imaginary unit [Sha62, Theorem 4.1]. We partly use the conventions of [Ott95, Section 5.3]. For any $f \in \mathcal{K}$ the Weyl operators $W(f)$ on $\Gamma_+(\mathcal{K})$ satisfy strong continuity: if $f_n \rightarrow f$ in \mathcal{K} then $\|(W(f_n) - W(f))\xi\| \rightarrow 0$ for every $\xi \in \Gamma_+(\mathcal{K})$.

Let $C^\infty(S^1, \mathbb{R})$ be the space of real-valued smooth function on S^1 . We define a seminorm on it by

$$\|f\| := \sum_{k \in \mathbb{N}} k |\hat{f}_k|^2. \quad (5.1)$$

We introduce a complex structure on $C^\infty(S^1, \mathbb{R})$ by means of the operator J :

$$J \left(\sum_{k \in \mathbb{Z} \setminus \{0\}} f_k e_k \right) := \sum_{k \in \mathbb{N}} (i f_k) e_k + \sum_{k \in \mathbb{N}} (-i f_{-k}) e_{-k},$$

where $e_k(e^{i\theta}) := e^{ik\theta}$. The space $C^\infty(S^1, \mathbb{R})$ quotiented by the null space with respect to the norm $\{f \in C^\infty(S^1, \mathbb{R}) : \|f\| = 0\}$ is equipped with the complex structure J . With J as the imaginary unit, the quotiented space becomes the complex Hilbert space \mathcal{H}_1 . This space admits the irreducible unitary representation U_1 of Möb = PSL(2, \mathbb{R}) with lowest weight 1. The action of Möb on $C^\infty(S^1, \mathbb{R})$

$$U_1(\gamma)(f) := f \circ \gamma^{-1}$$

extends to \mathcal{H}_1 . For a function $f \in C^\infty(S^1, \mathbb{R})$ we denote with $[f]$ its image in \mathcal{H}_1 . The seminorm is induced by the complex scalar product

$$\langle f, g \rangle := \frac{1}{2} \sum_{k \in \mathbb{N}} k (\widehat{\hat{f}}_k \hat{g}_k + \widehat{\hat{f}}_{-k} \widehat{\hat{g}}_{-k}).$$

The Möbius group acts on $\Gamma_+(\mathcal{H}_1)$ via the second quantization, and we denote it by $U(\gamma) := \Gamma_+(U_1(\gamma))$. The adjoint action of $U(\gamma)$ on the Weyl operators is particularly simple:

$$\text{Ad } U(\gamma) W([f]) = W(U_1(\gamma)[f]).$$

The family of von Neumann algebras

$$\mathcal{A}_{U(1)}(I) := \{W([f]) : f \in C^\infty(S^1, \mathbb{R}), \text{ supp}(f) \subset I\}''$$

with the Fock vacuum vector $\Omega \in \Gamma_+(\mathcal{H}_1)$ and the representation U is a Möbius covariant net [GLW98]. The representation U of PSL(2, \mathbb{R}) can be extended to a projective representation U of Diff $_+(S^1)$ in such a way that $\mathcal{A}_{U(1)}$ is actually a conformal net, see [PS86, Theorem 9.3.1]. We show that U can be extended to $\mathcal{D}^s(S^1)$, $s > 2$.

In the following, elements in the universal covering $\widetilde{\mathcal{D}^s(S^1)}$ are considered as maps $\tilde{\gamma}$ from $\mathbb{R} \rightarrow \mathbb{R}$ such that $\tilde{\gamma}(\theta + 2\pi) = \tilde{\gamma}(\theta) + 2\pi$.

Lemma 5.1. *Let $\gamma \in \mathcal{D}^s(S^1)$, $s > 3/2$, the image of $\tilde{\gamma} \in \widetilde{\mathcal{D}^s(S^1)}$ through the covering map and $\lambda_{m,n} := \frac{1}{2\pi} \int_0^{2\pi} e^{-im\theta} e^{in\tilde{\gamma}(\theta)} d\theta$, where m, n are either $m < 0, n > 0$ or $m > 0, n < 0$. Then there exists $C_{s,\gamma} \geq 0$ such that*

$$|\lambda_{m,n}| \leq \frac{C_{s,\gamma}}{(|m| + |n|)^{s-1}}.$$

Proof. As in the proof of [Seg81, Proposition 5.3], consider the path $\tilde{\gamma}_t$ in $\widetilde{\mathcal{D}^s(S^1)}$:

$$[0, 1] \ni t \mapsto \tilde{\gamma}_t := t\tilde{\gamma} + (1-t)\text{id} \in \widetilde{\mathcal{D}^s(S^1)}.$$

This is indeed a path in $\widetilde{\mathcal{D}^s(S^1)}$, because $\tilde{\gamma}'_t(\theta) > 0$.

For $0 \leq t \leq 1$, we have $(\tilde{\gamma}_t^{-1})' \in H^{s-1}(S^1)$ by [IKT13, Theorem B.2(ii), Lemma B.1]. From the definition of the norm $\|f\|_{s-1} = \left(\sum_k (1+k^2)^{s-1} |\hat{f}_k|^2 \right)^{\frac{1}{2}}$ where \hat{f}_k is the k -th Fourier component, it follows that

$$\left| \widehat{(\tilde{\gamma}_t^{-1})'}_{\mp(|m|+|n|)} \right| \leq \frac{\left\| (\tilde{\gamma}_t^{-1})' \right\|_{s-1}}{(|m|+|n|)^{s-1}} \leq \frac{2\pi C_{s,\gamma}}{(|m|+|n|)^{s-1}},$$

where $\sup_t \left\{ \left\| (\tilde{\gamma}_t^{-1})' \right\| \right\} =: 2\pi C_{s,\gamma}$ which is finite, because $t \mapsto \tilde{\gamma}_t$ is continuous in $\mathcal{D}^s(S^1)$ and whose first derivatives are uniformly separated from 0.

By setting $t = \frac{|n|}{|m|+|n|}$, with $+$ corresponding to the case $m < 0, n > 0$ and $-$ corresponding to $m > 0, n < 0$, we have

$$\begin{aligned} \lambda_{m,n} &= \frac{1}{2\pi} \int_0^{2\pi} e^{\pm i(|n|+|m|)\tilde{\gamma}_t(\theta)} d\theta = \frac{1}{2\pi} \int_0^{2\pi} e^{\pm i(|n|+|m|)\varphi} (\tilde{\gamma}_t^{-1})'(\varphi) d\varphi \\ &= \frac{1}{2\pi} \widehat{(\tilde{\gamma}_t^{-1})'}_{\mp(|m|+|n|)}, \end{aligned}$$

therefore, $|\lambda_{m,n}| \leq \frac{C_{s,\gamma}}{(|m|+|n|)^{s-1}}$ as desired. \square

Note that the map $V(\gamma)[f] := [f \circ \gamma^{-1}]$ for $\gamma \in \mathcal{D}^s(S^1)$ is well-defined, because the kernel of $[\cdot]$ is the constant functions and they remain constant after composition by γ^{-1} .

In order to estimate the Hilbert-Schmidt norm of $A_{V(\gamma)} := \frac{1}{2}J[V(\gamma), J]$, note that $A_{V(\gamma)}$ is anti-complex linear [Ott95, Section 5.3], namely, $A_{V(\gamma)}J = -JA_{V(\gamma)}$. Therefore, its Hilbert-Schmidt norm on the complex Hilbert space \mathcal{H}_1 is just the half of its Hilbert-Schmidt norm on \mathcal{H}_1 as a real Hilbert space. If we put the norm defined by (5.1) on $C^\infty(S^1, \mathbb{C})$, we obtain a complex Hilbert space $\mathcal{H}_1^\mathbb{C}$ which is naturally isomorphic to the direct sum of two copies of \mathcal{H}_1 , where the complex structure is given by J above. This space $\mathcal{H}_1^\mathbb{C}$ has the basis $\left\{ \frac{1}{\sqrt{k}} e_k \right\}_{k \in \mathbb{Z}, k \neq 0}$, where $e_m(\theta) = e^{im\theta}$, and the operator $A_{V(\gamma)}$ can be extended diagonally and its Hilbert-Schmidt norm on \mathcal{H}_1 as a real linear operator is the same as its Hilbert-Schmidt norm on $\mathcal{H}_1^\mathbb{C}$.

Proposition 5.2. *Let $\gamma \in \mathcal{D}^s(S^1)$, $s > 2$. Then there is a unitary operator $U(\gamma)$ that implements the action on the CCR algebra corresponding to the map $V(\gamma)$, namely, $\text{Ad}U(\gamma)(W([f])) = W([f \circ \gamma^{-1}])$, for $f \in C^\infty(S^1, \mathbb{R})$.*

Proof. Let $f, g \in C^\infty(S^1, \mathbb{R})$. The (real) symplectic bilinear form $\sigma([f], [g]) := \text{Im} \langle f, g \rangle$ can be written as follows:

$$\sigma([f], [g]) = \frac{1}{4\pi} \int_0^{2\pi} f'(e^{i\theta}) g(e^{i\theta}) d\theta.$$

As $\gamma \in \mathcal{D}^s(S^1)$, $s > 2$, γ is in $\text{Diff}_+^1(S^1)$ and the map $V(\gamma)$ preserves the symplectic form $\sigma(\cdot, \cdot)$.

Following [Vro13, Theorem 24], we only need to show that the Hilbert–Schmidt norm of the operator $[V(\gamma), J]$ is finite. As remarked above, we can compute it on $\mathcal{H}_1^{\mathbb{C}}$ with the basis $\left\{ \frac{1}{\sqrt{k}} e_k \right\}_{k \in \mathbb{Z}, k \neq 0}$. The scalar product $\left\langle \frac{1}{\sqrt{m}} e_m, \frac{1}{2} J[V(\gamma), J] \frac{1}{\sqrt{n}} e_n \right\rangle$ vanishes when $m > 0, n > 0$ or $m < 0, n < 0$. The remaining cases are $m < 0, n > 0$ and $m > 0, n < 0$ and

$$\begin{aligned} \left| \left\langle \frac{1}{\sqrt{m}} e_m, \frac{1}{2} J[V(\gamma), J] \frac{1}{\sqrt{n}} e_n \right\rangle \right|^2 &= \frac{|m|}{|n|} |\langle e_m, V(\gamma) e_n \rangle_{L^2(S^1)}|^2 \\ &= \frac{|m|}{|n|} |\lambda_{m,n}|^2. \end{aligned}$$

With $A_{V(\gamma)} = \frac{1}{2} J[V(\gamma), J]$, by Lemma 5.1 we have

$$\|A_{V(\gamma)}\|_{\text{HS}}^2 = \sum_{m>0, n<0} \frac{|m|}{|n|} |\lambda_{m,n}|^2 \leq \sum_{m>0, n<0} \frac{|m|}{|n|} \frac{C_{s,\gamma}^2}{(|m| + |n|)^{2(s-1)}}.$$

Let $p := |m| + |n|$, then

$$\begin{aligned} \sum_{m>0, n>0} \frac{m}{n(m+n)^{2(s-1)}} &= \sum_{p>1} \frac{1}{p^{2(s-1)}} \sum_{n=1}^{p-1} \frac{p-n}{n} \leq \sum_{p>1} \frac{p-1}{p^{2(s-1)}} \sum_{n=1}^{p-1} \frac{1}{n} \\ &\leq \sum_{p>1} \frac{(p-1)(2 + \log(p))}{p^{2(s-1)}} \end{aligned}$$

which converges if $s > 2$, therefore, $\|A_{V(\gamma)}\|_{\text{HS}}^2 < \infty$. \square

Theorem 5.3. *The map $\alpha : \mathcal{D}^s(S^1) \rightarrow \text{Aut}(\mathcal{B}(\Gamma_+(\mathcal{H}_1)))$ such that $\gamma \mapsto \alpha_\gamma := \text{Ad } U(\gamma)$ is pointwise strongly continuous if $s > 2$.*

Proof. Let $f \in C^\infty(S^1, \mathbb{R})$ and $\{\gamma_n\} \subset \mathcal{D}^s(S^1)$ a sequence converging to γ in $\mathcal{D}^s(S^1)$. Recall that $C^\infty(S^1, \mathbb{R}) \subset H^s(S^1)$ for every s and that if $f \in H^s(S^1)$, $s \geq 1/2$, then $\|f\| \leq \|f\|_s$, where $\|f\| := \sum_{k \in \mathbb{N}} k |\hat{f}_k|^2$. By Lemma 2.3, the map $(f, \gamma) \mapsto f \circ \gamma^{-1}$ is continuous for $s > 3/2$. Using Proposition 5.2 for $s > 2$ and the strong continuity of the Weyl operators, it follows that $\alpha_{\gamma_n}(W([f])) \rightarrow \alpha_\gamma(W([f]))$, $f \in C^\infty(S^1, \mathbb{R})$.

Let \mathcal{W} be the linear span of Weyl operators $W([f])$. By the previous paragraph, we have $\lim_{n \rightarrow \infty} \text{Ad } U(\gamma_n)(x) = \text{Ad } U(\gamma)(x)$ in the strong topology for every $x \in \mathcal{W}$, and \mathcal{W} is dense in $\mathcal{B}(\Gamma_+(\mathcal{H}_1))$ in the strong operator topology. Now let $\{\xi_n\} \subset \Gamma_+(\mathcal{H}_1)$ be a dense sequence. Let $A \in \mathcal{B}(\Gamma_+(\mathcal{H}_1))$. By Kaplanski's density theorem we can choose a sequence $\{A_m\} \subset \mathcal{W}$ such that $A_m \rightarrow A$ strongly. Thus we have for every ξ_n

$$\lim_{m \rightarrow \infty} \text{Ad } U(\gamma)(A_m) \xi_n = \text{Ad } U(\gamma)(A) \xi_n,$$

i.e. $f_n(\gamma) := \text{Ad } U(\gamma)(A) \xi_n$ is the pointwise limit of $f_{n,m}(\gamma) := \text{Ad } U(\gamma)(A_m) \xi_n$. Note that $\mathcal{D}^s(S^1)$ is a Baire space, since it is an open set of a complete metric space [IKT13, Lemma B.2, cf. Corollary 2.1(ii)]. By Baire–Osgood's theorem [Car00, Theorem 11.20][Pro] applied to the maps $f_{n,m}$ and f_n from a Baire space $\mathcal{D}^s(S^1)$ into the Hilbert space $\Gamma_+(\mathcal{H}_1)$, we get that the set

$$D(f_n) := \{\gamma \in \mathcal{D}^s(S^1) : f_n \text{ is not continuous in } \gamma\}$$

is meager. Thus also $\bigcup_n D(f_n)$ is meager. It follows that $\mathcal{D}^s(S^1) \setminus \bigcup_n D(f_n)$ is nonempty and hence there is $\gamma_0 \in \mathcal{D}^s(S^1)$ for which all f_n are continuous. Since $\{\xi_n\}$ is dense,

$$\gamma \mapsto \text{Ad } U(\gamma)(A)\xi =: f_\xi^A(\gamma)$$

is continuous at γ_0 for every $\xi \in \Gamma_+(\mathcal{H}_1)$. Set $\gamma_1 := \gamma_0^{-1}\gamma$, then

$$g_\xi^A(\gamma_1) := \text{Ad } U(\gamma_1)(A)\xi = \text{Ad}[U(\gamma_0)^*U(\gamma)](A)\xi = U(\gamma_0^{-1})f_{U(\gamma_0)\xi}^A(\gamma)$$

converges to $U(\gamma_0)^*f_{U(\gamma_0)\xi}^A(\gamma_0)$ as $\gamma \rightarrow \gamma_0$ for every $A \in \mathcal{B}(\Gamma_+(\mathcal{H}_1))$ and for every $\xi \in \Gamma_+(\mathcal{H}_1)$. In other words, the map $\gamma_1 \mapsto \text{Ad } U(\gamma_1)$ is pointwise continuous in the strong operator topology at the identity $\text{id} \in \mathcal{D}^s(S^1)$.

Since the map

$$\gamma \mapsto \text{Ad } U(\gamma) \in \text{Aut}(\mathcal{B}(\Gamma_+(\mathcal{H}_1)))$$

is a group homomorphism and is continuous at id , it is continuous for every $\gamma \in \mathcal{D}^s(S^1)$. \square

As we saw in Lemma 3.1, $\text{Diff}_+^{1,\text{ps}}(S^1) \subset \mathcal{D}^s(S^1)$ if $s < 5/2$.

Corollary 5.4. *The $U(1)$ -current net $\mathcal{A}_{U(1)}$ is continuously $\mathcal{D}^s(S^1)$ -covariant, $s > 2$, and in particular is $\text{Diff}_+^{1,\text{ps}}(S^1)$ -covariant.*

Proof. The proof is the same as in [CDIT, Proposition 4.1]. \square

Note that this is stronger than the general result, Proposition A.4, as here we have the continuity of $\text{Diff}_+^{1,\text{ps}}(S^1)$ -action as a subgroup of $\mathcal{D}^s(S^1)$.

Corollary 5.5. *The projective unitary representation U of $\text{Diff}_+(S^1)$ on $\Gamma_+(\mathcal{H}_1)$ extends continuously to $\mathcal{D}^s(S^1)$, $s > 2$.*

Proof. The map $\gamma \mapsto \text{Ad } U(\gamma)$ is continuous by Theorem 5.3, and this is equivalent to the continuity of $\gamma \mapsto U(\gamma)$ in $\mathcal{U}(\Gamma_+(\mathcal{H}_1))/\mathbb{T}$. \square

Corollary 5.6. *The Virasoro net Vir_1 with central charge $c = 1$ is continuously $\mathcal{D}^s(S^1)$ -covariant, $s > 2$, and in particular is $\text{Diff}_+^{1,\text{ps}}(S^1)$ -covariant.*

Proof. Let Vir_1 the Virasoro net of central charge $c = 1$, where

$$\text{Vir}_1(I) = \{U(\gamma) : \gamma \in \text{Diff}_+(S^1), \text{ supp } \gamma \subset I\}'',$$

which is a subnet of $\mathcal{A}_{U(1)}$. The subspace $\mathcal{H}_{\text{Vir}_1} := \overline{\bigcup_I \text{Vir}_1(I)\Omega}$ is invariant for $U(\gamma)$, $\gamma \in \mathcal{D}^s(S^1)$ by Corollary 5.5, hence the representation U restricts to $\mathcal{H}_{\text{Vir}_1}$, and the covariance follows. \square

Let $U_{1,0}$ the irreducible positive energy projective unitary representation of $\text{Diff}_+(S^1)$ with central charge 1 and lowest weight 0. The finite tensor product $U_n := \bigotimes_n U_{1,0}$, is a positive energy projective representation of $\text{Diff}_+(S^1)$ which contains $U_{n,0}$ as a subrepresentation. By Corollary 5.6, all the representations $U_{n,0}$ of $\text{Diff}_+(S^1)$ extend to $\mathcal{D}^s(S^1)$, $s > 2$. This is a partial improvement of the results of [CDIT], where all $U_{c,h}$ were extended to $\mathcal{D}^s(S^1)$, $s > 3$.

We now show that for these conformal nets the representations of B_0 constructed in Sect. 4.1 are strongly continuous.

Lemma 5.7. *Let $\dot{\gamma} \in B_0$, $\dot{v} \in \text{Diff}_+(S^1, -1)$ and $2 < s < 5/2$. The homomorphism $\alpha_{\dot{v}} : B_0 \longrightarrow \mathcal{D}^s(S^1)$, $\dot{\gamma} \mapsto \alpha_{\dot{v}}(\dot{\gamma}) := \dot{v} \circ \dot{\gamma} \circ \dot{v}^{-1}$, where B_0 is equipped with the C^∞ -topology, is continuous.*

Proof. Let $\{\gamma_n\} \subset B_0$ be a sequence converging to $\dot{\gamma} \in B_0$ with respect to the C^∞ -topology. We denote with v the lift to $\widetilde{\text{Diff}}_+^0(S^1)$ of \dot{v} and with γ_n and γ the lift to \widetilde{B}_0 of $\dot{\gamma}_n$ and $\dot{\gamma}$, respectively. We use the same strategy of Lemma 3.3. Namely, the convergence $v \circ \gamma_n \circ v^{-1} \rightarrow v \circ \gamma \circ v^{-1}$ in the $L^1(S^1)$ -topology (actually, even in the uniform topology) is straightforward. Then, by

$$\left| \left(v \circ \widehat{\gamma_n \circ v^{-1}} \right)_k \right| \leq \frac{\text{Var} \left((v \circ \gamma_n \circ v^{-1})'' \right)}{k^3}$$

it is sufficient to show that the right-hand side is uniformly bounded in n . The second derivative of $v \circ \gamma_n \circ v^{-1}$ is

$$\begin{aligned} \frac{d^2}{d\theta^2} (v \circ \gamma_n \circ v^{-1}) (\theta) &= v''(\gamma_n(v^{-1}(\theta))) \gamma_n'(v^{-1}(\theta))^2 \frac{1}{v'(v^{-1}(\theta))^2} \\ &\quad + v'(\gamma_n(v^{-1}(\theta))) \gamma_n''(v^{-1}(\theta)) \frac{1}{v'(v^{-1}(\theta))^2} \\ &\quad - v'(\gamma_n(v^{-1}(\theta))) \gamma_n'(v^{-1}(\theta)) \frac{v''(v^{-1}(\theta))}{v'(v^{-1}(\theta))^3}. \end{aligned} \quad (5.2)$$

To evaluate its total variation, we use the following facts: for every pair of functions f_1, f_2 with bounded variation, it holds [Pau15, Theorem 3.7] that

$$\begin{aligned} \text{Var}(f_1 \cdot f_2) &\leq \|f_1\|_\infty \text{Var}(f_2) + \|f_2\|_\infty \text{Var}(f_1) + 3\text{Var}(f_1)\text{Var}(f_2) \\ \text{Var}(f_1 \circ f_2) &\leq L_{f_1} \text{Var}(f_2), \end{aligned}$$

where f_1 is Lipschitz and L_{f_1} is the Lipschitz constant of f_1 . Now, the total variations of the second and the third terms are uniformly bounded in n since $L_{\gamma_n^{(k)}}$ are uniformly bounded in n . As for the first term, we have $\text{Var}(v'' \circ \gamma_n \circ v^{-1}) \leq 2\pi \left\| (v'' \circ \gamma_n \circ v^{-1})' \right\|_{L^\infty(0, 2\pi)} + |v''(2\pi) - v''(0)|$, and this is again uniformly bounded since v'' has a bounded derivative on the open interval $(0, 2\pi)$ and $L_{\gamma_n^{(k)}}$ are uniformly bounded in n . \square

Proposition 5.8. *Let $2 < s < 5/2$ and $\gamma \in \text{Diff}_+(S^1, -1)$. The map $U_\gamma := U \circ \alpha_\gamma$ is a strongly continuous unitary projective representation of B_0 when $U = U_{n,0}$, $n \in \mathbb{Z}_+$, or U is as in Corollary 5.5.*

Let \mathcal{A} be the $\text{U}(1)$ -current net or the Virasoro net Vir_c with $c \in \mathbb{Z}_+$ and $\gamma \in \text{Diff}_+(S^1, -1)$. Every soliton σ_γ of \mathcal{A} as in Sect. 3.2 is continuously B_0 -covariant with respect to the representation U_γ .

Proof. This is clear from Corollary 5.5 and Lemma 5.7. \square

6. Outlook

Let us collect some open problems.

- There appears to be no known soliton that is not dilation-covariant. Is dilation covariance automatic in solitons? This is not obvious, because we cannot implement dilations by cutting the generators as we did for translations.
- Is it possible to classify all solitons for some specific conformal nets? For example, for Virasoro nets any such soliton should give rise to a representation of the group of the diffeomorphisms of \mathbb{R} with compact support. Yet, the lack of any compact subgroup makes it difficult to classify such representations.
- As the action $\gamma_*(f)$ of a diffeomorphism on vector fields involves the derivative of γ , it may decrease the regularity, especially, it may have discontinuous derivative. On the other hand, if γ and $\text{Exp}(f)$ are implementable, so is $\text{Exp}(\gamma_*(f))$ by $U(\gamma)U(\text{Exp}(f))U(\gamma)^*$, hence implementability of $\text{Exp}(g)$ is not directly related with the regularity of g . What is the precise relationship? Is $\mathcal{D}(L_0)$ a core for such $T(g)$?
- Which is the smallest $s > 0$ for which conformal nets are $\mathcal{D}^s(S^1)$ -covariant? Does s depend on the net? For which s do (c, h) -representations of $\text{Diff}_+(S^1)$ extend to $\mathcal{D}^s(S^1)$?

Acknowledgements. We would like to thank Roberto Longo for suggesting the problem. We are grateful to Sebastiano Carpi, André Henriques, Karl-Hermann Neeb, and Stefano Rossi for various interesting discussions. S.D. and Y.T. acknowledge the MIUR Excellence Department Project awarded to the Department of Mathematics, University of Rome Tor Vergata, CUP E83C18000100006.

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A Piecewise Smooth C^1 -Diffeomorphisms

Here we show that any $\gamma \in \text{Diff}_+^{1,\text{ps}}(S^1)$ is implementable in any conformal net. The strategy is due to André Henriques. We thank him for permitting us to include this in the present paper.

An element $\gamma \in \text{Diff}_+^{1,\text{ps}}(S^1)$ has only finitely many nonsmooth points, hence if we show that any γ with one nonsmooth point is implemented, the thesis follows by composing such elements finitely many times. Furthermore, by composing with rotations and dilations, we may assume that the nonsmooth point is -1 and $\gamma(-1) = -1, \gamma'(-1) = 1$.

If $\gamma \in \text{Diff}_+^{1,\text{ps}}(S^1)$, let $\tilde{\gamma}$ be a lift of γ to the universal covering $\text{Diff}_+^1(S^1)$. As we observed in Sect. 3.2, there exists an open interval I of S^1 that contains -1 and $\gamma_{I_-}, \gamma_{I_+} \in \text{Diff}_+(S^1)$ such that γ agrees with γ_{I_-} in I_- and with γ_{I_+} in I_+ , where I_- and I_+ are the connected components of $I \setminus \{-1\}$. Denote the derivative of γ from the right and from the left by

$$\partial_{\pm}\gamma(-1) := \lim_{\theta \rightarrow \pi_{\pm}} \frac{\tilde{\gamma}_{I_{\pm}}(\theta) - \tilde{\gamma}_{I_{\pm}}(\pi)}{\theta - \pi}.$$

For $f \in C^{\infty}(S^1, \mathbb{R})$ and $\gamma \in \text{Diff}_+(S^1)$ we define

$$f^{(k)}(e^{i\theta}) := \frac{d^k}{d\theta^k} f(e^{i\theta})$$

and

$$\gamma^{(k)}(e^{i\theta}) := \frac{d^k}{d\theta^k} \tilde{\gamma}(\theta)$$

where $\tilde{\gamma}$ is the lift of γ in $\widetilde{\text{Diff}_+(S^1)}$ and $\widetilde{\text{Diff}_+(S^1)}$ is identified with the group of maps $\tilde{\gamma} : \mathbb{R} \rightarrow \mathbb{R}$ satisfying $\tilde{\gamma}(\theta + 2\pi) = \tilde{\gamma}(\theta) + 2\pi$.

Recall that $\text{Vect}(S^1)$ is a Lie algebra with the bracket $[f, g] := f'g - g'f$. As in [Tan10], for $0 \leq n \leq \infty$, consider the following Lie subalgebras of $\text{Vect}(S^1)$

$$\begin{aligned} \mathfrak{b}_n &:= \left\{ f \in C^\infty(S^1, \mathbb{R}) : f^{(k)}(-1) = 0, \text{ for } 0 \leq k \leq n \right\}, \\ \mathfrak{b}_\infty &:= \left\{ f \in C^\infty(S^1, \mathbb{R}) : f^{(k)}(-1) = 0, \text{ for all } k \in \mathbb{N} \right\}. \end{aligned}$$

To each algebra corresponds a Lie subgroup of $\text{Diff}_+(S^1)$,

$$\begin{aligned} B_0 &:= \left\{ \gamma \in \text{Diff}_+(S^1) : \gamma(-1) = -1 \right\}, \\ B_1 &:= \left\{ \gamma \in \text{Diff}_+(S^1) : \gamma(-1) = -1, \quad \gamma^{(1)}(-1) = 1 \right\}, \\ B_n &:= \left\{ \gamma \in \text{Diff}_+(S^1) : \gamma(-1) = -1, \quad \gamma^{(1)}(-1) = 1, \gamma^{(k)}(-1) = 0, \text{ for } 2 \leq k \leq n \right\}, \\ B_\infty &:= \left\{ \gamma \in \text{Diff}_+(S^1) : \gamma(-1) = -1, \gamma^{(1)}(-1) = 1, \gamma^{(k)}(-1) = 0, \text{ for all } k \geq 2 \right\}. \end{aligned}$$

By explicit calculations, \mathfrak{b}_n 's are normal Lie subalgebras of \mathfrak{b}_0 . Correspondingly, B_n is a normal subgroup of B_0 for every $n \geq 1$: indeed, if $\gamma \in B_n$ and $\gamma_0 \in B_0$, then $\gamma^{-1}\gamma_0\gamma$ has the same k -th derivatives at -1 as γ_0 for $k = 1, \dots, n$, hence $\gamma^{-1}\gamma_0\gamma\gamma_0^{-1} \in B_n$ and $\gamma_0\gamma\gamma_0^{-1} \in B_n$. From this, it is immediate that \mathfrak{b}_n is a normal Lie subalgebra of \mathfrak{b}_1 and B_n is a normal subgroup of B_1 .

The quotient Lie algebra $\mathfrak{b}_1/\mathfrak{b}_n$ is finite dimensional and every element $[g]$ can be identified with the $(n-1)$ -tuple of the real numbers $(g^{(2)}(-1), \dots, g^{(n)}(-1))$. Furthermore, it follows from straightforward computations that $\mathfrak{b}_1/\mathfrak{b}_n$ is nilpotent. Similarly, the quotient B_1/B_n is a finite-dimensional Lie group and an element $[\gamma] \in B_1/B_n$ can be identified with the $(n-1)$ -tuple of real numbers $(\gamma^{(2)}(-1), \dots, \gamma^{(n)}(-1))$. As we see below, $\mathfrak{b}_1/\mathfrak{b}_n$ is the Lie algebra of B_1/B_n , and the latter is connected and simply connected as it is diffeomorphic to \mathbb{R}^{n-1} , hence the exponential map is surjective [CG90, Theorem 1.2.1], which is the key of the following Lemma A.2.

Lemma A.1. *The Lie algebra of the group B_1/B_n is $\mathfrak{b}_1/\mathfrak{b}_n$. Let Exp_n be the exponential map from $\mathfrak{b}_1/\mathfrak{b}_n$ to B_1/B_n . With the natural quotient maps $[\cdot]$, the following diagram commutes.*

$$\begin{array}{ccc} \mathfrak{b}_1 & \xrightarrow{[\cdot]} & \mathfrak{b}_1/\mathfrak{b}_n \\ \text{Exp} \downarrow & & \text{Exp}_n \downarrow \\ B_1 & \xrightarrow{[\cdot]} & B_1/B_n \end{array}$$

Proof. We first prove that if $[f_1] = [f_2]$ in $\mathfrak{b}_1/\mathfrak{b}_n$, then $[\text{Exp}(f_1)] = [\text{Exp}(f_2)]$ in B_1/B_n . It is enough to show that $\text{Exp}(f_1)^{(k)}(-1) = \text{Exp}(f_2)^{(k)}(-1)$ for $0 \leq k \leq n-1$.

The case $k = 0$ is obvious by definition of Exp . We show this by induction in k . By [CL55, Theorem 7.2] we have that

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{\partial^k}{\partial \theta^k} \text{Exp}(tf_j)(e^{i\theta}) \right) \Big|_{e^{i\theta}=-1} &= \frac{\partial^k}{\partial \theta^k} \left(\frac{\partial}{\partial t} \text{Exp}(tf_j)(e^{i\theta}) \right) \Big|_{e^{i\theta}=-1} \\ &= \frac{\partial^k}{\partial \theta^k} f_j(\text{Exp}(tf_j)(e^{i\theta})) \Big|_{e^{i\theta}=-1} \end{aligned} \quad (6.1)$$

for $j = 1, 2$. By the chain rule, we observe that the last expression can be written in terms of $\frac{\partial^\ell}{\partial \theta^\ell} \text{Exp}(tf_j)(e^{i\theta}) \Big|_{e^{i\theta}=-1}$ with $0 \leq \ell \leq k-1$ and $f_j^{(\ell)}(-1)$ with $0 \leq \ell \leq k$, because $f_j^{(1)}(-1) = 0$ and $\frac{\partial^k}{\partial \theta^k} \text{Exp}(tf_j)(e^{i\theta}) \Big|_{e^{i\theta}=-1}$ does not appear.

Therefore, $\frac{\partial^k}{\partial \theta^k} \text{Exp}(tf_j)(e^{i\theta}) \Big|_{e^{i\theta}=-1}$, $j = 1, 2$ satisfy the same differential equation with respect to t (6.1) with the same initial data, and we conclude that $\text{Exp}(tf_1)^{(k)}(-1) = \text{Exp}(tf_2)^{(k)}(-1)$.

Let $f \in \mathfrak{b}_1$. Then $[\text{Exp}(tf)]$ is a one-parameter group in B_1/B_n and it does not depend on the representative in $[f]$ by the previous paragraph. The Lie bracket $[[f], [g]]$ can be computed from $[\text{Exp}(tf)]$, $[\text{Exp}(sg)]$ and it gives $[[f, g]] = [f'g - fg']$, namely, $\mathfrak{b}_1/\mathfrak{b}_n$ is the Lie algebra of B_1/B_n . \square

Lemma A.2. *Let $\{\lambda_n\}_{n \geq 2}$ be a sequence of real numbers. There exists $g \in C^\infty(S^1, \mathbb{R})$ such that $\text{Exp}(g)^{(n)}(-1) = \lambda_n$ for all $n \geq 2$.*

Proof. Note that $\mathfrak{b}_1/\mathfrak{b}_n \leftarrow \mathfrak{b}_1/\mathfrak{b}_{n+1}$ is a Lie algebra homomorphism, since $\mathfrak{b}_n \supset \mathfrak{b}_{n+1}$. Recall the inverse limit of the sequence of Lie algebras

$$\mathfrak{b}_1/\mathfrak{b}_2 \leftarrow \cdots \leftarrow \mathfrak{b}_1/\mathfrak{b}_n \leftarrow \mathfrak{b}_1/\mathfrak{b}_{n+1} \leftarrow \cdots$$

is by definition the Lie algebra of sequences (g_2, \dots, g_n, \dots) , $g_n \in \mathfrak{b}_1/\mathfrak{b}_n$ such that $g_n/\mathfrak{b}_{n-1} = g_{n-1}$. This is isomorphic to $\mathfrak{b}_1/\mathfrak{b}_\infty$, because any such sequence corresponds to a sequence $(\lambda'_2, \dots, \lambda'_n, \dots)$ where $g_n^{(k)}(-1) = \lambda'_k$ and by Borel's theorem [Hor90, Theorem 1.2.6] there is $g \in \text{Vect}(S^1)$ such that $g(-1) = g'(-1) = 0$ and $g^{(k)}(-1) = \lambda'_k$.

Similarly, the inverse limit of the sequence of groups

$$B_1/B_2 \leftarrow \cdots \leftarrow B_1/B_n \leftarrow B_1/B_{n+1} \leftarrow \cdots$$

is isomorphic to B_1/B_∞ , because to any sequence $(\lambda_2, \dots, \lambda_n, \dots)$ one can associate $\gamma \in \text{Diff}_+(S^1)$ such that $\gamma^{(k)}(-1) = \lambda_k$ by Borel's theorem, as we did in Sect. 3.2.

Since $\mathfrak{b}_1/\mathfrak{b}_n$ is a nilpotent Lie algebra, the exponential map $\text{Exp}_n : \mathfrak{b}_1/\mathfrak{b}_n \rightarrow B_1/B_n$ is surjective [CG90, Theorem 1.2.1]. It follows that the inverse limit Exp_∞ of the maps Exp_n is surjective, because any sequence $(\gamma_2, \dots, \gamma_n, \dots)$ has an inverse image $(\text{Exp}_2^{-1}(\gamma_2), \dots, \text{Exp}_n^{-1}(\gamma_n), \dots)$.

To a given sequence $\{\lambda_n\}_{n \geq 2}$, we take the element $(\lambda_2, \dots, \lambda_n, \dots) \in B_1/B_\infty$. Its inverse image with respect to Exp_∞ is a sequence $(\lambda'_2, \dots, \lambda'_n, \dots) \in \mathfrak{b}_1/\mathfrak{b}_\infty$. By Borel's theorem, there is $g \in \text{Vect}(S^1)$ such that $g(-1) = g'(-1) = 0$, $g^{(n)}(-1) = \lambda'_n$. This g has the desired property $\text{Exp}(g)^{(k)}(-1) = \lambda_k$ by Lemma A.1. \square

The inverse limit of $\mathfrak{b}_1/\mathfrak{b}_2 \leftarrow \cdots \leftarrow \mathfrak{b}_1/\mathfrak{b}_n \leftarrow \cdots$ is isomorphic to the Lie algebra of formal power series $x^2\mathbb{C}[[x]]$, where the Lie bracket is $[f, g] := f'g - g'f$, $f, g \in x^2\mathbb{C}[[x]]$. Similarly, the inverse limit of $B_1/B_2 \leftarrow \cdots \leftarrow B_1/B_n \leftarrow \cdots$ is the group $x + x^2\mathbb{C}[[x]]$ with product given by the composition of formal power series.

Lemma A.3. *Let $\gamma \in \text{Diff}_+^{1,\text{ps}}(S^1)$, smooth on $S^1 \setminus \{-1\}$ and $\gamma'(-1) = 1$. There exist g that is piecewise smooth, C^1 and possibly nonsmooth at $\{-1\}$ and $\underline{\gamma} \in \text{Diff}_+(S^1)$ such that $\gamma = \text{Exp}(g) \circ \underline{\gamma}$.*

Proof. Let us first show that, for two sequences of real numbers $\{\lambda_n^+\}_{n \geq 2}$, $\{\lambda_m^-\}_{m \geq 2}$, there exists $g \in C^1(S^1, \mathbb{R})$, such that

- $\text{Exp}(g)(-1) = -1$, $\text{Exp}(g)^{(1)}(-1) = 1$
- $\text{Exp}(g) \in \text{Diff}_+^{1,\text{ps}}(S^1)$
- $\text{Exp}(g)$ is smooth on $S^1 \setminus \{-1\}$
- $\partial_+^n \text{Exp}(g)(-1) = \lambda_n^+$, $\partial_-^m \text{Exp}(g)(-1) = \lambda_m^-$ for all $n, m \geq 2$.

By applying Lemma A.2 to $\{\lambda_n^+\}_{n \geq 2}$, $\{\lambda_m^-\}_{m \geq 2}$, there exist g_+ , $g_- \in C^\infty(S^1, \mathbb{R})$ such that $\text{Exp}(g_+)^{(n)}(-1) = \{\lambda_n^+\}$ and $\text{Exp}(g_-)^{(n)}(-1) = \{\lambda_n^-\}$, $m, n \geq 2$, $\text{Exp}(g_\pm)^{(n)}(-1) = -1$ and $\text{Exp}(g_\pm)^{(1)}(-1) = 1$.

We may assume that g_\pm has compact support around -1 . By gluing the restrictions of g_+ and g_- to I_+ , I_- respectively, we obtain g that is smooth on $S^1 \setminus \{-1\}$, is in $C^1(S^1, \mathbb{R})$ and $g|_{I_+} = g_+|_{I_+}$, $g|_{I_-} = g_-|_{I_-}$. As the only nonsmooth point of g is -1 , we have $\text{Exp}(g) \in \text{Diff}_+^{1,\text{ps}}(S^1)$ and $\text{Exp}(g)$ is smooth on $S^1 \setminus \{-1\}$.

For $n \geq 2$, we set $\lambda_n^+ := \partial_+^n \gamma(-1)$ and $\lambda_m^- := \partial_-^m \gamma(-1)$. By the observation above, there exists $g \in C^1(S^1, \mathbb{R})$, smooth on $S^1 \setminus \{-1\}$ such that $\text{Exp}(g) \in \text{Diff}_+^{1,\text{ps}}(S^1)$ and $\partial_+^n \text{Exp}(g)(-1) = \lambda_n^+$, $\partial_-^m \text{Exp}(g)(-1) = \lambda_m^-$ for all $n, m \geq 2$, $\text{Exp}(g)$ is smooth on $S^1 \setminus \{-1\}$ and $\text{Exp}(g)(-1) = -1$, $\text{Exp}(g)^{(1)}(-1) = 1$. It follows that $\underline{\gamma} := \gamma \circ \text{Exp}(-g)$ has $\partial_+^k \underline{\gamma}(-1) = \partial_-^k \underline{\gamma}(-1) = 0$ for all $k \geq 2$, therefore, $\underline{\gamma} \in B_\infty \subset \text{Diff}_+(S^1)$. \square

Proposition A.4. *In any conformal net (\mathcal{A}, U, Ω) , U can be extended to $\text{Diff}_+^{1,\text{ps}}(S^1)$ (not necessarily continuously) in such a way that the net is covariant with respect to U .*

Proof. We first show that $\gamma \in \text{Diff}_+^{1,\text{ps}}(S^1)$ is implementable if $\gamma(-1) = -1$ and $\gamma'(-1) = 1$ and γ is smooth elsewhere.

By Lemma A.3, we have $\gamma = \text{Exp}(g) \circ \underline{\gamma}$, where g is piecewise smooth and C^1 , and $\underline{\gamma} \in \text{Diff}_+(S^1)$. The smooth element $\underline{\gamma}$ is already implemented by U , therefore, to obtain the desired extension, it is enough to prove that $\text{Exp}(g)$ is implementable for g that is piecewise smooth and C^1 .

Any such g can be approximated by smooth $g(\theta, \mu) = g * h_\mu(\theta)$ with $0 < \mu \leq 1$, where h_1 is a smooth function with support in $[-1, 1]$ such that $h_1 \geq 0$, $\int h_1 = 1$ and $h_\mu(\theta) = \frac{1}{\mu} h_1\left(\frac{\theta}{\mu}\right)$. We set $g(\theta, 0) = g(\theta)$. Then it is clear that g is a continuous function of (θ, μ) and uniformly Lipschitz in θ , since $\partial_\theta g(\theta, \mu) = g' * h_\mu(\theta)$ and $\partial_\theta g(\theta, 0) = g'(\theta)$. By [CL55, Chapter1, Theorem 7.4], $\text{Exp}(tg_\mu)(\theta)$ is continuous in μ at each t, θ . Now, as g_μ is smooth for $\mu > 0$, we have $\text{Ad } U(\text{Exp}(tg_\mu))(\mathcal{A}(I)) = \mathcal{A}(\text{Exp}(tg_\mu)I)$. Note that $U(\text{Exp}(tg_\mu)) = e^{iT(g_\mu)}$ up to a scalar, where T is the stress-energy tensor for U , and since $g_\mu \rightarrow g$ in the $\mathcal{S}_2(S^1, \mathbb{R})$ -topology [CW05, Lemma 4.6],

$e^{itT(g_\mu)} \rightarrow e^{itT(g)}$ in the strong operator topology [CW05, Proposition 4.5]. Therefore, it holds that $\text{Ad } e^{itT(g)}(x) = \lim_{\mu \rightarrow 0} \text{Ad } U(\text{Exp}(tg_\mu))(x)$ for any $x \in \mathcal{A}(I)$, and by the continuity above, $\text{Ad } e^{itT(g)}(x) \in \mathcal{A}(\text{Exp}(tg)I)$. If we set $U(\text{Exp}(tg)) = e^{itT(g)}$, this acts covariantly on the net \mathcal{A} .

It remains to show that U gives a well-defined projective representation of $\text{Diff}_+^{1,\text{ps}}(S^1)$. Let us first define U and show the well-definedness.

- First consider γ that has only one nonsmooth point at -1 and $\gamma(-1) = -1$, $\gamma'(-1) = 1$. We take the decomposition $\gamma = \text{Exp}(g) \circ \underline{\gamma}$ and define $U(\gamma) = U(\text{Exp}(g))U(\underline{\gamma})$. This is well-defined as a projective representation. Indeed, its adjoint action on $\mathcal{A}(I)$, $-1 \notin \bar{I}$ is determined by U as the representation of $\text{Diff}_+(S^1)$, and such $\mathcal{A}(I)$'s generate $\mathcal{B}(\mathcal{H})$, therefore, if we take another decomposition $U(\gamma) = U(\text{Exp}(g_1))U(\gamma_1)$, the difference must be a scalar.
- Second, if γ has only one nonsmooth point, then $\gamma = \gamma_L \circ \gamma_0 \circ \gamma_R$ with γ_0 such that $\gamma_0(-1) = -1$, $\gamma_0'(-1) = 1$ and smooth elements γ_L, γ_R . By the well-definedness above and the fact that U is already defined on smooth elements, $U(\gamma) = U(\gamma_L)U(\gamma_0)U(\gamma_R)$ is also well-defined.
- If γ has finitely many nonsmooth points, we decompose it into $\gamma = \gamma_0 \cdot \check{\gamma}$, where γ_0 fixes all these nonsmooth points, has derivative 1 and $\text{supp } \gamma_0$ is a disjoint union of intervals around these nonsmooth points. As the nonsmooth part has disjoint unions, if we take the product of U defined above on each component, this does not depend on the order of the product. If we consider two such decompositions, the nonsmooth parts cancel each other up to a smooth element which is already defined, hence U is well-defined.

That U is a projective representation is shown as follows.

- For two elements γ_1, γ_2 such that $\gamma_j(-1) = -1$, $\text{Ad } U(\gamma_1\gamma_2)$ and $\text{Ad } U(\gamma_1)U(\gamma_2)$ implement the same action of $\mathcal{A}(I)$ such that $-1 \notin \bar{I}$ as before, hence the difference between $U(\gamma_1\gamma_2)$ and $U(\gamma_1)U(\gamma_2)$ must be a scalar.
- By rotation, the homomorphism property $U(\gamma_1\gamma_2) = U(\gamma_1)U(\gamma_2)$, up to a scalar, follows also when γ_1 and γ_2 has only one and same nonsmooth point.
- For two elements γ_1, γ_2 with finitely many nonsmooth points, take the decompositions as above: $\gamma_j = \gamma_{j,0} \cdot \check{\gamma}_j$. We may assume that the components of $\check{\gamma}_1\gamma_{2,0}\check{\gamma}_1^{-1}$ is either disjoint from the components of $\gamma_{1,0}$, or have a common nonsmooth point. If they are disjoint, their representation by U commute. If they have a common nonsmooth point, we can merge them to a single element and we have shown the homomorphism property above. In this way, we have the decomposition $\gamma_1\gamma_2 = (\gamma_{1,0}\check{\gamma}_1\gamma_{2,0}\check{\gamma}_1^{-1}) \cdot (\check{\gamma}_1\check{\gamma}_2)$, where $\gamma_{1,0}\check{\gamma}_1\gamma_{2,0}\check{\gamma}_1^{-1}$ is supported around the nonsmooth points and $\check{\gamma}_1\check{\gamma}_2$ is smooth and we have $U(\gamma_1)U(\gamma_2) = U(\gamma_{1,0})U(\check{\gamma}_1)U(\gamma_{2,0})U(\check{\gamma}_2) = U(\gamma_{1,0}\check{\gamma}_1\gamma_{2,0}\check{\gamma}_1^{-1})U(\check{\gamma}_1\check{\gamma}_2) = U(\gamma_1\gamma_2)$ up to a scalar.

We have seen the covariance of the net with respect to γ_0 such that $\gamma_0(-1) = -1$ and $\gamma_0'(-1) = 1$ and smooth elsewhere. Any element $\gamma \in \text{Diff}_+^{1,\text{ps}}(S^1)$ can be decomposed as a product of such elements and smooth elements, and for each of them we have shown the covariance, hence the covariance holds also for γ . \square

References

- [Bau95] Hellmut, B.: Operator Algebraic Methods in Quantum Field Theory. Akademie Verlag, Berlin (1995) <https://books.google.com/books?id=rq3vAAAAMAAJ>
- [BCL98] Bertozzini, P., Conti, R., Longo, R.: Covariant sectors with infinite dimension and positivity of the energy. Commun. Math. Phys. **193**(2), 471–492 (1998). [arXiv:hep-th/9704007](https://arxiv.org/abs/hep-th/9704007)
- [BE98] Böckenhauer, J., Evans, D.E.: Modular invariants, graphs and α -induction for nets of subfactors. I. Commun. Math. Phys. **197**(2), 361–386 (1998). [arXiv:hep-th/9801171](https://arxiv.org/abs/hep-th/9801171)
- [BE99a] Böckenhauer, J., Evans, D.E.: Modular invariants, graphs and α -induction for nets of subfactors. II. Commun. Math. Phys. **200**(1), 57–103 (1999). [arXiv:hep-th/9805023](https://arxiv.org/abs/hep-th/9805023)
- [BE99b] Böckenhauer, J., Evans, D.E.: Modular invariants, graphs and α -induction for nets of subfactors. III. Commun. Math. Phys. **205**(1), 183–228 (1999). [arXiv:hep-th/9812110](https://arxiv.org/abs/hep-th/9812110)
- [Buc74] Buchholz, D.: Product states for local algebras. Commun. Math. Phys. **36**, 287–304 (1974) <http://projecteuclid.org/euclid.cmp/1103859773>
- [Car98] Carpi, S.: Absence of subsystems for the Haag–Kastler net generated by the energy-momentum tensor in two-dimensional conformal field theory. Lett. Math. Phys. **45**(3), 259–267 (1998) https://drive.google.com/file/d/0B0qlHPab_Xp5c044QTJzbHAWsVE/view
- [Car00] Carothers, N.L.: Real Analysis. Cambridge University Press (2000) <https://books.google.com/books?id=4VFDVv1NFIAC>
- [Car04] Carpi, S.: On the representation theory of Virasoro nets. Commun. Math. Phys. **244**(2), 261–284 (2004). [arXiv:math/0306425](https://arxiv.org/abs/math/0306425)
- [CDIT] Carpi, S., Del Vecchio, S., Iovieno, S., Tanimoto, Y.: Positive energy representations of Sobolev diffeomorphism groups of the circle. [arXiv:1808.02384](https://arxiv.org/abs/1808.02384)
- [CG90] Corwin L.J., Greenleaf, F.P.: Representations of Nilpotent Lie Groups and Their Applications. Part I: Basic Theory and Examples. Cambridge University Press (1990) <https://books.google.com/books?id=H2e1tVW8WrMC>
- [CHK+15] Carpi, S., Hillier, R., Kawahigashi, Y., Longo, R., Xu, F.: $N = 2$ superconformal nets. Commun. Math. Phys. **336**(3), 1285–1328 (2015). [arXiv:1207.2398](https://arxiv.org/abs/1207.2398)
- [CKL08] Carpi, S., Kawahigashi, Y., Longo, R.: Structure and classification of superconformal nets. Ann. Henri Poincaré, **9**(6), 1069–1121 (2008). [arXiv:0705.3609](https://arxiv.org/abs/0705.3609)
- [CL55] Coddington, E.A., Levinson, N.: Theory of Ordinary Differential Equations. McGraw-Hill Book Company, Inc., New York (1955) <https://books.google.com/books?id=2PJQAAAAMAAJ>
- [CW05] Carpi, S., Weiner, M.: On the uniqueness of diffeomorphism symmetry in conformal field theory. Commun. Math. Phys., **258**(1), 203–221 (2005). [arXiv:math/0407190](https://arxiv.org/abs/math/0407190)
- [DFK04] D’Antoni, C., Fredenhagen, K., Köster, S.: Implementation of conformal covariance by diffeomorphism symmetry. Lett. Math. Phys. **67**(3), 239–247 (2004). [arXiv:math-ph/0312017](https://arxiv.org/abs/math-ph/0312017)
- [DMS97] Di Francesco, P., Mathieu, P., Sénéchal, D.: Conformal Field Theory. Springer, New York (1997). <https://books.google.com/books?id=keUrdME5rhIC>
- [FH05] Fewster, C.J., Hollands, S.: Quantum energy inequalities in two-dimensional conformal field theory. Rev. Math. Phys. **17**(5), 577–612 (2005). [arXiv:math-ph/0412028](https://arxiv.org/abs/math-ph/0412028)
- [FJ96] Fredenhagen, K., Jörß, M.: Conformal Haag–Kastler nets, pointlike localized fields and the existence of operator product expansions. Commun. Math. Phys. **176**(3), 541–554 (1996) <https://projecteuclid.org/euclid.cmp/1104286114>
- [Fre93] Fredenhagen, K.: Superselection sectors in low-dimensional quantum field theory. J. Geom. Phys. **11**(1–4), 337–348 (1993). [https://doi.org/10.1016/0393-0440\(93\)90062-J](https://doi.org/10.1016/0393-0440(93)90062-J)
- [Fro76] Fröhlich, J.: New super-selection sectors (“soliton-states”) in two dimensional Bose quantum field models. Commun. Math. Phys. **47**(3), 269–310 (1976) <http://projecteuclid.org/euclid.cmp/1103899761>
- [GF93] Gabbiani, F., Fröhlich, J.: Operator algebras and conformal field theory. Commun. Math. Phys. **155**(3), 569–640 (1993). <https://projecteuclid.org/euclid.cmp/1104253398>
- [GLW98] Guido, D., Longo, R., Wiesbrock, H.-W.: Extensions of conformal nets and superselection structures. Commun. Math. Phys. **192**(1), 217–244 (1998). [arXiv:hep-th/9703129](https://arxiv.org/abs/hep-th/9703129)
- [GW84] Goodman, R., Wallach, N.R.: Structure and unitary cocycle representations of loop groups and the group of diffeomorphisms of the circle. J. Reine Angew. Math. **347**, 69–133 (1984) <https://eudml.org/doc/152598>
- [Hen17a] Henriques, A.: Bicommunt categories from conformal nets (2017) [arXiv:1701.02052](https://arxiv.org/abs/1701.02052)
- [Hen17b] Henriques, A.: Loop groups and diffeomorphism groups of the circle as colimits (2017) [arXiv:1706.08471](https://arxiv.org/abs/1706.08471)
- [Hor90] Hörmander, L.: The Analysis of Linear Partial Differential Operators. I, 2nd edn. Springer, Berlin (1990). <https://books.google.com/books?id=aaLrCAAAQBAJ>
- [IKT13] Inci, H., Kappeler, T., Topalov, P.: On the regularity of the composition of diffeomorphisms. Mem. Am. Math. Soc. **226**(1062), vi+60 (2013). [arXiv:1107.0488](https://arxiv.org/abs/1107.0488)

- [Kac90] Kac, V.G.: Infinite-Dimensional Lie Algebras, 3rd edn. Cambridge University Press, Cambridge (1990) <https://books.google.com/books?id=kuEjSb9teJwC>
- [Kat04] Katznelson, Y.: An Introduction to Harmonic Analysis. Cambridge University Press, Cambridge (2004) https://books.google.com/books?id=gkpUE_m5vvsC
- [Kos02] Köster, S.: Conformal transformations as observables. *Lett. Math. Phys.* **61**(3), 187–198 (2002). [arXiv:math-ph/0201016](https://arxiv.org/abs/math-ph/0201016)
- [Kos03] Köster, S.: Structure of coset models. Ph.D. Thesis, Göttingen University (2003) [arXiv:math-ph/0308031](https://arxiv.org/abs/math-ph/0308031)
- [KR87] Kac, V.G., Raina, A.K.: Bombay lectures on highest weight representations of infinite-dimensional Lie algebras. World Scientific Publishing Co. Inc., Teaneck, NJ (1987). <https://books.google.com/books?id=0P23OB84eqUC>
- [KLX05] Kac, V., Longo, R., Xu, F.: Solitons in Affine and Permutation Orbifolds. *Commun. Math. Phys.* **253**, 723–764 (2005). [arXiv:math/0312512](https://arxiv.org/abs/math/0312512)
- [LR95] Longo, R., Rehren, K.-H.: Nets of subfactors. *Rev. Math. Phys.* **7**(4), 567–597 (1995). Workshop on Algebraic Quantum Field Theory and Jones Theory (Berlin, 1994) [arXiv:hep-th/9411077](https://arxiv.org/abs/hep-th/9411077)
- [LX04] Longo, R., Xu, F.: Topological sectors and a dichotomy in conformal field theory. *Commun. Math. Phys.* **251**(2), 321–364 (2004). [arXiv:math/0309366](https://arxiv.org/abs/math/0309366)
- [Mat74] Matherm, J.N.: Commutators of diffeomorphisms. *Comment. Math. Helv.*, **49**, 512–528 (1974). <https://doi.org/10.1007/BF02566746>
- [Mil84] Milnor, J.: Remarks on infinite-dimensional Lie groups. In: *Relativity, groups and topology, II* (Les Houches, 1983), pages 1007–1057. North-Holland, Amsterdam, (1984) <https://books.google.com/books?id=QK-HXwAACAAJ>
- [MTW18] Morinelli, V., Tanimoto, Y., Weiner, M.: Conformal covariance and the split property. *Commun. Math. Phys.* **357**(1), 379–406 (2018). [arXiv:1609.02196](https://arxiv.org/abs/1609.02196)
- [Mug99] Müger, M.: On soliton automorphisms in massive and conformal theories. *Rev. Math. Phys.* **11**(3), 337–359 (1999). [arXiv:hep-th/9803057](https://arxiv.org/abs/hep-th/9803057)
- [Ott95] Ottesen, J.T.: Infinite-Dimensional Groups and Algebras in Quantum Physics. Springer, Berlin (1995) <https://books.google.com/books?id=7Cn6CAAQBAJ>
- [Pau15] Pausinger, F.: A Koksma–Hlawka inequality for general discrepancy systems. *J. Complex* **31**(6), 773–797 (2015) <https://www.sciencedirect.com/science/article/pii/S0885064X15000606?via=ihl3vAAAAAAAJ>
- [Pro] ProofWiki. Baire–Osgood theorem. https://proofwiki.org/wiki/Baire-Osgood_Theorem
- [PS86] Pressley A., Segal, G.: Loop Groups. The Clarendon Press Oxford University Press, New York, (1986) <https://books.google.com/books?id=MbFBXyuxLKgC>
- [Reh98] Rehren, K.-H.: Spin statistics and CPT for solitons. *Lett. Math. Phys.* **46**(2), 95–110 (1998) <https://www.researchgate.net/publication/226210205>
- [Rob74] Roberts, J.E.: Some applications of dilatation invariance to structural questions in the theory of local observables. *Commun. Math. Phys.* **37**, 273–286 (1974). <https://projecteuclid.org/euclid.cmp/1103859929>
- [RS75] Reed, M., Simon, B.: Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-Adjointness. Academic Press, New York (1975). <https://books.google.com/books?id=Kz7s7bgVe8gC>
- [Sch96] Schlingemann, D.: On the existence of kink (soliton) states. *Rev. Math. Phys.* **8**(8), 1187–1203 (1996). [arXiv:hep-th/9512100](https://arxiv.org/abs/hep-th/9512100)
- [Sch98] Schlingemann, D.: Construction of kink sectors for two-dimensional quantum field theory models—an algebraic approach. *Rev. Math. Phys.* **10**(6), 851–891 (1998). [arXiv:math-ph/9902028](https://arxiv.org/abs/math-ph/9902028)
- [Seg81] Segal, G.: Unitary representations of some infinite-dimensional groups. *Commun. Math. Phys.* **80**(3), 301–342 (1981). <https://projecteuclid.org/euclid.cmp/1103919978>
- [Sha62] Shale, D.: Linear symmetries of free boson fields. *Trans. Am. Math. Soc.* **103**, 149–167 (1962). <https://doi.org/10.2307/1993745>
- [Str81] Strătilă S.: Modular theory in operator algebras. Editura Academiei Republicii Socialiste România, Bucharest. Abacus Press, Tunbridge Wells (1981). <https://books.google.com/books?id=li3vAAAAAAAJ>
- [Tan10] Tanimoto, Y.: Representation theory of the stabilizer subgroup of the point at infinity in $\text{Diff}(S^1)$. *Int. J. Math.* **21**(10), 1297–1335 (2010). [arXiv:0905.0875](https://arxiv.org/abs/0905.0875)
- [Tan11] Tanimoto, Y.: Ground state representations of loop algebras. *Ann. Henri Poincaré*, **12**(4), 805–827 (2011). [arXiv:1005.0270](https://arxiv.org/abs/1005.0270)
- [Tan18] Tanimoto, Y.: Ground state representations of some non-rational conformal nets. *Symmetry*, **10**(9), (2018). [arXiv:1807.11723](https://arxiv.org/abs/1807.11723)
- [TL99] Toledano Laredo, V.: Integrating unitary representations of infinite-dimensional Lie groups. *J. Funct. Anal.* **161**(2), 478–508 (1999). [arXiv:math/0106195](https://arxiv.org/abs/math/0106195)
- [Vro13] Vromen, L.J.D.: Circle diffeomorphisms acting on fermionic and bosonic fock space. Master thesis, Utrecht University (2013). <https://dspace.library.uu.nl/handle/1874/282847>

- [Wei05] Weiner, M.: Conformal covariance and related properties of chiral qft. (2005). Ph.D. thesis, Università di Roma “Tor Vergata”. [arXiv:math/0703336](https://arxiv.org/abs/math/0703336)
- [Wei06] Weiner, M.: Conformal covariance and positivity of energy in charged sectors. Commun. Math. Phys. **265**(2), 493–506 (2006). [arXiv:math-ph/0507066](https://arxiv.org/abs/math-ph/0507066)

Communicated by Y. Kawahigashi