KMS states with respect to conformal Hamiltonian

Yoh Tanimoto

(joint with Roberto Longo)
University of Rome "Tor Vergata", JSPS fellow for research abroad

August 4th 2016, MSJ-SI, Sendai

Introduction

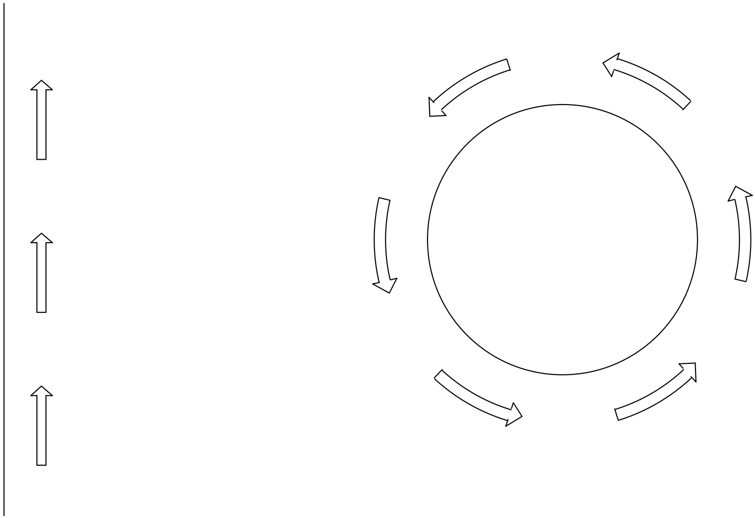
2d CFT

- Important observables decompose into chiral components: $\phi(x, t) = \phi_+(x + t) + \phi_-(x - t)$.
- $\phi_\pm$ are first defined on $\mathbb{R}$, and by conformal invariance of the vacuum, extend to S^1.
- Their algebras are very restricted: stress-energy tensor (Virasoro algebra), currents (Heisenberg algebra, Kac-Moody algebras).

Thermal states. Which Hamiltonian?

- Time translations: the natural Hamiltonian as $1 + 1$-dim QFT (c.f. Camassa-Longo-T.-Weiner '12,'13): uniqueness of KMS states on completely rational CFTs, classification for Virasoro algebra with $c = 1$ and the $\mathrm{U}(1)$-current, different states on other examples.
- **Conformal Hamiltonian** L_0: the generator of rotations. **They are in many cases Gibbs states**. Motivations from 3d quantum gravity.

$\mathbb{R}$ and S^1 pictures



Connected by the Cayley transform $t = i\frac{1+z}{1-z}$.

Thermal states

Gibbs states on a finite system

- $M_n(\mathbb{C})$: $(n \times n)$-matrix algebra, H: self-adjoint, $\sigma_t = \mathrm{Ad}\, e^{itH}$.
- A state (positive normalized linear functional) on $M_n(\mathbb{C})$ given by

$$\varphi(x) = \frac{\mathrm{Tr}(e^{-\beta H}x)}{\mathrm{Tr}(e^{-\beta H})}$$

has the **KMS condition**: $f(t) = \varphi(\sigma_t(x)y), f(t+i\beta) = \varphi(y\sigma_t(x))$.

KMS states on a C^*-dynamical system

- The KMS condition can be considered also for infinite systems: $\mathfrak{A}$: C^*-algebra, σ_t: one-parameter automorphisms group.
- For a CFT $\mathcal{A}$, there is the universal C^*-algebra $C^*(\mathcal{A})$.
- **Main result**: KMS states on $C^*(\mathcal{A})$ with respect to rotations are the convex combination of Gibbs states for important conformal nets $\mathcal{A}$.

Chiral components of 2d CFT

Conformal field theory in $d = 2$ Minkowski space is covariant with respect to the group $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$.
Important observables live on the lightrays:

- The stress-energy tensor
  $T^{00} + T^{01} = 2T_-(t - x), \quad T^{00} - T^{01} = 2T_-(t + x),$
- Conserved current $\partial_\mu j^\mu = 0$,
  $j^0 - j^1 = 2j_+(t + x), \qquad j^0 + j^1 = 2j_-(t - x),$

and mutually commute: $[X_+, Y_-] = 0$.

One can consider a QFT on a lightray $X(x_+)$, $x_+ := x + t$ and its local algebras $\mathcal{A}(I) = \{e^{iX_+(f)} : \mathrm{supp}\, f \subset I\}''$, $I \subset S^1$. The collection of local algebras $\{\mathcal{A}(I)\}$ is called a **conformal net**.

Chiral conformal net

Definition

A **conformal net** on S^1 is $(\mathcal{A}, U, \Omega)$, where $\mathcal{A}$ is a map from the set of intervals in S^1 into the set of von Neumann algebras on $\mathcal{H}$ which satisfies

- Isotony: $I \subset J \Rightarrow \mathcal{A}(I) \subset \mathcal{A}(J)$.
- Locality: $I \cap J \Rightarrow [\mathcal{A}(I), \mathcal{A}(J)] = \{0\}$.
- Diffeomorphism covariance: U is a projective unitary representation of $\mathrm{Diff}(S^1)$ such that $\mathrm{Ad}\, U(g)(\mathcal{A}(I)) = \mathcal{A}(gI)$ and if $\mathrm{supp}\, g \cap I = \emptyset$, then $\mathrm{Ad}\, U(g)$ acts trivially on $\mathcal{A}(I)$.
- Positive energy: the restriction of U to rotations has the positive generator L_0.
- Vacuum: there is a unique (up to a scalar) unit vector Ω such that $U(g)\Omega = \Omega$ for $g \in \mathrm{PSL}(2, \mathbb{R})$ and cyclic for $\mathcal{A}(I)$.

Many examples: $U(1)$-current (free massless boson), Free massless fermion, Virasoro nets (stress energy tensor), Loop group nets.

The universal C^*-algebra

$\mathcal{A}$: a conformal net on S^1. A **representation** of the net $\mathcal{A}$ is a family $\{\rho_I\}$ of representations of the algebras $\{\mathcal{A}(I)\}$ such that $\rho_{I_2}|_{\mathcal{A}(I_1)} = \rho_{I_1}$ for $I_1 \subset I_2$. The equivalence classes of representations of $\mathcal{A}$ are called **sectors**.

$\bigvee_{I \subset S^1} \mathcal{A}(I) = \mathcal{B}(\mathcal{H})$ is not interesting, and contains only the information on the vacuum sector.

One can consider an abstract C^*-algebra which contains the information of all charged sectors: for $x \in A(I)$, $\rho_{\mathrm{univ}}(x) := \bigoplus_\mu \rho_\mu(x)$, where the direct sum runs all cyclic (locally normal) representations generated by the GNS representations on the free $*$-algebra generated by $\mathcal{A}(I)$'s.
We define the **universal C^*-algebra** of $\mathcal{A}$ (Fredenhagen, Guido-Longo) by

$$C^*(\mathcal{A}) := \overline{\{\rho_{\mathrm{univ}}(x) : x \in \mathcal{A}(I) \text{ for some } I \subset S^1\}}^{\|\cdot\|}.$$

There is a natural action of rotations σ_t.
Any representation of the net $\mathcal{A}$ lifts to $C^*(\mathcal{A})$.
We study **locally normal** states and representations.

Example: the Virasoro algebra

The stress-energy tensor satisfies the commutation relations in terms of the Fourier components $-2\pi T(z) = \sum L_n z^{-n-2}$ (in the S^1-picture),

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}.$$

- A unitary irreducible representation is charaterizedd by c and h, where $L_0\Omega = h\Omega$, $L_n\Omega = 0$ for $n \geq 1$. Integrates to $\mathrm{Diff}(S^1)$ (Goodman-Wallach).
- Possible values are $c = 1 - \frac{6}{m(m+1)}, m = 2, 3, \cdots, h = \frac{((m+1)r-ms)^2-1}{4m(m+1)}, r = 1, \cdots, m-1, s = 1, \cdots, r$ and $c \geq 1, h \geq 0$ (Friedan-Qiu-Shenker, Goddard-Kent-Olive).
- **For a fixed** $c, h = 0$, $\mathrm{Vir}_c(I) = \overline{\{e^{iT_{c,0}(f)} : \mathrm{supp}\, f \subset I\}}^{\mathrm{vN}}$ is a conformal net. Any net includes Vir_c for some $c > 0$.
- **For a fixed** c, any value h corresponds to a sector of Vir_c (Buchholz & Schulz-Mirbach, Carpi, Weiner).
- We need to study **non-irreducible representations** of $C^*(\mathrm{Vir}_c)$.

GNS representation and the extremal decomposition

- For a state φ on a C^*-algebra $C^*(\mathcal{A})$, it is natural to consider the **GNS representation** ρ_φ of $C^*(\mathcal{A})$.
- In the GNS representation, it is natural to consider the weak closure $\rho_\varphi(C^*(\mathcal{A}))''$.
- If φ is a KMS state, then it extends to a KMS state on $\rho_\varphi(C^*(\mathcal{A}))''$

For a conformal net $\mathcal{A}$, the **split property** is automatic (Morinelli-T.-Weiner '16): for each $\overline{I_1} \subset I_2$, there is an intermediate type I factor $\mathcal{A}(I_1) \subset \mathcal{N}(I_1, I_2) \subset \mathcal{A}(I_2)$.

Split property $\Longrightarrow$ GNS representation of a locally normal KMS state φ is defined on a separable Hilbert space.

$\Longrightarrow \rho_\varphi(C^*(\mathcal{A}))''$ is disintegrated into factors.

von Neumann algebras and types

A general von Neumann algebra $\mathcal{M}$ (a weak-operator-topology-closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$) on a separable Hilbert space is a direct integral of **factors** (von Neumann algebras with trivial center, or "simple" algebras). Factors can be classified into

- Type I: isomorphic to $\mathcal{B}(\mathcal{H})$.
- Type II: admits a tracial state/weight.
- Type III: all projections are equivalent, typical for **local** algebras in QFT.

Correspondingly, one can decompose the state φ. The case where $\rho_\varphi(\mathfrak{A})$ is type I is tractable.

Gibbs states

Examples: take Virasoro net Vir_c, a representation $\pi_{c,h}$ with $h > 0$. $e^{-\beta L_0^h}$ is trace class $\Longrightarrow$ one can define the Gibbs state

$$\varphi_{h,\beta}(x) = \frac{\mathrm{Tr}(\pi_{c,h}(x)e^{-\beta L_0^h})}{\mathrm{Tr}(e^{-\beta L_0^h})}, \ x \in C^*(\mathrm{Vir}_c).$$

The GNS representation with respect to $\varphi_{h,\beta}$ is an infinite direct sum of $\pi_{c,h}$ (the multiplicity corresponds to the eigenvectors of L_0^h). Especially, it is of type I.

In general, for a conformal net $\mathcal{A}$ and a KMS state φ, if the GNS representation is factorial of type I, then it is given as a Gibbs state in an irreducible representation.

Classification of KMS states: no type III representations

Let φ be a locally normal KMS state on $C^*(\mathcal{A})$.

- Rotations can be implemented by elements in $C^*(\mathcal{A})$ (D'Antoni-Fredenhagen-Köster).
- Rotations = the modular group is inner $\Longrightarrow$ no type III component in $\rho_\varphi(C^*(\mathcal{A}))''$.
- Using the tracial weight τ and the Radon-Nikodym derivative, the KMS state φ can be written as

$$\varphi(x) = \frac{\tau(e^{-\beta L_0^{\rho_\varphi}} \rho_\varphi(x))}{\tau(e^{-\beta L_0^{\rho_\varphi}})},$$

where $L_0^{\rho_\varphi}$ is the generator of rotations affliated to $\rho_\varphi(C^*(\mathcal{A}))''$, different from the standard GNS implementation.

Question: type II component possible?

Classification of KMS states: nets with only type I representations

We will see later that, if there are only countably many **irreducible** sectors of $\mathcal{A}$ for a given value l_0 of the lowest eigenvalue of L_0^ρ, then $\mathcal{A}$ admits only type I representations.

Theorem

Any rotation β-KMS state on a conformal net $\mathcal{A}$ which admits only type I representations can be written as $\int \varphi_\lambda d\lambda$, where

$$\varphi_\lambda(x) = \frac{\mathrm{Tr}(\rho_\lambda(x) e^{-\beta L_0^{\rho_\lambda}})}{\mathrm{Tr}(e^{-\beta L_0^{\rho_\lambda}})},$$

and ρ_λ is an irreducible representation of $\mathcal{A}$.

Examples:

- the $\mathrm{U}(1)$-current net, the Virasoro nets Vir_c by the above criterion.
- finite tensor products by a general argument.

The structure of the universal C^*-algebra: the rational case

Let $\mathcal{A}$ be a **completely rational** net (there are finitely many irreducible sectors and the conjugate sector exists, Kawahigashi-Longo-Müger, Longo-Xu).

Then any representation of $C^*(\mathcal{A})$ is a **direct sum** (not a direct integral) of irreducible representations and (Carpi-Conti-Hillier-Weiner)

$$C^*(\mathcal{A}) = \bigoplus_{\mu=1}^{n} \mathcal{B}(\mathcal{H}_\mu).$$

Any locally normal representation is the direct sum of irreducible representations (Kawahigashi-Longo-Müger).

Non-rational case

There are important non-rational nets.

- The Virasoro nets with $c \geq 1$.
- The $\mathrm{U}(1)$-current (the Heisenberg algebra): $[J_m, J_n] = n\delta_{m+n,0}$.
- Finite tensor products of them.

Their irreducible representations have been classified.

- The Virasoro nets with c: lowest weight representations with $h \geq 0$.
- The $\mathrm{U}(1)$-current: lowest weight representations $q \in \mathbb{R}$.
- Finite tensor products of them.

And there are corresponding Gibbs states.

Problem: Are there non-type I representations for these nets?

Answer: No.

Some nets with only type I sectors

Let $\mathcal{A}$ be a conformal net with **split property** and assume that **for each $l_0 \geq 0$, there are only countably many irreducible sectors where the lowest eigenvalue of L_0^μ is l_0.**

Let ρ any factorial representation of $\mathcal{A}$. $e^{itL_0^\rho}$ is contained in $\rho(C^*(\mathcal{A}))$ (D'Antoni-Fredenhagen-Köster), hence $e^{i2\rho L_0^\rho}$ is a scalar, hence the spectrum of L_0 is $l_0 + \mathbb{N}$.
ρ can be further disintegrated into irreducible representations (in general this is not unique) and each of the component has the lowest eigenvalue $l_0 + n$, hence **countably many** irreducible representations.
On the other hand, **if ρ were non-type I, there must appear uncountably many irreducible representations**
(Kawahigashi-Longo-Müger), which contradicts the above countability.

Examples:

- h for the Virasoro nets with $c \geq 1$.
- $\frac{q^2}{2}$ for the $\mathrm{U}(1)$-current.

Asymptotically AdS_3 black holes

Consider the classical black hole solutions of the Einstein equation which are asymptotically AdS_3 (the BTZ black holes). These solutions can be parametrized by $\mathrm{Diff}(S^1)/S^1 \times \mathrm{Diff}(S^1)/S^1$.

In the (hypothetical) quantum theory, an asymptotically AdS_3 black hole should be in a thermal state with the Bekenstein-Hawking temperature. The Virasoro algebra, the Lie algebra of $\mathrm{Diff}(S^1)$, should acts on the physical Hilbert space as the symmetry of the system.

Hence, **the Virasoro algebra should be in a thermal state**.
The time-translations correspond to the action of S^1. Hence we need to study thermal states on the Virasoro algebra with respect to S^1.

Open problems

- Is there any conformal net with type II_∞ or III representations?
  - cyclic orbifold: $(\mathcal{A} \otimes \cdots \otimes \mathcal{A})^{\mathbb{Z}_n}$, a classification of sectors not available in general (if $\mathcal{A}$ is not rational).
  - infinite tensor product $\mathcal{A} \otimes \mathcal{A} \cdots$, not conformal, $e^{-\beta L_0}$ not trace class...
- Should the trace class property of $e^{-\beta L_0^\rho}$ follow for a factorial representation ρ from any general assumption on the vacuum representation?
  $\Longrightarrow \rho$ is type I.
- Which KMS states and which additional observables should appear in the BTZ black hole?
- Can we make sense of Black hole entropy?