KMS states on conformal QFT

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Abstract.

Some recent results on KMS states on chiral components of two-dimensional conformal quantum field theories are reviewed. A chiral component is realized as a conformal net of von Neumann algebras on a circle, and there are two natural choices of dynamics: rotations and translations.

For rotations, the natural choice is the universal C^*-algebra. We classify KMS states on a large class of conformal nets by their superselection sectors. They can be decomposed into Gibbs states with respect to the conformal Hamiltonian.

For translations, one can consider the quasilocal C^*-algebra and we construct a distinguished geometric KMS state on it, which results from diffeomorphism covariance. We prove that this geometric KMS state is the only KMS state on a completely rational net. For some non-rational nets, we present various different KMS states.

§1. Introduction

The theory of operator algebras has been developed in a particularly close relationship with its application to physics. Among intersting connections, let us focus on the KMS condition.

In statistical physics (see e.g. [17]), one considers a system of interest in contact with a heat bath. The heat bath is a system much larger than the system of interest and has a fixed temperature. They can only exchange energy, and after some time, the whole system arrives at an equilibrium state. As the system of interest is small compared with the heat bath, we may assume that the temperature in the equilibrium is that of the heat bath, say $1/\beta$. If the system of interest is a quantum mechanical system on a Hilbert space $\mathcal{H}$ with the Hamiltonian H (such that $e^{-\beta H}$ is of trace class), the equilibrium state restricted to it can be

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represented by the density matrix

$$\rho = \frac{e^{-\beta H}}{\mathrm{Tr}(e^{-\beta H})}.$$

Let us write $\sigma_t = \mathrm{Ad}\, e^{itH}$. It is immediate that this state above satisfies the following KMS condition: for $x, y \in \mathcal{B}(\mathcal{H})$, there is an analytic function $f_{x,y}$ on $\mathbb{R} + i(0, \beta)$ such that

$$f_{x,y}(t) = \frac{\mathrm{Tr}(\rho y \sigma_t(x))}{\mathrm{Tr}(e^{-\beta H})}, \quad f_{x,y}(t + i\beta) = \frac{\mathrm{Tr}(\rho \sigma_t(x) y)}{\mathrm{Tr}(e^{-\beta H})} \text{ for } t \in \mathbb{R}.$$

Now, if we are to consider a continuum system, the Hamiltonian is not necessarily of trace class. Yet the KMS condition can be stated for any C^*-algebra and a one-parameter automorphism group σ_t. Indeed, the KMS condition can be thought as a characterization of thermal equilibrium states in general [15].

We are interested in low-dimensional quantum field theory. In addition to that it is natural to consider thermal states on quantum field theory, there is also a particular motivation to study two-dimensional conformal field theory: when one considers a black hole in the AdS background in $(2+1)$-dimensions, the classical solutions can be parametrized by two copies of the group $\mathrm{Diff}(S^1)$. At quantum level, $\mathrm{Diff}(S^1)$ should be unitarily represented, hence also its Lie algebra, the Virasoro algebra, and it should be in a thermal state with the Hawking temperature. Therefore, it is natural to consider thermal states on the Virasoro algebra, or in the operator-algebraic setting, the Virasoro net (see the next section). See [14] and the references therein.

Furthermore, this KMS condition appears in a completely different context (see, e.g. [23, 2]). Let $\mathcal{M}$ be a von Neumann algebra and Ω a cyclic separating vector. The antilinear map

$$S_0 : x\Omega \longmapsto x^*\Omega$$

is well-defined and densely defined, and turns out to be closable. The closure S has the polar decomposition $S = J\Delta^{\frac{1}{2}}$. Then, for $x \in \mathcal{M}$, $\sigma_t(x) := \mathrm{Ad}\,\Delta^{it}(x) \in \mathcal{M}$, namely σ_t defines a one-parameter automorphism group, called the modular automorphisms [23]. Then, for $x, y \in \mathcal{M}$, there is an analytic function $f_{x,y}$ on $\mathbb{R} + i(-1, 0)$ such that

$$f_{x,y}(t) = \langle \Omega, y\sigma_t(x)\Omega \rangle, \quad f_{x,y}(t - i) = \langle \Omega, \sigma_t(x) y\Omega \rangle \text{ for } t \in \mathbb{R}.$$

In this sense, the state $\langle \Omega, \cdot\, \Omega \rangle$ is said to satisfy the KMS condition with respect to the modular automorphism group at the inverse temperature $\beta = -1$.

§2. Conformal nets, the universal algebra and representations

Let us turn to low-dimensional conformal field theory in thermal states. In the operator-algebraic approach, chiral components of a two-dimensional conformal field theory are realized as a net $\{\mathcal{A}(I)\}$ of von Neumann algebras on the circle S^1 satisfying the analogue of Haag-Kastler axioms [13, 12]. Especially, such a net should satisfy isotony, namely $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)$ if $I_1 \subset I_2$, and Möbius covariance, that is, there is a strongly continuous unitary representation U of $\mathrm{PSU}(1,1)$ (which acts naturally on S^1) such that $\mathrm{Ad}\, U(g)(\mathcal{A}(I)) = \mathcal{A}(gI)$. If U can be further extended to a projective unitary representation of $\mathrm{Diff}(S^1)$ and the covariance holds and $\mathrm{Ad}\, U(g)$ acts trivially on $\mathcal{A}(I)$, where I and $\mathrm{supp}\, g$ are disjoint, the net $\mathcal{A}$ is said to be a conformal net.

Given a net $\{\mathcal{A}(I)\}$, one can consider a representation $\{\rho_I\}$, which is a family of representations ρ_I of $\mathcal{A}(I)$ on a common Hilbert space $\mathcal{H}_\rho$ with the compability condition $\rho_{I_2}|_{\mathcal{A}(I_1)} = \rho_{I_1}$ for $I_1 \subset I_2$. One can consider a universal C^*-algebra $C^*(\mathcal{A})$ of the net which includes each local algebra $\mathcal{A}(I)$, and any representation ρ can be factored through $C^*(\mathcal{A})$ [11, 19]. Actually, we are interested in locally normal representations (i.e. each ρ_I is σ-weakly continuous) and one can consider universal algebra $C^*_{\mathrm{ln}}(\mathcal{A})$ for such locally normal representations [8]. A representation ρ is said to be irreducible if $\rho(C^*(\mathcal{A}))'' = \mathcal{B}(\mathcal{H})$. It is important to note that a general representation of a C^*-algebra cannot be meaningfully decomposed into irreducible representations, and even if it is possible and an irreducible representations appears in a decomposition, it may not appear in another decomposition [10, 16].

A natural dynamics to consider here is the rotation group: $\mathrm{PSU}(1,1)$ contains a subgroup $\left\{\begin{pmatrix} e^{\frac{i\theta}{2}} & 0 \\ 0 & e^{-\frac{i\theta}{2}} \end{pmatrix}\right\}$, whose elements are represented by $e^{i\theta L_0}, \theta \in \mathbb{R}$. The adjoint action $\mathrm{Ad}\, e^{i\theta L_0}$ passes to an automorphism σ_θ of the universal algebra $C^*(\mathcal{A})$, and our task is to classfy the KMS states with respect to $\{\sigma_\theta\}$.

A KMS state φ on the universal algebra can be studied through its GNS representation ρ_φ: one can introduce a (pre-)Hilbert space structure on $C^*(\mathcal{A})$ with the inner product $\langle y, x\rangle_\varphi = \varphi(y^*x)$ and $C^*(\mathcal{A})$ itself acts naturally from the left. In the GNS representation, it is natural to take the weak closure of $\rho_\varphi(C^*(\mathcal{A}))$, which is a von Neumann algebra. Any von Neumann algebra can be decomposed into a direct integral of so-called factors, and factors are classified into type I, II and III [22, Chapter V]. Following this, the state φ gets also decomposed as a convex combination (possibly an integral), so that the weak closure in the GNS representation of each component is a factor. The KMS state naturally extends to the weak closure, where the Tomita-Takesaki modular theory [23] can be exploited.

In some concrete cases, representations may be completely classified, and the classification of KMS states follows, as we see in the next section.

§3. KMS state with respect to rotations

If we assume conformal covariance, any representation ρ is Möbius covariance [9, Theorem 5]: in particular there is a positive self-adjoint operator L_0^ρ such that $\rho(\sigma_\theta(x)) = \mathrm{Ad}\, e^{i\theta L_0^\rho}(\rho(x))$. Furthermore, for irreducible representations of some nets, $e^{-\beta L_0^\rho}$ turns out to be of trace class. In that case, one can define the Gibbs state by

$$\varphi_\rho(x) = \frac{\mathrm{Tr}(e^{-\beta L_0^\rho} x)}{\mathrm{Tr}(e^{-\beta L_0^\rho})}.$$

This clearly satisfies the KMS condition.

Conversely, for a given KMS state φ, one can consider its GNS representation ρ_φ and its factorial decomposition, as explained above. First observation is that no type III component appears, because by [9] the modular group, in this case the rotation automorphisms, is inner. Therefore, the weak closure $\rho_\varphi(C^*(\mathcal{A}))''$ is a factor of either type I or type II, and it admits the canonical trace Tr. One has furthermore the following.

Theorem 1. *Let φ be a rotational β-KMS state on $C^*(\mathcal{A})$ such that $\rho_\varphi(C^*(\mathcal{A}))''$ is of type I, then* $\mathrm{Tr}(e^{-\beta L_0^{\rho_\varphi}}) < \infty$ *and*

$$\varphi(x) = \frac{\mathrm{Tr}(e^{-\beta L_0^{\rho_\varphi}} x)}{\mathrm{Tr}(e^{-\beta L_0^{\rho_\varphi}})}.$$

Then a natural question arises whether (locally normal) type II representation can appear in the GNS representation. It turns out that we can even exclude type II representations for many conformal nets.

The first class is the so-called competely rational nets [16, Definition 8]. Complete rationality is defined through certain analytic properties of the net, yet it can be characterized by their representation theory: a completely rational net admits only finitely many irreducible representations up to unitary equivalence and each irreducible representation has its conjugate [20, Theorem 4.9]. In this case, any representation can be decomposed into a direct sum of irreducible representations [16, Corollary 39], in particular, they are of type I.

Let us consider non-rational conformal nets. The U(1)-current net [4] is generated by the derivative of the massless free field, and it admits a family of irreducible representations $\{\rho_q\}$ parametrized by the charge density $q \in \mathbb{R}$. In each ρ_q, the lowest eigenvalue of the conformal Hamiltonian $L_0^{\rho_q}$ is $\frac{q^2}{2}$. Another family is called the Virasoro nets generated by the diffeomorphism covariance itself. They are classified by the so-called central charge c, and their irreducible representations are classified by the lowest eigenvalue h of the conformal Hamiltonian [7, 24]. In both cases, for each fixed value of the lowest eigenvalue of the conformal Hamiltonian, there are only finitely many inequivalent irreducible representations. With this property we can prove that the conformal net admits only type I reprensetations. One can also prove that any finite tensor product of such nets have the same property.

As a straightforward corollary, we obtain the following.

Theorem 2. *Let $\mathcal{A}$ be either completely rational, the* U(1)*-current net, the Virasoro net* Vir_c *or a finite tensor product of them. Then any rotational β-KMS state φ on $C^*(\mathcal{A})$ is a convex combination of Gibbs states in irreducible representations.*

It is an open problem whether there is any conformal net with representations not of type I. A simple class of nets whose representations have not been classified are the fixed point nets by finite groups acting on non-rational nets.

§4. KMS state with respect to translations

Another natural choice of dynamics, when the two-dimensional CFT is considered as the full system, is time-translations, as we studied in [5, 6]. We take again the chiral components of the full CFT.

To study translational KMS states, we come back to the real line picture $\mathbb{R} = S^1 \setminus \{\infty\}$, and restrict the conformal net on S^1 to $\mathbb{R}$. There is the one-parameter subgroup in the $\mathrm{PSU}(1,1)$, which acts on $\mathbb{R}$ as the usual translations. The C^*-algebra of the interest is the quasilocal C^*-algebra

$$\mathcal{A} = \overline{\bigcup_{I \Subset \mathbb{R}} \mathcal{A}(I)}^{\|\cdot\|}.$$

A conformal net $\mathcal{A}$ has the distingished vacuum state $\langle \Omega, \cdot\, \Omega \rangle$, and it has the Bisognano-Wichmann property: the modular automorphism group of $\mathcal{A}(\mathbb{R}_+)$ with respect to the vacuum is dilations. This implies that the vacuum is the KMS state with respect to dilations.

By exploiting the Bisognano-Wichmann property and conformal covariance, one can construct a canonical, geometric KMS state: by a local diffeomorphism, the action of dilations can be locally intertwined to the action of translations. By composing this and a KMS state of the half-line algebra $\mathcal{A}(\mathbb{R}_+)$, namely the vacuum, one obtains a KMS state of the quasilocal algebra with respect to translations. The temperature $1/\beta$ can be tuned by further composing it with a dilation.

Theorem 3. *Any conformal net has a translational β-KMS state φ_{geo}, the geometric KMS state.*

The next natural question is whether there are other KMS states. As for completely rational nets, the answer turns out to be no. The reason is, if one has a translational KMS state, one can naturally obtain a Möbius covariant net [18, 25], and by complete rationality, this can be identified as an extension of $\mathcal{A}$ as a conformal net. Yet, again by complete rationality, such extensions are severely restricted and one can actually prove that the KMS state must be the geometric KMS state φ_{geo}.

Non-rational nets can admit non-geometric KMS states. The U(1)-current net has irreducible representations parametrized by $q \in \mathbb{R}$, and correspondingly, we can construct different KMS states φ_q. When these states are restricted to the Virasoro net Vir_1, they are different states for different q^2. We can prove that they exhaust all the KMS states whose GNS representations are factorial (i.e. extremal KMS states): This is based on the fact that Vir_1 is the fixed point net of a completely rational net $\mathrm{LSU}(2)_1$, the SU(2)-loop group net at level 1 [21]. Any extremal KMS state on Vir_1 can be extended to an extremal KMS state on $\mathrm{LSU}(2)_1$, but the dynamics is modified by a one-parameter automorphism group (by the argument of [1]: while the original statement and a textbook argument [3] contain a flaw, for translational KMS states on conformal nets one can adjust it). This one-parameter group specifies a subnet of $\mathrm{LSU}(2)_1$ isomorphic to the U(1)-current net, which is the fixed point subnet under the given one-parameter group. Therefore, a translational KMS state on Vir_1 extends to a translational KMS state on the U(1)-current net. For other Vir_c, $c > 1$, we can construct a continuous family of translationa KMS states. It is open whether they exhaust the extremal KMS states on Vir_c.

As we pointed out above, from these KMS states, one can construct a Möbius covariant net [18]. It is also an interesting problem to idenfity these nets with know nets.

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