Wedge-local fields in integrable QFT with bound states (partly with D. Cadamuro, arXiv:1502.01313, to appear in *Comm. Math. Phys.*)

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Towards more 2d QFTs

Construct Haag-Kastler nets (local observables) for integrable models with bound states (factorizing S-matrices with **poles**).
Non-perturbative, interacting quantum field theories in $d = 2$.
Study duality, solitons, bound states,...

- Sine-Gordon, Bullough-Dodd, $Z(N)$-Ising, Toda field theories...

Methods and partial results

Take the conjectured S-matrix with **poles** as an input, construct first **observables localized in wedges**, then prove the existence of local observables indirectly.

- **Weakly commuting** fields: $\tilde{\phi}(f) = z^\dagger(f^+) + \chi(f) + z(J_1 f^-)$ (c.f. Lechner '08, $\phi(f) = z^\dagger(f) + z(J_1 f^-)$ for S-matrix without poles).
- Problem: $\tilde{\phi}(f)$ and the reflected field $\tilde{\phi}'(g)$ strongly commute?
- Arguments for local operators (modular nuclearity).

Nonperturbative Quantum Field Theory

Computing all correlation functions of pointlike fields

- $F(x_1, \cdots, x_n) = \langle\Omega, \phi(x_1)\cdots\phi(x_n)\Omega\rangle$: Wightman functions
- Constructive QFT: $\mathcal{P}(\phi)_2$ models (Glimm-Jaffe), Sine-Gordon (Fröhlich-Seiler),...

Integrable QFT

- Factorizing S-matrix (Zamolodchikov-Zamolodchikov): Sine-Gordon, Sinh-Gordon, nonlinear σ-models, Toda field theories...
- Compute form factors: $\langle p_1, \cdots, p_n|\phi(x)|q_1, \cdots q_m\rangle$ and expand $F(x_1, x_2) = \sum \int \langle\Omega, \phi(x_1)|p_1, \cdots p_n\rangle\langle p_1, \cdots, p_n|\, \phi(x_2)\Omega\rangle$.
- Example of form factors ($Z(3)$-Ising): contains a factor $\prod F(\theta_{ij})$:

$$F(\theta) = c\int_0^\infty \frac{dt}{t}\frac{2\cosh\frac{1}{3}t\sinh\frac{2}{3}t}{\sinh^2 t}\left(1 - \cosh t\left(1 - \frac{\theta}{i\pi}\right)\right)$$

- Problem: convergence of the expansion.

Alternative strategy

- Pointlike field are hard. Larger regions have better observables.
- **(right-)Wedge**: $W_{\mathrm{R}} := \{(t, x) : x > |t|\}$.

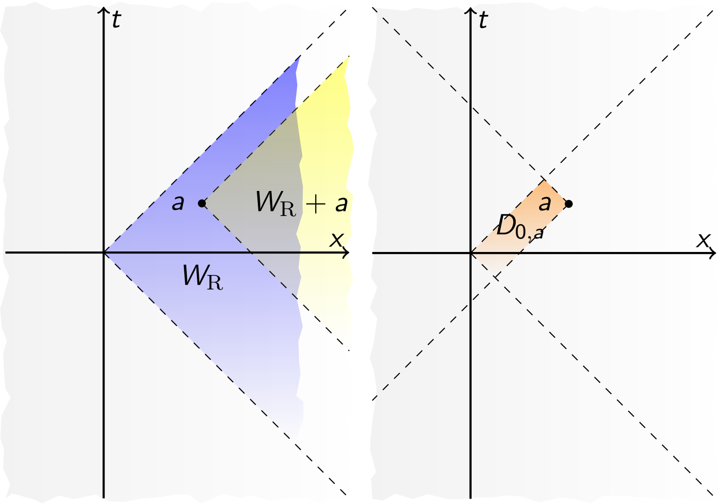
Wedge-local fields in integrable models (Schroer, Lechner)

- S: factorizing S-matrix (without poles).
- $z^\dagger, z$: Zamolodchikov-Faddeev algebra (creation and annihilation operators defined on S-**symmetric Fock space**).
- $\phi(f) = z^\dagger(f^+) + z(J_1 f^-)$, $\mathrm{supp}\, f \subset W_{\mathrm{L}}$, is localized in W_{L}.

The full QFT (without bound states)

- The observables $\mathcal{A}(W_{\mathrm{L}})$ in W_{L} are generated by $\phi(f)$.
- For diamonds $D_{a,b}$, define $\mathcal{A}(D_{a,b}) = \mathcal{A}(W_{\mathrm{L}} + a) \cap \mathcal{A}(W_{\mathrm{R}} + b)$.
- Examine the **boost operator** in order to show the existence of local operators (modular nuclearity (Buchholz, D'antoni, Longo, Lechner)).

Standard wedge and double cone



Overview of the strategy

- Haag-Kastler net $(\{\mathcal{A}(O)\}, U, \Omega)$: local observables $\mathcal{A}(O)$, spacetime symmetry U and the vacuum Ω.
- Wedge-algebras first: construct $\mathcal{A}(W_{\mathrm{R}}), U, \Omega$ from **wedge-local fields**, then take the intersection

$$\mathcal{A}(D_{a,b}) = U(a)\mathcal{A}(W_{\mathrm{R}})U(a)^* \cap U(b)\mathcal{A}(W_{\mathrm{R}})'U(b)^*$$

  The intersection is large enough if **modular nuclearity** or wedge-splitting holds.
- **Wedge-local fields**: a pair of operator-valued distributions ϕ, ϕ' such that $[e^{i\phi(f)}, e^{i\phi'(g)}] = 0$ if $\mathrm{supp}\, f \subset W_{\mathrm{L}}, \mathrm{supp}\, g \subset W_{\mathrm{R}}$.

Examples: scalar analytic factorizing S-matrix (Lechner '08), twisting by inner symmetry (T., '14), diagonal S-matrix (Alazzawi-Lechner '15)...

More example? **S-matrices with poles**.

Factorizing S-matrix models (Lechner, Schroer)

- Input: **analytic** function $S : \mathbb{R} + i(0, \pi) \to \mathbb{C}$,

$$\overline{S(\theta)} = S(\theta)^{-1} = S(-\theta) = S(\theta + \pi i), \quad \theta \in \mathbb{R}.$$

- S-symmetric Fock space: $\mathcal{H}_1 = L^2(\mathbb{R}, d\theta)$, $\mathcal{H}_n = P_n \mathcal{H}_1^{\otimes n}$, where P_n is the projection onto S-**symmetric** functions:

$$\Psi_n(\theta_1, \cdots, \theta_n) = S(\theta_{k+1} - \theta_k)\Psi_n(\theta_1, \cdots, \theta_{k+1}, \theta_k, \cdots, \theta_n).$$

- Zamolodchikov-Faddeev algebra: S-symmetrized creation and annihilation operators $z^\dagger(\xi) = Pa^\dagger(\xi)P, z(\xi) = Pa(\xi)P, P = \bigoplus_n P_n$.
- **Wedge-local field**: $\phi(f) = z^\dagger(f^+) + z(J_1 f^-)$,

$$f^\pm(\theta) = \int dx e^{\pm i x \cdot p(\theta)} f(x), \quad p(\theta) = (m \cosh \theta, m \cosh \theta),$$

J_1 is the one-particle CPT operator, $\phi'(g) = J\phi(g_j)J$, $g_j(x) = \overline{g(-x)}$.
If $\mathrm{supp}\, f \subset W_\mathrm{L}, \mathrm{supp}\, g \subset W_\mathrm{R}$, then $[e^{i\phi(f)}, e^{i\phi'(g)}] = 0$.

S-matrix with poles

If S has a pole:

$$[\phi(f), \phi'(g)]\Psi_1(\theta_1) =$$
$$-\int d\theta \left(f^+(\theta)g^-(\theta)S(\theta_1 - \theta) - f^+(\theta + \pi i)g^-(\theta + \pi i)S(\theta_1 - \theta + \pi i)\right)$$
$$\times \Psi_1(\theta_1)$$

obtains the **residue** of S and does not vanish.

- Example (Bullough-Dodd models): poles at $\theta = \frac{\pi i}{3}, \frac{2\pi i}{3}$, residues $-R, R$

$$S_B(\theta) = \frac{\tanh\frac{1}{2}\left(\theta + \frac{2\pi i}{3}\right)}{\tanh\frac{1}{2}\left(\theta - \frac{2\pi i}{3}\right)} \cdot \frac{\tanh\frac{1}{2}\left(\theta + \frac{(B-2)\pi i}{3}\right)}{\tanh\frac{1}{2}\left(\theta - \frac{(B-2)\pi i}{3}\right)} \frac{\tanh\frac{1}{2}\left(\theta - \frac{B\pi i}{3}\right)}{\tanh\frac{1}{2}\left(\theta + \frac{B\pi i}{3}\right)},$$

where $0 < B < 2, B \neq 1$. $S(\theta) = S\left(\theta + \frac{\pi i}{3}\right) S\left(\theta - \frac{\pi i}{3}\right)$.

New wedge-local field?

The bound state operator

S: two-particle S-matrix, poles $\theta = \frac{\pi i}{3}, \frac{2\pi i}{3}$, $S(\theta) = S\left(\theta + \frac{\pi i}{3}\right) S\left(\theta - \frac{\pi i}{3}\right)$

P_n: S-symmetrization, $\mathcal{H} = \bigoplus P_n \mathcal{H}_1^{\otimes n}$, $\mathcal{H}_1 = L^2(\mathbb{R})$,

$$\mathrm{Dom}(\chi_1(f)) := H^2\left(-\tfrac{\pi}{3}, 0\right)$$
$$(\chi_1(f))\xi(\theta) := \sqrt{2\pi|R|} f^+\left(\theta + \tfrac{\pi i}{3}\right) \xi\left(\theta - \tfrac{\pi i}{3}\right),$$

where $H^2\left(-\frac{\pi}{3}, 0\right)$ is the space of analytic functions in $\mathbb{R} + i\left(-\frac{\pi}{3}, 0\right)$ such that $\xi(\cdot - \gamma i)$ is uniformly bounded in L^2-norm, $\gamma \in \left(-\frac{\pi}{3}, 0\right)$, and f^+ is analytic.

$$\chi_n(f) = nP_n\left(\chi_1(f) \otimes I \otimes \cdots \otimes I\right) P_n,$$
$$\chi(f) := \bigoplus \chi_n(f),$$
$$(\chi_1'(g)\xi)(\theta) := (J_1\chi(g_j)J_1)(\theta) = \sqrt{2\pi|R|} g^+\left(\theta - \frac{\pi i}{3}\right) \xi\left(\theta + \frac{\pi i}{3}\right),$$
$$\chi_n'(g) := J_n\chi_n(g_j)J_n, \quad \chi'(g) := J\chi(g_j)J.$$

Wedge-local fields and weak commutativity

New fields:

$$\widetilde{\phi}(f) := \phi(f) + \chi(f) \qquad (= z^\dagger(f^+) + \chi(f) + z(J_1 f^-)),$$
$$\widetilde{\phi}'(g) := J\widetilde{\phi}(g_j)J, \qquad \chi'(g) = J\chi(g_j)J.$$

Theorem (Cadamuro-T. arXiv:1502.01313)

f, g real, $\mathrm{supp}\, f \subset W_{\mathrm{L}}, \mathrm{supp}\, g \subset W_{\mathrm{R}}$*,* $\Phi, \Psi \in \mathrm{Dom}(\widetilde{\phi}(f)) \cap \mathrm{Dom}(\widetilde{\phi}'(g))$*,*

then $\langle \widetilde{\phi}(f)\Phi, \widetilde{\phi}'(g)\Psi \rangle = \langle \widetilde{\phi}'(g)\Phi, \widetilde{\phi}(f)\Psi \rangle$*.*

Namely, the fields $\widetilde{\phi}(f), \widetilde{\phi}'(g)$ are ***weakly*** *wedge-local.*

Proof)

$$\langle \chi(f)\Phi_1, \chi'(g)\Psi_1 \rangle = 2\pi i R \int d\theta\, f^+\left(\theta + \frac{\pi i}{3}\right) g^+\left(\theta - \frac{2\pi i}{3}\right) \overline{\Phi(\theta)}\Psi_1(\theta)$$
$$= 2\pi i R \int d\theta\, f^+\left(\theta + \frac{\pi i}{3}\right) g^-\left(\theta + \frac{\pi i}{3}\right) \overline{\Phi(\theta)}\Psi_1(\theta) \ldots$$

Guesswork for the bound state operator

- Form factor $F_n(\underline{\theta})$: expansion coefficients of local operator ψ.
- S-matrix + LSZ-reduction formula $\Longrightarrow$ necessary conditions for F_n (**Form factor axioms**)
- Form factor axioms $\Longrightarrow$ formal commutation relations of ψ, **locality**.

How can one modify the field ϕ? Add a **Simplest** term X.

Proposition

If the formal expansion of the weak commutator

$$\langle(\phi(f)+X)^*\Psi_1, \psi(0)\Psi_2\rangle - \langle\Psi_1, \psi(0)(\phi(f)+X)\Psi_2\rangle$$

vanishes and X preserves the particle number, then $X = \chi(f)$.

Proof) The action of $\phi(f)$ is explicitly known. Expand the weak commutator in terms of form factors and use form factor axioms.

Compatibility with the form factor program

- Form factor $F_n(\underline{\theta})$: expansion coefficients of local operator ψ.
- S-matrix + LSZ-reduction formula $\Longrightarrow$ necessary conditions for F_n (**Form factor axioms**)
- Form factor axioms $\Longrightarrow$ formal commutation relations of ψ, **locality**.

Do our candidate $\widetilde{\phi}(f)$ and local operator ψ commute?

Proposition

Formal expansion of the weak commutator

$$\langle \Psi_1, [\widetilde{\phi}(f), \psi(0)]\Psi_2\rangle = \langle \widetilde{\phi}(f)^* \Psi_1, \psi(0)\Psi_2\rangle - \langle \Psi_1, \psi(0)\widetilde{\phi}(f)\Psi_2\rangle$$

vanishes.

Proof) The action of $\widetilde{\phi}(f)$ is explicitly known. Expand the weak commutator in terms of form factors and use form factor axioms.

The one-particle bound state operator

- $\mathcal{H}_1 = L^2(\mathbb{R})$
- $f^+(\zeta)$: analytic in $\mathbb{R} + i(0,\pi)$, $\overline{f^+(\theta + \frac{2\pi i}{3})} = f^+(\theta + \frac{\pi i}{3})$.
- $\mathrm{Dom}(\chi_1(f)) = H^2(-\frac{\pi}{3}, 0)$: analytic functions in $\mathbb{R} + i(-\frac{\pi}{3}, 0)$
- $(\chi_1(f))\xi(\theta) := f^+\left(\theta + \frac{\pi i}{3}\right)\xi\left(\theta - \frac{\pi i}{3}\right) \quad \left(= \overline{f^+(\theta + \frac{2\pi i}{3})}\xi\left(\theta - \frac{\pi i}{3}\right)\right)$

Question

What are self-adjoint extensions of $\chi_1(f)$?

- Write $\chi_1(f) = M_{f^+(\cdot + \frac{\pi i}{3})}\Delta_1^{\frac{1}{6}}$, $(\Delta_1^{\frac{1}{6}}\xi)(\theta) = \xi(\theta - \frac{\pi i}{3})$
- **Many extensions**: $n_\pm(\chi_1(f)) =$ "half of the zeros" of f^+
- **Choose** $f = \bar{h} * h$, consider the Beurling decomposition of h^+. There is an extension of the form $M_{u_h}^*\Delta_1^{\frac{1}{6}}M_{u_h}$, M_{u_h} is unitary.
- $\widetilde{\phi}$ is **no longer distribution**.

Towards proof of strong commutativity

If $\chi(f) + \chi'(g)$ is self-adjoint...

- $\chi(f) + \chi'(g) + cN$ is self-adjoint.
- $T(f, g) := \widetilde{\phi}(f) + \widetilde{\phi}'(g) + cN$ is self-adjoint by Kato-Rellich. $(= \chi(f) + \chi'(g) + cN + \phi(f) + \phi'(g))$
- $[T(f, g), \widetilde{\phi}(f)] = [cN, \widetilde{\phi}(f)] = [cN, \phi(f)]$ is small, $\|\widetilde{\phi}(f)\Psi\| \leq \|T(f, g)\Psi\|$.
- use Driessler-Fröhlich theorem with $T(f, g)$ **as the reference operator** to show **strong commutativity**.

Why is self-adjointness of $\chi(f) + \chi'(g)$ difficult?

- $\chi(f)$ should have different domain of self-adjointness, depending on f.

$$(\chi_1(f))\xi(\theta) := \sqrt{2\pi|R|}f^+\left(\theta + \frac{\pi i}{3}\right)\xi\left(\theta - \frac{\pi i}{3}\right).$$

  ξ might have poles at zeros of f^+.
- From two particles on, the operator is of the form PAP...

Existence of local operators: modular nuclearity

- $\mathcal{N} \subset \mathcal{M}$: inclusion of von Neumann algebras, Ω: cyclic and separating for both, Δ: the modular operator for $\mathcal{M}$.
- **Modular nuclearity** (Buchholz-D'antoni-Longo): if the map

$$\mathcal{N} \ni A \longmapsto \Delta^{\frac{1}{4}} A\Omega \in \mathcal{H}$$

  is nuclear, then the inclusion $\mathcal{N} \subset \mathcal{M}$ is split.
- (sketch of proof) By assumption, the map

$$\mathcal{N} \ni \mathcal{A} \longmapsto \langle JA\Omega, \cdot\, \Omega\rangle = \langle \Delta^{\frac{1}{2}} A^* \Omega, \cdot\, \Omega\rangle \in \mathcal{M}_*$$

  is nuclear. $\langle JBJ\Omega, A\Omega\rangle = \sum \varphi_{1,n}(A)\varphi_{2,n}(B)$ and one may assume that $\varphi_{k,n}$ are normal. This defines a normal state on $\mathcal{N} \otimes \mathcal{M}'$ which is equivalent to $\mathcal{N} \vee \mathcal{M}'$.
- **Bisognano-Wichmann property**: for $\mathcal{M} = \mathcal{A}(W_{\mathrm{R}})$, Δ^{it} is Lorentz boost (follows if one assumes strong commutativity)

Choose $f = \bar{h} * h$, assume strong commutativity ($\Longrightarrow \Delta^{it} =$ boosts)...

Consider $\mathcal{A}(W_{\mathrm{R}} + a) \subset \mathcal{A}(W_{\mathrm{R}})$, where $a = (0, a_1)$ and the vacuum Ω.
Modular nuclearity: $\mathcal{A}(W_{\mathrm{R}}) \ni A \mapsto \Delta^{\frac{1}{4}} U(a) A\Omega \in \mathcal{H}$,

$$(\Delta^{\frac{1}{4}} U(a) A\Omega)_n(\theta_1, \cdots, \theta_n) =$$
$$e^{-ia_1 \sum_k \sinh(\theta_k - \frac{\pi i}{2})} (A\Omega)_n \left(\theta_1 - \frac{\pi i}{2}, \cdots, \theta_n - \frac{\pi i}{2}\right),$$

which contains a strongly damping factor $e^{-c \sum_k \cosh \theta_k}$.

- (1) Bounded analytic extension. (2) Cauchy integral.

$A \in \mathcal{A}(W_{\mathrm{R}}) \Longrightarrow A\Omega \in \mathrm{Dom}(\widetilde{\phi}(f)) \Longrightarrow (A\Omega)_n \in \mathrm{Dom}(\chi_n(f))$, where
$\chi_1(f) = M_{u_h}^* \Delta_1^{\frac{1}{6}} M_{u_h}$.

$$\langle \chi_n(f)(A\Omega)_n, (A\Omega)_n \rangle = n \|(\Delta_1^{\frac{1}{12}} M_{u_h} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}) \cdot (A\Omega)_n\|^2$$
$$= \langle (\widetilde{\phi}(f) - \phi(f))(A\Omega)_n, (A\Omega)_n \rangle$$
$$= \langle (Af^+ - \phi(f)A\Omega)_n, (A\Omega)_n \rangle \leq 3\sqrt{n+1} \|f^+\| \cdot \|A\Omega\|^2$$

Towards modular nuclearity

Choose a **nice** h so that $|h^+(\zeta)| > |e^{-ia_1 \sinh \frac{\zeta}{2}}|$ for $-\mathrm{Im}\,\zeta > \epsilon > 0$.

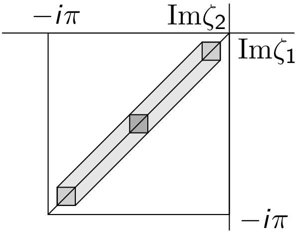
$\Longrightarrow$ Estimate of $(U(\frac{a}{2})A\Omega)_n$ around $\left(\theta_1 - \frac{\pi i}{6}, \theta_2, \cdots, \theta_n\right)$ by $\|A\|$

$\Longrightarrow$ By S-symmetry and the flat tube theorem, $(U(\frac{a}{2})A\Omega)_n$ has an analytic continuation in all variables in the cube.

- Since $A\Omega \in \mathrm{Dom}(\Delta) = \mathrm{Dom}(\Delta_1^{\otimes n})$, it is analytic on the diagonal.
- By $\Delta^{\frac{1}{2}} A\Omega = JA^*\Omega$, $(U(\frac{a}{2})A\Omega)_n$, it is analytic on the lower cube.

$\Longrightarrow$ Estimate of $(U(\frac{a}{2})A\Omega)_n$ around $\left(\theta_1 - \frac{\pi i}{2}, \cdots, \theta_n - \frac{\pi i}{2}\right)$ by $\|A\|$

$\Longrightarrow$ nuclearity for minimal distance (Alazzawi-Lechner '15).



Some features of the models

- No Reeh-Schlieder property for polynomials, but for the von Neumann algebra.

$$\widetilde{\phi}(f)\Omega = f^+ \text{ is not in the domain of } \widetilde{\phi}(f).$$

- No energy bound for $\widetilde{\phi}$ ($\Rightarrow$ no pointlike field?).

$$\widetilde{\phi}(f) = \phi(f) + \chi(f), \;\; \chi_1(f) = M_{f^+(\cdot + \frac{\pi i}{3})}\Delta_1^{\frac{1}{6}}.$$

- Non-temperate polarization-free generator (c.f. Borchers-Buchholz-Schroer '01).

$$(\chi_1(f)U_1(a)\Psi_1)(\theta) = \sqrt{2\pi|R|}f^+\left(\theta + \frac{\pi i}{3}\right)e^{ia\cdot p\left(\theta - \frac{\pi i}{3}\right)}\Psi_1\left(\theta - \frac{\pi i}{3}\right),$$

which grows exponentially.

- Non-distribution: self-adjoint extension of $\widetilde{\phi}$ does not always exist.
- Bound states?

Open problem: self-adjointness of n-particle bound state operators

Two-particle case

- $\overline{P_2(u_h^*\Delta^{\frac{1}{6}}u_h \otimes \mathbb{1})P_2} = u_h^* \otimes u_h^* \cdot P_2(\Delta^{\frac{1}{12}} \otimes \mathbb{1}) \cdot (\Delta^{\frac{1}{12}} \otimes \mathbb{1})P_2 \cdot u_h \otimes u_h$
- $P_2(\Delta^{\frac{1}{12}} \otimes \mathbb{1}) \cdot (\Delta^{\frac{1}{12}} \otimes \mathbb{1})P_2$ is self-adjoint.
- it is enough to show that

$$\overline{P_2(\Delta^{\frac{1}{12}} \otimes \mathbb{1})} \cdot (\Delta^{\frac{1}{12}} \otimes \mathbb{1})P_2 = P_2(\Delta^{\frac{1}{12}} \otimes \mathbb{1}) \cdot (\Delta^{\frac{1}{12}} \otimes \mathbb{1})P_2$$

- P_2 (strongly) commutes with $\Delta \otimes \Delta$
- The problem reduces further to $P_2(\Delta^{\frac{1}{12}} \otimes \Delta^{-\frac{1}{12}})P_2$, which has essentially only one variable.
- Prove that $(\Delta^{\frac{1}{12}} \otimes \Delta^{-\frac{1}{12}}) + M_S(\Delta^{-\frac{1}{12}} \otimes \Delta^{\frac{1}{12}})M_S^*$ is self-adjoint.

From three particles on, extracting $\Delta \otimes \Delta \otimes \Delta$ is not enough.

More examples: $Z(N)$-Ising models, Sine-Gordon models

- $Z(N)$-Ising model:
  - $N-1$ species of particles, diagonal S-matrix.
  - The first and $(N-1)$-th are "elementary".
  - k-th and l-th form the $(k+l(\mathrm{mod}\ N))$-th "bound state".
  - We can generalize $\chi(f)$, so that $\chi_1(f)\Psi_1$ is the bound state between f and Ψ_1.
  - For $N=3$, $\chi_1(f) = \begin{pmatrix} 0 & M_{f_2^+(\cdot+\frac{\pi i}{3})}\Delta^{\frac{1}{6}} \\ M_{f_1^+(\cdot+\frac{\pi i}{3})}\Delta^{\frac{1}{6}} & 0 \end{pmatrix}$
  - $\widetilde{\phi}(f) = \phi(f) + \chi(f)$ is weakly wedge-local, where f corresponds to "elementary" particles.
- Sine-Gordon model:
  - soliton and antisoliton, breathers (depending on the coupling constant). S-matrix non diagonal on solitons.
  - Take solitons if there is only one breather. Take breathers if there are more. Consider $\chi(f)$ for the corresponding species.
  - $\widetilde{\phi}(f) = \phi(f) + \chi(f)$ is weakly wedge-local, where f corresponds to chosen spiecies.

Summary

- input: two-particle factorizing S-matrix with **poles**
- **new field** $\widetilde{\phi}(f) = \phi(f) + \chi(f)$
- weak commutativity
- modular nuclearity (by assuming strong commutation)
- features of $\widetilde{\phi}(f)$: no polynomial Reeh-Schlieder property, no energy bound, non-temperateness, non-distribution

Open problems

- **strong commutativity**
- for non-scalar models (Sine-Gordon, Z(N)-Ising...) strong commutativity and modular nuclearity more difficult