Renormalization group according to Balaban–Dimock 2025.05.05 Roma Tre.

1. Models, methods, results.

Balaban ~1982 U(1)-Higgs $d=2,3$, ~1989 Yang-Mills $d=3,4$ (4 incomplete) UV stability.
King 1986 U(1)-Higgs $d=2,3$ convergence of correlation functions, some OS axioms.
Dimock 2011~ ϕ^4_3, QED_3, Gross-Neveu$_2$ UV stability.
Lattice regularization, counterterms in such a way that, after RG, the model looks "the same" up to "irrelevant" terms. UV stability: bounded partition functions. Convergence?

2. The ϕ^4_3-model.

$L, M, N \in \mathbb{N}$. $\mathbb{T}^N_M = L^{-N}\mathbb{Z}^3 / L^M\mathbb{Z}^3$. $\phi: \mathbb{T}^N_M \to \mathbb{R}$ fields.
$S^N(\phi) = \frac{1}{2}\|\partial\phi\|^2 + \frac{\bar\mu}{2}\|\phi\|^2 + V^N(\phi)$, $V^N(\phi) = \varepsilon^N \mathrm{Vol}(\mathbb{T}_M) + \frac{\mu^N}{2}\int \phi(x)^2 dx + \frac{\lambda}{4}\int \phi(x)^4 dx$.
Want to find ε^N, μ^N s.t. after RG, the action looks like $S^0(\phi)$.
We scale up to the unit lattice $\mathbb{T}^0_{M+N}$. Φ_0: field. $S^N_0(\Phi) = S_0(\Phi)$: action.
Averaging. N fixed. For $\tilde\Phi_1: \mathbb{T}^1_{M+N} \to \mathbb{R}$, $\tilde\rho_1(\tilde\Phi_1) = \bullet \int \exp(-\frac{a}{2L^2}\|\tilde\Phi_1 - Q\Phi_0\|^2)\rho_0(\Phi_0)\, d\Phi_0$,
$(Qf)(Y) = \sum_{x\in B(Y)} f(x)$. $\boxed{Y \cdot x}\,L$ Scale back to $\mathbb{T}^0_{M+N-1}$, $\rho_1(\Phi_1)$. Repeat.

The free flow

Take $\lambda = 0$. After k RG steps, $\rho_k^{\mathrm{free}}(\Phi_k) = Z_k \exp(-S_k(\Phi_k, \phi_k))$, where $\phi_k = a_k G_k Q_k^T \Phi_k$
$G_k = (-\Delta + \bar\mu_k + a_k Q_k^T Q_k)^{-1}$, $S_k(\Phi_k, \phi) = \frac{a_k}{2}\|\Phi_k - Q_k\phi\|^2 + \frac{1}{2}\langle \phi, (-\Delta + \bar\mu_k)\phi\rangle$, $Q_k = Q^k$, $a_k = \frac{1-L^{-2k}}{1-L^{-2}}$
$\bar\mu_k = \bar\mu L^{-2(N-k)}$

Small fields

Fix $p \in \mathbb{N}$, $p_k = (-\log(L^{-(N-k)}))^p$. $\Phi_k: \mathbb{T}^0_{M+N-k} \to \mathbb{R}$, $\phi_k: a_k G_k Q_k^T \Phi_k$.
$\mathcal{S}_k$: the set of Φ_k s.t. $|\Phi_k - Q_k\phi_k| \le p_k$, $|\partial\phi_k| \le p_k$, $|\phi_k| \le (\lambda L^{-(N-k)})^{-\frac{1}{4}} p_k$.

The interacting flow

Define $\tilde\rho^p_{k+1}(\tilde\Phi_{k+1}) = \bullet \int \exp(-\frac{a}{2L^2}\|\tilde\Phi_{k+1} - Q\Phi_k\|^2)\chi^w_k(\bullet)\chi_k(\Phi_k)\rho^p_k(\Phi_k)\, d\Phi_k$
χ_k: $\Phi_k \in \mathcal{S}_k$. χ^w_k: Φ_k is close to the minimum of the free quadratic form.
Thm (Dimock) For L (, M) large, λ small, $\rho^p_k(\Phi_k) = Z_k \exp(-S_k(\Phi_k, \phi_k) - V_k(\phi_k) + E_k(\phi))$
where $E_k(\phi) = \sum_X E_k(X, \phi)$, X: M-block, $E_k(X,\phi)$ depends only on $\phi(x), x \in X$.
There are ε^N_0, μ^N_0 s.t. $\varepsilon^N_N = \mu^N_N = 0$. Sketch) Extract the min Ψ_k. $\phi_k(\Psi_k + Z) = \phi^0_{k+1} + a_k G_k Q_k^T Z$.
Cluster expansion $\int \exp(-\sum_X H(X, Z, \phi))\, dZ = \exp(-\sum_Y H^\#(Y, \phi))$. $\Rightarrow V_{k+1}, E_{k+1}$.
UV stability: $\exp(-c\mathrm{Vol}(\mathbb{T}_M)) \le Z_{M,N} \le \exp(c\mathrm{Vol}(\mathbb{T}_M))$. Convergence? compare $\rho^{N+1}_{k+1}, \rho^N_k$. Large field get $e^{-p_k^2}$ for every $\square$ violating $\mathcal{S}_k$.

3. O(4)-sigma model

$\phi: \mathbb{R}^2 \to S^3 = SU(2)$. $A[\phi] = \int d^2x \sum_\nu \partial_\nu\phi(x)^2$, $|\phi(x)|^2 = 1$. $Z = \int \mathcal{D}\phi\, e^{-\frac{1}{g^2}A[\phi]}\delta(|\phi(x)|^2 - 1)$.
$U: \mathbb{Z}^2 \to SU(2)$, $A(U) = \sum_{b \subset \mathbb{T}_M} \mathrm{Tr}(1 - \mathrm{Tr}\,\partial U(b))$. $\partial U(b) = U(b_-)U(b_+)^*$ b_- —b— b_+
Block averaging $C(U) = \frac{1}{L^2}\sum_{x\in B(Y)} U(x) / \left|\frac{1}{L^2}\sum_{x\in B(Y)} U(x)\right|$. RG $e^{-\frac{1}{g_1^2}A_1[V]} = \int dU \delta(C(U)V^{-1})\chi e^{-\frac{1}{g^2}A(U)}$
Need to find the minimum of $A(U)$ under $C(U) = V$.
Thm (Dybalski–Stottmeister–T.) If $\|\partial V - 1\| < \varepsilon^2$ small, then $A(U)$ has an extremum under $C(U) = V$, $\|\partial U - 1\| < \varepsilon$.

Renormalization group according to Balaban-Dimock (details) 2025.05.05

1. Lattice regularization for QFT

Want to construct gauge theories in the Euclidean signature.
Need regularization. Lattices preserve gauge invariance and reflection positivity.
Define the model first on discrete tori, take the UV limit, then IR.
Add counterterms in such a way that, after RG, the action looks the "same".

<u>Models so far considered</u>

Balaban : 1982 $U(1)$-Higgs $d=2,3$, YM $d=3,4$ UV stability.
King : 1986 $U(1)$-Higgs $d=2,3$, convergence of correlation functions. Some OS axioms.
Dimock : 2011 ~ ϕ^4_3, QED_3, Gross-Neveu$_2$. UV stability.
This talk: ϕ^4_3, $O(4)$-sigma model.

2. The ϕ^4_3-model.

2.1 the model.

$M, L \in \mathbb{N}$. For each $N \in \mathbb{N}$, let $\mathbb{T}^{-N}_M = L^{-N}\mathbb{Z}^3/L^M\mathbb{Z}^3$ be the discrete torus.
For $\phi : \mathbb{T}^{-N}_M \to \mathbb{R}$, write $\|\phi\|^2 = \langle\phi,\phi\rangle$, where $\langle u, v\rangle = L^{-3N}\sum_{x\in\mathbb{T}^{-N}_M} u(x)v(x)$.
For $\nu = 1,2,3$, $\partial_\nu\phi(x) = L^{-N}(\phi(x+L^{-N}e_\nu) - \phi(x))$, e_ν unit vectors.
$\Delta = -\partial^*_\nu\partial^\nu$.
The density for the model is given by

$$\rho^N(\phi) = \exp\left(-\frac{1}{2}\langle\phi, (-\Delta+\bar{\mu})\phi\rangle + V^N(\phi)\right),$$
$$V^N(\phi) = \varepsilon^N \mathrm{Vol}(\mathbb{T}_M) + \frac{1}{2}\mu^N\int\phi(x)^2dx + \frac{1}{4}\lambda\int\phi(x)^4dx,$$

where $\bar{\mu}$ is the (physical) mass, λ the coupling constant, ε^N, μ^N counterterms.

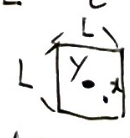
$$Z_{M,N} = \int \rho^N(\phi)\, d\phi \qquad \mathbb{R}^{|\mathbb{T}^{-N}_M|}$$

UV stability : there are "nice" choices of ε^N, μ^N, behaving well under RG,
s.t. $\exp(-c\mathrm{Vol}(\mathbb{T}_M)) \leq Z_{M,N}/Z_{M,N}(0) \leq \exp(c\mathrm{Vol}(\mathbb{T}_M))$, $Z_{M,N}(0)$ the free partition function.

2.2 Scaled up to the unit lattice $\mathbb{T}^0_{M+N}$

$\rho_0(\Phi_0) = \exp(-\frac{1}{2}\|\partial\Phi_0\|^2 + \bar{\mu}_0\|\Phi\|^2 + V_0(\Phi_0))$, N omitted from the notation.
$V_0(\Phi_0) = \varepsilon_0|\mathbb{T}^0_{M+N}| + \frac{1}{2}\mu_0\sum_x \Phi_0(x)^2 + \frac{1}{4}\lambda_0\sum\Phi_0(x)^4$,
$\bar{\mu}_0 = L^{-2N}\bar{\mu}$, $\lambda_0 = L^{-N}\lambda$, $\mu_0 = L^{-2N}\mu^N$, $\varepsilon_0 = L^{-3N}\varepsilon^N$.

2.3 General RG transforms.



Q: block averaging $(Qf)(y) = L^{-3}\sum_{x\in B(y)} f(x)$,
$\tilde{\rho}_{k+1}(\tilde{\Phi}_{k+1}) = \mathcal{N}\int \exp\left(-\frac{a}{2L^2}\|\tilde{\Phi}_{k+1} - Q\Phi_k\|^2\right)\rho_k(\Phi_k)\,d\Phi_k$, $k = 0,\cdots N-1$. $\Phi_k : \mathbb{T}^0_{M+N-k} \to \mathbb{R}$.
$\rho_{k+1}(\Phi_{k+1}) = \tilde{\rho}_{k+1}(\Phi_{k+1,L})\cdot L^{-|\mathbb{T}^0_{M+N-k}|/2}$ $\Phi_{k+1,L}(x) = L^{-\frac{1}{2}}\Phi_{k+1}(x/L)$.

2.4 The free flow.

Take $\lambda=0$. $\rho_0^{\text{free}}(\Phi_0) = \exp(-\frac{1}{2}\langle\Phi_0, (-\Delta+\bar{\mu}_0)\Phi_0\rangle)$.

After k RG steps, $\rho_k^{\text{free}}(\Phi_k) = Z_k \exp(-S_k(\Phi_k, \phi_k))$, where

$\phi_k = a_k G_k Q_k \Phi_k : \mathbb{T}^{-k}_{M+N-k} \to \mathbb{R}$, $G_k = (-\Delta + \bar{\mu}_k + a_k Q_k^T Q_k)^{-1} \in \mathcal{B}(\mathcal{L}^2(\mathbb{T}^{-k}_{M+N-k}))$

$a_k = a\frac{1-L^{-2}}{1-L^{-2k}}$, $S_k(\Phi_k, \phi) = \frac{a_k}{2}\|\Phi_k - Q_k\phi\|^2 + \frac{1}{2}\langle\phi, (-\Delta+\bar{\mu}_k)\phi\rangle$.

2.5 Small fields

For $\Phi_k : \mathbb{T}_{M+N-k} \to \mathbb{R}$, set $\phi_k := a_k G_k Q_k \Phi_k$.

Fix $p \in \mathbb{N}$, $p_k := (-\log\lambda_k)^p = (-\log(\lambda L^{N-k}))^p$.

S_k: the set of Φ_k s.t. $|\Phi_k - Q_k\phi_k| \le p_k$, $|\partial\phi_k| \le p_k$, $|\phi_k| \le (\lambda L^{-(N-k)})^{-\frac{1}{4}} p_k$

If any of these conditions is violated, $\rho_k(\Phi_k)$ gets $O(e^{-p_k^2})$.

2.6 The interacting flow.

Define, for fixed p, $\tilde{\rho}^p_{k+1}(\tilde{\Phi}_{k+1}) = N\int \exp(-\frac{a}{2L^2}\|\tilde{\Phi}_{k+1} - Q\Phi_k\|^2)\chi^w_k(C_k^{-\frac{1}{2}}(\Phi_k - \Psi_k))\chi_k(\Phi_k)\rho_k(\Phi_k)d\Phi_k$

$C_k = (\Delta_k + \frac{a}{L^2}Q^T Q)^{-1}$, $\Delta_k = a_k - a_k^2 Q_k G_k Q_k^T$

$\chi^w_k(W)$: $|W| \le p_k$, Ψ_k: the minimum of $\frac{a}{2L^2}\|\tilde{\Phi}_{k+1} - Q\Phi_k\|^2 + S_k(\Phi_k, \phi_k)$ in Φ_k, $\chi_k(\Phi_k)$: $\Phi_k \in S_k$

<u>Thm</u> (Dimock 2013) For L (M for random walk expansion) large, λ small, one can write

$\rho^p_k(\Phi_k) = Z_k \exp(-S_k(\Phi_k, \phi_k) - V_k(\phi_k) + E_k(\phi_k))$,

where $E_k(\phi) = \sum_X E_k(X, \phi)$, X: M-blocks, $E_k(X, \phi)$ depends only on $\phi(x)$, $x \in X$.

sketch). For fixed $\tilde{\Phi}_{k+1}$, extract the minimum Ψ_k in Φ_k of the quadratic part,

$\phi_k(\Psi_k + Z) = \phi^0_{k+1} + a_k G_k Q_k^T Z$.

Cluster expansion $\int \exp(-\sum_X H(X, \Phi, \phi))d\Phi = \exp(-\sum_Y H^\#(Y, \phi))$.

$\Rightarrow S_{k+1}, V_{k+1}, E_{k+1}$. There are ε^N_0, μ^N_0 s.t. $\varepsilon^N_N = \mu^N_N = 0$.

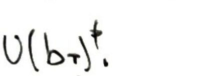
2.7 Convergence ?

Do $Z_{M,N} = \int \rho_k(\Phi_k)d\Phi_k$ $\quad Z_{M,N}/Z_{M,N}(0)$ converge as $N \to \infty$?

Compare ρ^{N+1}_{k+1}, ρ^N_k. $\quad V^{N+1}_{k+1}(\phi^{N+1}_{k+1})$, $V^N_k(\phi^N_k)$.

Large field contributions get $O(e^{-p_k^2})$ for each block. Take $k \sim N$.

3 O(4)-sigma model.



$\phi : \mathbb{R}^2 \to S^{4-1} \cong SU(2)$. $A[\phi] = \int d^2x\, \partial_\mu\phi(x)\partial^\mu\phi(x)$, $|\phi(x)| = 1$.

$U : \mathbb{Z}^2 \to SU(2)$ $\quad A(U) = \sum_{b \subset \mathbb{T}_M} \operatorname{Re}\operatorname{Tr}(\mathbb{1} - \partial U(b))$, $\partial U(b) = U(b_-)U(b_+)^\dagger$.

<u>Block averaging.</u> $C(U)(Y) = \frac{1}{L^2}\sum_{x \in B(Y)} U(x) \Big/ \left|\frac{1}{L^2}\sum_{x \in B(Y)} U(x)\right|$ (polar decomposition)

<u>RG</u> $e^{-\frac{1}{g_1}A_1(V)} = \int dU\, \delta(C(U)V^{-1})\chi\, e^{-\frac{1}{g_0}A(U)}dU$.

We need to find the minimum of $A(U)$ under the constraint $C(U)V^{-1} = \mathbb{1}$.

<u>Thm</u> If V satisfies $\|\partial V - \mathbb{1}\| < \varepsilon^2$, then under $\|\partial U - \mathbb{1}\| < \varepsilon$, $C(U) = V$, $A(U)$ has an extremum

(Dubalski-Stattmeister-T.)

Cluster expansion.

$\Phi : \mathbb{T}^0_{\bullet} \to \mathbb{R}$, X : M-polymer $\Xi = \int \exp\left(\sum_X H(X, \Phi, \phi)\right) d\mu(\Phi)$.
where $d\mu(\Phi)$ is ultralocal. ϕ plays no role, so $H(X, \Phi)$.
want $\Xi = \exp\left(\sum_Y H^\#(Y)\right)$ ⋯ (✱)

<u>Thm</u> (Dimock 2013 Theorem 27). Assume that $|H(X, \Phi)| \leq H_0 e^{-\kappa d_M(X)}$.
Then (✱) holds, with $H^\#(Y) \leq O(1) H_0 e^{-(\kappa - 3\kappa_0 - 3) d_M(Y)}$.

proof) ① Mayer expansion.

$$\exp\left(\sum_X H(X, \Phi)\right) = \prod_X \left((e^{H(X,\Phi)} - 1) + 1\right) = \sum_{\{X_i\}} \prod_i \left(e^{H(X_i, \Phi)} - 1\right)$$
$$= \sum_{\{Y_j\}} \prod_j K(Y_j, \Phi),$$

where $\sum_{\{X_i\}}$ is a sum over collections over distinct polymers, $\sum_{\{Y_j\}}$ over collections of disjoint polymers

$$K(Y, \Phi) = \sum_{\{X_i\} : \cup\{X_i\} = Y} \prod_i \left(e^{H(X_i, \Phi)} - 1\right)$$
$$= \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(X_1, \ldots, X_n) : \cup X_i = Y} \prod_i \left(e^{H(X_i, \Phi)} - 1\right)$$

②. $\int \sum_{\{Y_j\}} \prod_j K(Y_j, \Phi) \, d\mu(\Phi) = \sum_{\{Y_j\}} \prod_j K^\#(Y_j)$.

$K^\#(Y) = \int K(Y, \Phi) \, d\mu(\Phi)$, because Y_j's are disjoint and $d\mu$ is ultralocal.

③ $\sum_{\{Y_i\}} \prod_i K^\#(Y_i) = \exp\left(\sum_Y H^\#(Y)\right)$, where

$H^\#(Y) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(Y_1, \ldots, Y_n) : \cup_j Y_j = Y} \rho^T(Y_1, \ldots, Y_n) \prod_i K^\#(Y_i)$,

$\rho^T(Y_1, \ldots, Y_n) = \sum_G \prod_{\{i,j\} \in G} \left(\zeta(Y_i, Y_j) - 1\right)$, G: graph with n vertices,

$\zeta(Y_i, Y_j) = 1$ if $Y_i \cap Y_j = \phi$, $\zeta(Y_i, Y_j) = 0$ else. giving a partition $\{I_1, \ldots, I_k\}$ of $(1, \ldots, n)$

This follows from $\sum_{\{Y_i\}} \prod_i K^\#(Y_i) = 1 + \sum_n \frac{1}{n!} \sum_{(Y_1 \ldots Y_n) : Y_i \cap Y_j = \phi} \prod_i K^\#(Y_i)$,

$f(N) = \sum_{(Y_1 \ldots Y_n)} \rho^T(Y_1, \ldots, Y_n) \prod_i K^\#(Y_i)$ $= \exp\left(\sum_N \frac{1}{N!} f(N)\right)$.

Fixed point argument.

By Thm 14, for small μ, and E_k, one has.

$\varepsilon_{k+1} = L^3 \varepsilon_k + \mathcal{L}_1 E_k + \varepsilon_k^*(\lambda_k, \mu_k, E_k)$, $\mu_{k+1} = L^2 \mu_k + \mathcal{L}_2 E_k + \mu_k^*(\lambda_k, \mu_k, E_k)$, $E_{k+1} = \mathcal{L}_3 E_k + E_k^*(\lambda_k, \ldots$

We want ε_0, μ_0, s.t. $\varepsilon_N = 0$, $\mu_N = 0$. $E_0 = 0$.. Rewrite the equation

$\mu_k = L^{-2}\left(\mu_{k+1} - \mathcal{L}_2 E_k - \mu_k^*\right)$, $E_k = \mathcal{L}_3 E_{k-1} + E_{k+1}^*$.

Consider the space of sequences of parameters satisfying $\varepsilon_N = \mu_N = E_0 = 0$, with the norm

$\|\xi\| = \sup_{0 \leq k \leq N} \{\lambda_k^{-\frac{1}{2} - \beta} |\mu_k|, \lambda_k^{-\beta} \|E_k\|_{k,k}\}$. complete.

Set the map. $\xi' = T\xi$ by $\mu_k' = L^{-2}\left(\mu_{k+1} - \mathcal{L}^2 E_k - \mu_k^*\right)$, $E_k' = \mathcal{L}_3 E_{k-1} + E_{k-1}^*$.

This is contraction. $\Rightarrow$ fixed point satisfies the flow.

Large field treatment.

The RG was defined, under the small field conditions, by
$\tilde{\rho}_1^{P}(\hat{\Phi}_1) = \int \exp(-\frac{1}{2}\frac{a}{L^2}\|\tilde{\Phi}_1 - Q\Phi_0\| - S_0(\Phi_0) - V_0(\Phi_0))\,\chi_0^w(C_0^{\frac{1}{2}}(\Psi_0 - \phi_0))\,\chi_0(\Phi_0)\,d\Phi_0$.
We want $\hat{\rho}_1$ without χ_0^w, χ_0, so that $\int \rho_1(\Phi_1)d\Phi_1 = \int \rho_0(\Phi_0)d\Phi_0$.
We insert in the integral $1 = \sum_{\Omega_1} \zeta(\Omega_1^c, \Phi_0, \tilde{\Phi}_1)\,\chi(\Omega_1, \Phi_0, \tilde{\Phi}_1)$ where the sum is over regions (M-blocks) Ω_1, and $\chi(\Omega_1, \Phi_0, \tilde{\Phi})$ is the characteristic function of small field condition: $|\tilde{\Phi}_1 - Q\Phi_0| \le p_0$, $|\partial\Phi_0| \le p_0$, $|\Phi_0| \le \lambda_0^{-\frac{1}{4}} p_0$, and $p_0 = (-\log(\lambda L^{-N}))^{p}$. $\zeta(\Omega_1^c, \Phi_0, \Phi_1)$ is the characteristic function of fields which violate at least one of these conditions at some point in each $\square \subset \Omega_1^c$.
Now we define $\hat{\rho}_1$ by

$$\tilde{\rho}_1(\Phi_1) = \sum_{\Omega_1} d\Phi_{0,\Omega_1^c}\,\zeta(\Omega_1^c, \Phi_0)\exp\left(-\frac{1}{2}\frac{a}{L^2}\|\tilde{\Phi}_1 - Q\Phi_0\|^2_{\Omega_1^c} - S_0(\Omega_1^c, \Phi_0) - V_0(\Omega_1^c, \Phi_0)\right)\cdot$$
$$\left[\int d\Phi_{0,\Omega_1}\,\chi(\Omega_1, \Phi_0)\exp\left(-\frac{1}{2}\frac{a}{L^2}\|\tilde{\Phi}_1 - Q\Phi_0\|^2_{\Omega_1} - S_0(\Omega_1, \Phi_0) - V_0(\Omega_1, \Phi_0)\right)\right]$$

The integral $[\ ]$ can be analysed using the small field conditions, so
$[\ ] = \exp(-S_1(\Omega_1, \Phi_1) - V_1(\Omega_1, \Phi_1) + \sum_{X \subset \Omega_1} E_1(X, \Phi_1))$.
In each M-block $\square \subset \Omega_1^c$, the exp factor gets $e^{-O(1)p_0^2}$, so altogether it gets $e^{-O(1)p_0^2|\Omega_1^c|_M}$. This helps control over the sum over Ω_1.
We repeat this analysis taking $\Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega_k, \cdots \supset \Omega_N$.

Sigma model, the fixed point problem.

- The constraint $C(U) = V$. Write $U(x) = U'(x)V(Y_x)$, $x \in B(Y)$.
- The action becomes $A(U') = \sum_{b \in \Lambda'} \mathrm{Re}\,\mathrm{Tr}(\mathbb{1} - U'(b_-)\partial V(Y_b)U'(b_+)^*)$.
  and the constraint $\frac{1}{L^2}\sum_{x \in B(Y)} \mathrm{Im}\,U'(x) = 0$.
- Write $U'(x) = A_0(x)\mathbb{1} + i\vec{A}(x)\cdot\vec{\sigma}$, $|A| = 1$. Then the constraint is $\frac{1}{L^2}\sum_{x \in B(Y)}\vec{A}(x) = 0$
- The critical point equation is $(\mathcal{L}_X A)(U_0) = 0$, where X is on the tangent plane.
  This translates into $-\Delta\vec{A} + \partial^* v_A = R_{\vec{A}} C_A$, or $\vec{A} = (G R_{\vec{A}}^{\dagger} Q^*(Q G R_{\vec{A}}^{\dagger} Q^*)^{-1} Q - 1) G \partial^* \vec{v}_A$,
  using the block-constant vector $\vec{C}_A = (R_{\vec{A}}^{-1})^* \partial^* \vec{W}_A$.
- Write $T = (G R_{\vec{A}}^{\dagger} Q^*(Q G R_{\vec{A}}^{\dagger} Q^*)^{-1} Q - 1) G \partial^* v_{\vec{A}}$. Solve the fixed point equation $\vec{A} = T\vec{A}$ in the metric space $Y_\varepsilon = \{\vec{A} : Q(\vec{A}) = 0,\ \sup_{b \in \Lambda'}\|\partial U(b) - \mathbb{1}\| \le \varepsilon\}$.
  with the metric $d(A_1, A_2) = \|A_1 - A_2\|_\infty$, ε uniform in RG step k.
- For this, we need $\|Gf\|_\infty \le c\|f\|_\infty$, $\|(QGQ^*)^{-1} f\|_\infty \le c\|f\|$,
  c independent of k.