Towards integral perturbation of two-dimensional CFT

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This work is very much incomplete and in progress!

Constructive QFT: old and new

- (Glimm, Jaffe...) Start with the free field on the **Minkowski space**, take the interacting Hamiltonian, define the new dynamics and take a new representation of the algebras.
- (Barata, Jäkel, Mund) Start with the free field on the **de Sitter space**, take a new vacuum in the same Hilbert space, and let the modular groups generate the dynamics and a new Haag-Kastler net.

- There are many two-dimensional CFTs. One can put them on the de Sitter space easily. Can one **change the dynamics** to obtain a **Haag-Kastler net**?
- **Hopefully**. Take a primary field, add it to the Lorentz generators, let them generate a new dynamics.

This work will (possibly) contain...

- Two-dimensional non-chiral CFT.
- Charged primary fields in the CFT, the representation theory, the brading.
- Analysis of operators (**incomplete!**). Time-zero fields, generators of the Lorentz group.
- W^*-perturbation of the KMS state. Buchholz-Borchers axioms for the de Sitter space.

- Infinite volume (Minkowski) limit? Scaling limit (back to CFT)?
- Integrable perturbations of (rational) CFT?

	Const. QFT	new program	example
unperturbed	massive free field	2d CFT	U(1)-current (2d ext)
perturbation	Wick polynomial	primary field	weight α field, $\alpha < \frac{1}{\sqrt{2}}$
model	$:P(\phi):$	integrable??	??

(Algebraic) Haag's theorem

- The **Haag's theorem** roughly says that, if a QFT on the Minkowski space is unitarily equivalent to the free field at time 0, then it is unitarily equivalent to the free field for all t. $\Longrightarrow$ need to change the representation
- An algebraic version (Weiner '11): if two Haag-Kastler nets on the Minkowski space (with the split property, the Bisognano-Wichmann property and the Haag duality) have the same wedge algebras at time 0, then the two nets are the same, including the vacuum.
- This can be circumvented on the **de Sitter space** (Barata-Jäkel-Mund '13). Keep the algebras at time 0, change the dynamics and the vacuum.
- Why not start with a 2d CFT?

A general construction strategy

- A general strategy (Jäkel-Mund '18).
- Fix a Haag-Kastler net on the de Sitter space (isotony, locality, Lorentz covariance, the vacuum, the KMS property for wedges (Buchholz-Borchers '99)).
- On the same Hilbert space, construct a new representation of the Lorentz group whose restriction to rotations remains the same.
- Under certain conditions (finite speed of propergation, existence of vacuum), one can generate a new Haag-Kastler net: keep the algebras of wedges at time zero, and the rest is defined by covariance with respect to the new representation.
- Examples: $\mathcal{P}(\phi)_2$-models.

CFT as a Haag-Kastler net on de Sitter space

- Any conformal (Möb $\times$ Möb-covariant in $2d$) field theory extends to the Einstein cylinder (Guido-Longo '03)
- The de Sitter space is conformally equivalent to part of the cylinder.
- By composing these maps, any CFT can be considered as a QFT on the de Sitter space.
- Lorentz transformations are contained in the Möb $\times$ Möb group. Indeed, the spacelike rotations $r_t \times r_{-t}$ and Lorentz boosts $\delta_s \times \delta_{-s}$ generate a three-dimensional Lie group, the $(2+1)$-dimensional Lorentz group (consider the Lie algebra generated by $L_1 \otimes \mathbb{1} - \mathbb{1} \otimes L_{-1}, L_0 \otimes \mathbb{1} - \mathbb{1} \otimes L_0, L_{-1} \otimes \mathbb{1} - \mathbb{1} \otimes L_1$).
- These generators are smearing of the field $L(z) \otimes \mathbb{1} - \mathbb{1} \otimes L(z^{-1})$ by the combination of $\sin, \cos, 1$.

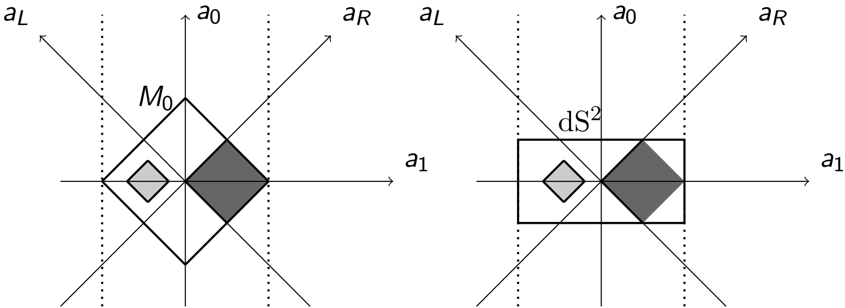

Figure: The Minkowski space M_0 and the de Sitter space dS^2 conformally embedded in $\mathbb{R}^2$. The cylinder is obtained by identifying the dotted lines. The dark grey region is a wedge W and the light grey region is a double cone.

Chiral components and primary fields in CFT

- Two-dimensional CFT has the following structure: there are **chiral components** that are QFT living on the left and right lightrays, and the whole CFT is an **extension** (in the sense close to that of Doplicher-Roberts).
- Chiral components are called conformal nets and there are many examples. They contain the Virasoro algebra $[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n}$, which extends the Möb group.
- The extension can be studied through the DHR representation theory.
- Extensions are generated by charged **primary fields** ψ, which satisfy $[L_m, \psi_n] = ((d-1)m - n)\psi_{m+n}$, where d is called the conformal dimension of ψ.
- Primary fields satisfy the so-called **braiding relations**.
- Some fields in a CFT have very specific and explicit commutation relations.

The U(1)-current 2d CFT

- Consider the derivative of the massless scalar field.
- It decomposes into the left and right chiral components: the $\mathrm{U}(1)$-current. $[J_m, J_n] = n\delta_{m+n}$.
- The current field is given by $J(z) = \sum_n z^{-n-1} J_n$.
- The $\mathrm{U}(1)$-current net admits a family of representations $\mathcal{H}_\alpha$ parametrized by $\alpha \in \mathbb{R}$, associated with the field Y_α.
- There are **two-dimensional extensions** of the tensor product of $\mathrm{U}(1)$-current nets, parametrized by α defined on $\bigoplus_{n\in\mathbb{Z}} \mathcal{H}_{n\alpha} \otimes \mathcal{H}_{n\alpha}$, (Morinelli-T.-Weiner '18).
- Each of these extensions is generated by the field $Y_\alpha \bar{Y}_\alpha$, the product of left and right charged fields.

Primary fields in the $\mathrm{U}(1)$-current algebra

- There is a primary field $Y_\alpha(z)$ acting on $\bigoplus_n \mathcal{H}_{n\alpha}$ and $Y_\alpha(z)$ shifts the charge by α.
- $E^\pm(\alpha, z) = \exp\left(\mp \sum_{n>0} \frac{\alpha J_{\pm n}}{n} z^{\mp n}\right)$
- $Y_\alpha(z) = c_\alpha E^-(z)E^+(z)z^{\alpha(0)}$, where c_α is the unitary shift $\mathcal{H}_\beta \to \mathcal{H}_{\alpha+\beta}$, $\alpha(0)$ on $\mathcal{H}_\beta$ gives $\alpha \cdot \beta$.
- The commutation relation is given by $Y_\alpha(z)Y_\beta(\zeta) = e^{i\alpha\beta}Y_\beta(\zeta)Y_\alpha(z)$ if $\arg z > \arg \zeta$.
- The factor $e^{i\alpha\beta}$ is called the braiding: $Y_\alpha(z)$, defined as a charged field of the $\mathrm{U}(1)$-current algebra, is not local.

Two-dimensional local fields

- Consider the Hilbert space $\bigoplus_n \mathcal{H}_{n\alpha} \otimes \mathcal{H}_{n\alpha}$ and the charged fields acting on the left and right components, respectively: $Y_\alpha(z)$, $\bar{Y}_\alpha(\zeta)$.
- The product $Y_\alpha(z)\bar{Y}_\alpha(\zeta)$ can be considered as a two-dimensional field.
- As the braiding relation holds, this field is **local in the two-dimensional sense**: for the spacelike separation, we have $z_1 > z_2, \zeta_1 < \zeta_2$.
- It is also relatively local with respect to the currents (both left and right).

Time-zero fields

- We would like to take the restriction of $Y_\alpha(z)\bar{Y}_\alpha(\zeta)$ to the time-zero plane $z = \frac{1}{\zeta}$, define the field $\varphi(z)$, and construct the new generator from the field $L(z) \otimes \mathbb{1} - \mathbb{1} \otimes L(z) + \varphi(z)$. Does this make sense?
- Consider the Fourier components of the charged fields $Y_{\alpha,n}$, $\bar{Y}_{\alpha,n}$.
- The restriction of the two-dimensional field to the time-zero circle corresponds to the Fourier components $\sum_n Y_{\alpha,n}\bar{Y}_{\alpha,n+m}$ for $m \in \mathbb{Z}$.
- Cf. the normal product $\sum_n Y_{\alpha,n}\bar{Y}_{\alpha,m-n}$, defined on finite energy vectors because of the positive energy condition.
- Estimates $\|Y_{\alpha,-n}\Omega\|^2 = \binom{2d+n-1}{n} \sim n^{2d-1}$ (Carpi-Kawahigashi-Longo-Weiner '18).
- For $d < \frac{1}{4}$, the Fourier components of the time-zero field makes sense as operators, because they can be applied to $\Omega \otimes \Omega$ (and other vectors).
- Moreover, $Y_\alpha(z)\bar{Y}_\alpha(1/z)$ is **commutative**.

New Lorentz generators

Lemma

*With $\lambda \in \mathbb{R}$, the Lorentz relations are **formally** satisfied for*

$$L_1 \otimes \mathbb{1} - \mathbb{1} \otimes L_{-1} + i\lambda \left(\sum_k Y_{\alpha,k} \bar{Y}_{\alpha,-1+k} + \mathrm{h.c.} \right)$$

$$L_0 \otimes \mathbb{1} - \mathbb{1} \otimes L_0$$

$$L_{-1} \otimes \mathbb{1} - \mathbb{1} \otimes L_1 - i\lambda \left(\sum_k Y_{\alpha,k} \bar{Y}_{\alpha,k+1} + \mathrm{h.c.} \right)$$

Proof.

As $Y_\alpha(z)$ is primary, it holds that $[L_m, Y_{\alpha,n}] = ((d-1)m - n)Y_{\alpha,m+n}$,

$$[L_m \otimes \mathbb{1} - \mathbb{1} \otimes L_{-m}, \sum_k Y_{\alpha,k} \bar{Y}_{\alpha,-n+k}] = ((2d-1)m - n) \sum_k Y_{\alpha,k} \bar{Y}_{\alpha,-m-n+k}$$

□

Open problems

- Are new Lorentz generators self-adjoint on a nice domain?
- Do the Lorentz relations hold as operators?
- Do the Lorentz relations extend to a group representation?
- Is there a rotation-invariant KMS state?

Further directions

- Is there a Lagrangian for this model?
- What about other CFT with charged fields?
- Cf. "integrable perturbation" by Zamolodchikov.
- The Minkowski limit? The S-matrix?
- Classifying possible interactions? "Relevant fields"?