Strong locality beyond linear energy bounds

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Constructing Haag-Kastler net from quantum field

- Wightman field $\phi(x)$ (operator-valued distribution): Poincaré/conformal covariance, positive energy, vacuum, **locality**.
- Haag-Kastler net $\{\mathcal{A}(O)\}$ (family of von Neumann algebras): Poincaré/conformal covariance, positive energy, vacuum, **locality**.
- $\mathcal{A}(O) = \{e^{i\phi(f)} : \mathrm{supp}\, f \subset O\}''$.

Strong commutativity of fields

Does locality of $\mathcal{A}$, $[\mathcal{A}(O_1), \mathcal{A}(O_2)] = \{0\}$ for O_1, O_2 spacelike separated, follow from locality of ϕ, $[\phi(f), \phi(g)] = 0$ for $\mathrm{supp}\, f, \mathrm{supp}\, g$ spacelike separated?
Yes, **in some cases**.

New examples:

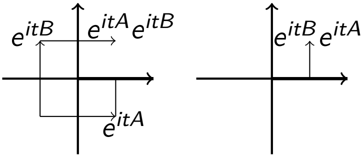
- $\mathcal{W}_3$-**algebra** with $c \geq 2$ (2d CFT) joint with S. Carpi and W. Weiner
- **Bullough-Dodd model** (2d massive integrable QFT) joint with H. Bostelmann and D. Cadamuro

Strong commutativity

Nelson's counterexample

- $L^2(X)$, where X is the Riemann surface obtained by glueing two cutted $\mathbb{R}^2$.
- $\mathscr{D}$:the set of smooth functions whose supports do not contain 0
- A be the derivative in x, B the derivative in y.

A and B commute on $\mathscr{D}$, while e^{itA} and e^{isB} are translations on X which do not commute **globally**.



When strong commutativity fails, there is a good reason.

A sufficient condition for strong commutativity

Theorem (Driessler-Fröhlich)

Let T be a positive self-adjoint operator, A, B symmetric operators on $\mathrm{Dom}(T)$ such that for $\Psi, \Phi \in \mathrm{Dom}(T)$

- $\|A\Psi\| \leq C\|T\Psi\|, \|B\Psi\| \leq C\|T\Psi\|$ *for* $\Psi \in \mathrm{Dom}(T)$.
- $|\langle A\Psi, T\Phi\rangle - \langle T\Psi, A\Phi\rangle| \leq C\|T\Psi\|\|\Phi\|$,
  $|\langle B\Psi, T\Phi\rangle - \langle T\Psi, B\Phi\rangle| \leq C\|T\Psi\|\|\Phi\|$.
- $|\langle A\Psi, T\Phi\rangle - \langle T\Psi, A\Phi\rangle| \leq C\|T^{\frac{1}{2}}\Psi\|\|T^{\frac{1}{2}}\Phi\|$,
  $|\langle B\Psi, T\Phi\rangle - \langle T\Psi, B\Phi\rangle| \leq C\|T^{\frac{1}{2}}\Psi\|\|T^{\frac{1}{2}}\Phi\|$.
- $\langle A\Psi, B\Phi\rangle = \langle B\Psi, A\Phi\rangle$

Then A and B strongly commute.

The difficult part is estimating $[H, A], [H, B]$ by T.
In Quantum Field Theory, there is a standard way: **linear energy bound**.

Linear energy bound

- ϕ a Wightman field: for each test function f, $\phi(f)$ is a symmetric operator.
- $[\phi(f), \phi(g)] = 0$ if $\operatorname{supp} f, \operatorname{supp} g$ are spacelike separated (weak locality).
- **Hamiltonian**: $[H, \phi(f)] = i\phi(f')$ (translation covariance).

Linear energy bound

$\|\phi(f)\Psi\| \leq C_f \|(H + r_f \mathbb{1})\Psi\|$ for all f.

In this case, $\|[H, \phi(f)]\Psi\| = \|\phi(f')\Psi\| \leq C_{g'} \|(H + r_{g'} \mathbb{1})\Psi\|$ and one can apply the Driessler-Fröhlich theorem with $T = H$ (Glimm-Jaffe). Many interacting scalar fields (including $\mathscr{P}(\phi)_2$ models) have a corresponding Haag-Kastler net.

Does linear energy bound fails in interesting examples?

Primary fields in 2d CFT

- $\phi(z) = \sum \phi_n z^{-n-d}$: primary (diffeomorphism covariant) field on S^1 with conformal dimension d.
- $L(z) = \sum L_n z^{-n-2}$: Virasoro algebra (Lie algebra of $\mathrm{Diff}(S^1)$).

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0},\\
[L_m, \phi_n] &= ((d-1)m-n)\phi_{m+n},
\end{aligned}
$$

(Conformal) Hamiltonian $H = L_0 = L(1)$.

- Bad news: A primary field with dimension $d > 2$ **never** satisfies linear energy bound.
- Good news: for arbitrariy f, $[\phi(f^{d-1}), L(f)] = 0$.

Can $L(f)$ be used for "local" Hamiltonian?

Local energy bounds

Theorem

A primary field ϕ with conformal dimension d can satisfy **at best** *the following bound:*

$$\|\phi_0\Psi\| \leq C\|(L_0 + r)^{d-1}\mathbb{1}\Psi\|$$

If this holds, then it satisfy the following **local energy bound***:*

$$\|\phi(f^{d-1})\Psi\| \leq \tilde{C}\|(L(f) + r\mathbb{1})^{d-1}\Psi\|$$

for **non-negative** *test function f.*

Proof: we have $U(\gamma)\phi(g)U(\gamma)^* = \phi((\gamma' \circ \gamma^{-1})^{d-1}(g \circ \gamma^{-1}))$ for test function f and $\gamma \in \mathrm{Diff}(S^1)$ and $U(\gamma)L_0U(\gamma)^* = L(\gamma' \circ \gamma^{-1}) + r_\gamma$.
$\gamma' \circ \gamma^{-1} = g$ must satisfy $\int \frac{1}{g} = 2\pi$.
To extend this to general nonnegative f, we need the optimal estimate.

Example: the $\mathcal{W}_3$-algebra

A non-Lie algebraic extension of the Virasoro algebra:

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \\
[L_m, W_n] &= (2m-n)W_{m+n}, \\
[W_m, W_n] &= \frac{c}{3\cdot 5!}(m^2-4)(m^2-1)m\delta_{m+n,0} \\
&\quad + b^2(m-n)\Lambda_{m+n} + \frac{1}{20}(m-n)(2m^2-mn+2n^2-8)L_{m+n},
\end{aligned}$$

The lowest weight representations are parametrized by $(c, h, w) \in \mathbb{C}$.
When $c \geq 2, h = w = 0$, it is **unitary** (Carpi-T.-Weiner arXiv:1910.08334)

A conformal net be associated to it if the generating fields commute strongly (**strong locality** of Vertex Operator Algebra, Carpi-Kawahigashi-Longo-Weiner '18)

The $\mathcal{W}_3$-net

- unitary vacuum representation for $c \geq 2$.
- W-field has conformal dimension 3.
- W **satisfies the optimal bound** $\|W(f^2)\Psi\| \leq C\|(L(f) + r_f \mathbb{1})^2\Psi\|$.
- $[W(f^2), L(f)] = 0$. $\Longrightarrow$ Driessler-Fröhlich theorem with $T = (L(f) + L(g) + r_{f,g})^2$ for nonnegative f, g.
- For general f, g, $C^\infty(L_0)$ is a nice core.

Theorem

The $\mathcal{W}_3$-algebra for $c \geq 2$ has an associated unitary simple VOA which is strongly local. One can construct the corresponding conformal Haag-Kastler net.

Lesson: avoid the condition on $[H, A], [H, B]$.

Wedge-local construction of massive 2d integrable QFT

- Haag-Kastler net $(\{\mathcal{A}(O)\}, U, \Omega)$: local observables $\mathcal{A}(O)$, spacetime symmetry U and the vacuum Ω.
- Wedge-algebras first (Lechner, Schroer): construct $\mathcal{A}(W_{\mathrm{R}}), U, \Omega$ from **wedge-local fields**, then take the intersection

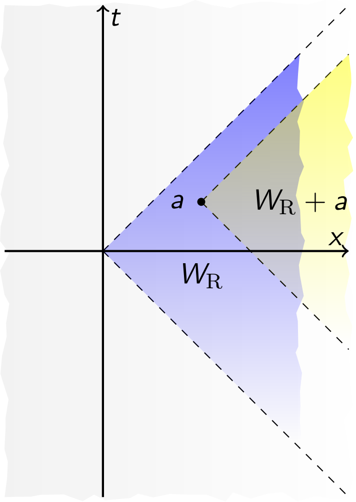
$$\mathcal{A}(D_{a,b}) = U(a)\mathcal{A}(W_{\mathrm{R}})U(a)^* \cap U(b)\mathcal{A}(W_{\mathrm{R}})'U(b)^*$$

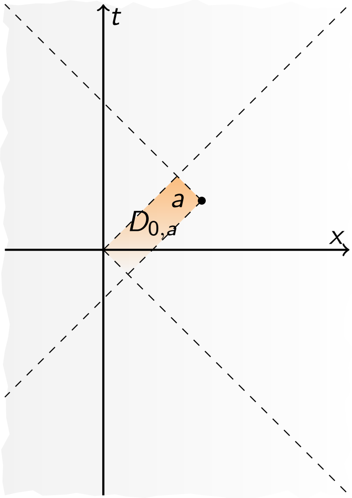
- Need to construct wedge-local observables: $\widetilde{\phi}, \widetilde{\phi}'$ such that $[e^{i\widetilde{\phi}(\xi)}, e^{i\widetilde{\phi}'(\eta)}] = 0$.
- two-particle S-matrix of the Bullough-Dodd model

$$S_\varepsilon(\theta) = \frac{\tanh\frac{1}{2}\left(\theta + \frac{2\pi i}{3}\right)}{\tanh\frac{1}{2}\left(\theta - \frac{2\pi i}{3}\right)} \cdot \frac{\tanh\frac{1}{2}\left(\theta - \frac{(1-\varepsilon)\pi}{3}\right)}{\tanh\frac{1}{2}\left(\theta + \frac{(1-\varepsilon)\pi i}{3}\right)} \frac{\tanh\frac{1}{2}\left(\theta - \frac{(1+\varepsilon)\pi i}{3}\right)}{\tanh\frac{1}{2}\left(\theta + \frac{(1+\varepsilon)\pi i}{3}\right)},$$

$0 < \varepsilon < \frac{\pi}{6}$.

Standard wedge and double cone





Wedge observables for the Bullough-Dodd S-matrix

- S-symmetric Fock space: $\mathcal{H}_1 = L^2(\mathbb{R}, d\theta)$, $\mathcal{H}_n = P_n \mathcal{H}_1^{\otimes n}$, where P_n is the projection onto S-**symmetric** functions:

$$\Psi_n(\theta_1, \cdots, \theta_n) = S(\theta_{k+1} - \theta_k)\Psi_n(\theta_1, \cdots, \theta_{k+1}, \theta_k, \cdots, \theta_n).$$

- S-symmetrized creation and annihilation operators (ZF-algebra): $z^\dagger(\xi) = Pa^\dagger(\xi)P, z(\xi) = Pa(\xi)P, P = \bigoplus_n P_n$.
- Bound state operator: $(\chi_1(\xi))\Psi_1(\theta) := \sqrt{2\pi|R|}\xi\left(\theta + \frac{\pi i}{3}\right)\Psi_1\left(\theta - \frac{\pi i}{3}\right), R = \mathrm{Res}_{\zeta=\frac{2\pi i}{3}} S(\zeta)$
- $\chi(\xi) := \bigoplus \chi_n(\xi), \qquad \chi_n(\xi) = nP_n\left(\chi_1(\xi) \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}\right) P_n$

Wedge-local fields (Lechner '03, Cadamuro-T. '16 arXiv:1502.01313)

$$\begin{aligned} \widetilde{\phi}(\xi) &:= \phi(\xi) + \chi(\xi) \qquad (= z^\dagger(\xi) + \chi(\xi) + z(\xi)), \\ \widetilde{\phi}'(\eta) &:= J\widetilde{\phi}(J_1\eta)J, \qquad \chi'(\eta) = J\chi(J_1\eta)J, \end{aligned}$$

where J is the CPT operator.

The one-particle bound state operator

- $\xi(\zeta)$: analytic in $\mathbb{R} + i(0,\pi)$, $\overline{\xi(\theta+\pi i)} = \xi(\theta)$ ("real").
- $\mathcal{H}_1 = L^2(\mathbb{R})$
- $\mathscr{D}_0 = H^2(-\frac{\pi}{3}, 0)$: L^2-analytic functions in $\mathbb{R} + i(-\frac{\pi}{3}, 0)$
- $(\chi_1(\xi))\Psi_1(\theta) := \sqrt{2\pi|R|}\xi(\theta + \frac{\pi i}{3})\Psi_1(\theta - \frac{\pi i}{3})$

What are self-adjoint extensions of $\chi_1(\xi)$? (T. arXiv:1508.06402)

- **Many extensions**: $n_\pm(\chi_1(\xi)) =$ "half of the zeros" of ξ
- **Choose** $\xi = \xi_0^2$.
  $\chi_1(\xi) := M_{\xi_+}^* \Delta_1^{\frac{1}{6}} M_{\xi_+}$ is self-adjoint and a natural extension of the above, M_{ξ_+} is unitary, $(\Delta_1^{\frac{1}{6}}\Psi_1)(\theta) = \Psi_1(\theta - \frac{\pi i}{3})$.

Strong commutativity

Note: $\chi_1(\xi) = M^*_{\xi_+}\Delta_1^{\frac{1}{6}} M_{\xi_+}$ **have different domains for different** ξ.

$$\chi(\xi) := \bigoplus \chi_n(\xi), \qquad \chi_n(\xi) = nP_n\left(\chi_1(\xi)\otimes \mathbb{1}\otimes\cdots\otimes\mathbb{1}\right)P_n$$

$\chi(\xi)+\chi'(\eta)$ **is self-adjoint**, hence...

- $\chi(\xi)+\chi'(\eta)+cN$ is self-adjoint.
- $T(\xi,\eta) := \widetilde{\phi}(\xi)+\widetilde{\phi}'(\eta)+cN$ is self-adjoint by Kato-Rellich.
- use Driessler-Fröhlich theorem (weak $\Rightarrow$ **strong commutativity**: $[e^{i\widetilde{\phi}(\xi)}, e^{i\widetilde{\phi}'(\eta)}]=0$) with $T(\xi,\eta)$ as the reference operator.
- $\Longrightarrow$ wedge-local observables ($\Longrightarrow$ Haag-Kastler net)