Construction of two-dimensional quantum field models through Longo-Witten endomorphisms

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Constructing QFT

What is quantum field theory? (cf. quantum mechanics, classical field)

- Lagrangian approach: perturbation theory is divergent.
- Need to compute n-point (Wightman) functions $\Longrightarrow$ S-matrix.
- Hilbert space and **local observables** can be reconstructed.

Form factor programme (Babujian, Karowski, Smirnov...)

- (Factorizing) S-matrix is conjectured from general requirements.
- Form factors ${}^{\mathrm{out}}\langle q_1, \cdots, q_m | O(x) | p_1, \cdots, p_n \rangle^{\mathrm{in}}$ are obtained.
- n-point functions $\langle 0 | O(x) O(0) | 0 \rangle =$ $\sum_n \int dp_1 \cdots dp_n \, \langle 0 | O(x) | p_1, \cdots, p_n \rangle^{\mathrm{in}} \, {}^{\mathrm{in}}\langle p_1, \cdots, p_n | O(0) | 0 \rangle$.
- Convergence? Locality in e.g. the sine-Gordon model?

von Neumann algebraic approach (Schroer, Lechner, T.,...)

Find simpler observables **in wedges**, then local observables **abstractly**!

Algebraic QFT

Haag-Kastler axioms

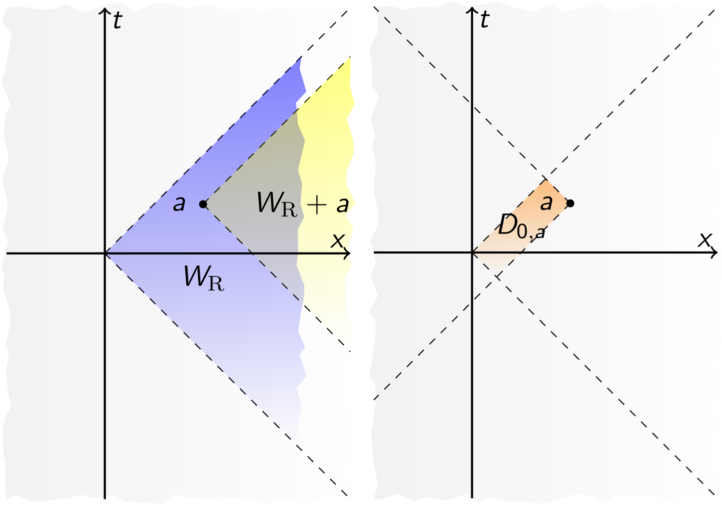
- Concerned with algebras of observables $\mathcal{A}(O)$ in spacetime regions O.
- **Isotony**, locality, Poincaré covariance, positivity of energy, existence of vacuum
- ϕ: quantum (Wightman) field $\Longrightarrow \mathcal{A}(O) := \overline{\{e^{i\phi(f)} : \mathrm{supp}\, f \subset O\}}^{\mathrm{vN}}$
- Examples: $P(\phi)_2$, Yukawa$_2$, $\phi^4_3, \cdots$ (many other models in the Wightman axioms, Haag-Kastler net? c.f. Glimm-Jaffe)

Interacting QFT is difficult because pointlike field $\phi(x)$ is complicated...
Isotony: $O_1 \subset O_2 \Longrightarrow \mathcal{A}(O_1) \subset \mathcal{A}(O_2)$ means that **larger** regions contain **more** observables, also **simpler** ones. **Wedge**: $W_{\mathrm{R}} := \{(t, x) : x > |t|\}$.

A strategy for constructing Haag-Kastler nets

- Construct observables in W_{R}
- Local observables by $\mathcal{A}(D_{a,b}) = \mathcal{A}(W_{\mathrm{R}} + a) \cap \mathcal{A}(W_{\mathrm{L}} + b)$

Standard wedge and double cone



Wedge-observables in integrable models (Schroer, Lechner)

- **analytic** S-matrix (e.g. the sinh-Gordon model) $S : \mathbb{R} + i(0,\pi) \to \mathbb{C}$,
  $$\overline{S(\theta)} = S(\theta)^{-1} = S(-\theta) = S(\theta + \pi i), \quad \theta \in \mathbb{R}.$$
- S-symmetric Fock space: $\mathcal{H}_1 = L^2(\mathbb{R}, d\theta)$, $\mathcal{H}_n = P_n \mathcal{H}_1^{\otimes n}$, where P_n is the projection onto S-symmetric functions:
  $\Psi_n(\theta_1, \cdots, \theta_n) = S(\theta_{k+1} - \theta_k)\Psi_n(\theta_1, \cdots, \theta_{k+1}, \theta_k, \cdots, \theta_n)$.
- Zamolodchikov-Faddeev algebra: S-symmetrized creation and annihilation operators $z^\dagger(\xi) = Pa^\dagger(\xi)P, z(\xi) = Pa(\xi)P, P = \bigoplus_n P_n$.
- **Wedge-local field**: with J_1: CPT, $f^\pm(\theta) = \int dx e^{\pm ix\cdot p(\theta)} f(x)$,
  $$\phi(f) = z^\dagger(f^+) + z(J_1 f^-).$$

Wedge-localization (Lechner '03, Bostelmann-Cadamuro '15)

If $\mathrm{supp}\, f, \mathrm{supp}\, g \subset W_\mathrm{R}$, then $[e^{i\phi(f)}, e^{iJ\phi(g)J}] = 0$. $\phi(f)$ formally commutes with observables coming from form factors localized at 0.

Local observables

- Define $\mathcal{A}(W_{\mathrm{R}} + a) := \overline{\{e^{i\phi(f)} : \mathrm{supp}\, f \subset W_{\mathrm{R}} + a\}}^{\mathrm{vN}}$, $\mathcal{A}(W_{\mathrm{L}} + b) := \overline{\{e^{iJ\phi(f)J} : \mathrm{supp}\, f \subset W_{\mathrm{R}} - b\}}^{\mathrm{vN}}$.
- Question: is there any nontrivial observable in $\mathcal{A}(D_{a,b}) = \mathcal{A}(W_{\mathrm{R}} + a) \cap \mathcal{A}(W_{\mathrm{L}} + b)$?
- Let $\Delta^{\frac{1}{4}} = U(\Lambda(\frac{\pi i}{2}))$ (imaginary Lorentz boost), consider the map

$$\mathcal{A}(W_{\mathrm{R}}) \ni x \longmapsto \Delta^{\frac{1}{4}} U(a) x\Omega,$$

  $a \in W_{\mathrm{R}}$. If this is a **nuclear** map (approximated "well" by finite matrices), then there are **many** local observables (Buchhoz-D'Antoni-Longo '90).

Theorem (Lechner '08)

If S is analytic, satisfies a regularity condition and and $S(0) = -1$, the map above is indeed nuclear for sufficiently large a, therefore, there are local observables in $\mathcal{A}(D_{a,0})$ for such a, and hence a **Haag-Kastler net**.

S-matrices with poles (bound states)

If S has a pole (e.g. the Bullough-Dodd model, the sine-Gordon model), $\phi(f) = z^\dagger(f^+) + z(J_1 f^-)$ is **no longer wedge-local**.

S: scalar, poles at $\theta = \frac{\pi i}{3}, \frac{2\pi i}{3}$, $S(\theta) = S\left(\theta + \frac{\pi i}{3}\right) S\left(\theta - \frac{\pi i}{3}\right)$

P_n: S-symmetrization, $\mathcal{H} = \bigoplus P_n \mathcal{H}_1^{\otimes n}$, $\mathcal{H}_1 = L^2(\mathbb{R})$,

$$(\chi_1(f))\xi(\theta) := \sqrt{2\pi|R|} f^+\left(\theta + \frac{\pi i}{3}\right) \xi\left(\theta - \frac{\pi i}{3}\right),$$

$$\chi_n(f) := nP_n\left(\chi_1(f) \otimes I \otimes \cdots \otimes I\right) P_n, \quad \chi(f) := \bigoplus \chi_n(f).$$

Theorem (Cadamuro-T. arXiv:1502.01313, CMP)

$\widetilde{\phi}(f) := \phi(f) + \chi(f)$ *commute weakly with* $\widetilde{\phi}'(g)$ *on a dense domain.*

Problem: self-adjointness (problem of domain)! ($\Longrightarrow$ Haag-Kastler net)
Work in progress: A_n-affine Toda, sine-Gordon...

Twisting the massive free field

Let ϕ be the massive **complex** free field, with the charge operator Q, and $\mathcal{M} = \overline{\{e^{i\phi(f)} : \mathrm{supp}\, f \subset W_{\mathrm{R}}\}}^{\mathrm{vN}}$ be the wedge-algebra. Take $\mathcal{H} \otimes \mathcal{H}$.

Theorem (T. arXiv:1301.6090, FOMS)

From the algebra generated by $\mathcal{M} \otimes \mathbb{C}\mathbb{1}$ and $e^{tQ\otimes Q}(\mathbb{C}\mathbb{1} \otimes \mathcal{M})e^{-tQ\otimes Q}$, one can obtain an interacting Haag-Kastler net, $t \in \mathbb{R} \setminus 2\pi\mathbb{Z}$. Federbush?

Proof of wedge-localization: $e^{sQ} \cdot e^{-sQ}$ is an **automorphism** of $\mathcal{M}$.
A variation: Let ϕ be the massive **real** free field, φ an inner symmetric function, $\mathcal{M}$ the wedge-algebra. One can construct an operator $\tilde{R}_\varphi$:

Theorem (T., Alazzawi-Lechner '17)

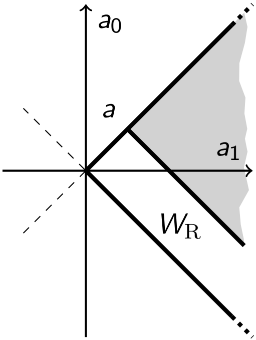
From the algebra generated by $\mathcal{M} \otimes \mathbb{C}\mathbb{1}$ and $\tilde{R}_\varphi(\mathbb{C}\mathbb{1} \otimes \mathcal{M})\tilde{R}_\varphi^*$, one can obtain an interacting Haag-Kastler net.

Proof of wedge-localization: $\Gamma(\varphi(P_1)) \cdot \Gamma(\varphi(P_1))^*$ implements an **endomorphism** of $\mathcal{M}$. $\tilde{R}_\varphi \cdot \tilde{R}_\varphi^*$ maps $\mathcal{M} \otimes \mathbb{C}\mathbb{1}$ into $\mathcal{M} \otimes \mathcal{B}(\mathcal{H})$.

Relations to CFT?

ϕ: massive free field, $\mathcal{M}$: the wedge algebra.

- One obtaines a CFT (the Heisenberg algebra) by restricting to the lightray.
- $\Gamma(\varphi(P_1))$ implements a **Longo-Witten endomorphism**: $\Gamma(\varphi(P_1))\mathcal{M}\Gamma(\varphi(P_1))^* \subset \mathcal{M}$ and $\Gamma(\varphi(P_1))$ commutes with the lightlike translations.



Take an interacting (integrable) QFT.
Consider the wedge algebra $\mathcal{M}$ and the restriction to the lightray $a \in W_{\mathrm{R}}$.
Question: How large is the lightlike intersection $\mathcal{M} \cap U(a)\mathcal{M}'U(a)^*$?
If this is nontrivial, one obtains a chiral component of a CFT (Guido-Longo-Wiesbrock '98).
Integrable perturbation of CFT?

Summary and open problems

- Some integrable QFT, including the sinh-Gordon model and other models with diagonal S-matrices (with CDD facctors), have been constructed in a mathematically satisfactory way.
- Some of them can be realized on the same Hilbert space as the free field, by twisting the observables in wedges, thus by "perturbing" the Heisenberg algebra (CFT).

- Complete the proof of modular nuclearity for nondiagonal S-matrices.
- Self-adjointness of the bound state operators ($\Longrightarrow$ models with bound states).
- Study the lightlike intersection ($\Longrightarrow$ CFT from integrable models?).
- Wedge-algebras by free product (Longo-T.-Ueda '17), local observables?
- Nets on the de Sitter spacetime (cf. Barata-Jäkel-Mund), conserved charges, proof of integrability?