Construction of wedge-local nets of observables through Longo-Witten endomorphisms

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Introduction

Historical problem

- Long standing open problem: interacting models in 4-dim
- Construction of nets of von Neumann algebras

Recent progress:

- Wedge-local net in 2-dim, based on a single von Neumann algebra and the modular theory (Borchers '92)
- Factorizing S-matrix models (Lechner '08).

Present approach:

- Chiral conformal net on S^1: many examples
- Endomorphisms of the half-line algebra (Longo-Witten '11)

Main result

Interacting massless wedge-local nets in 2-dim

Wedge-local nets (Borchers '92)

- Local net: von Neumann algebras $\mathcal{A}(O)$ parametrized by open regions O
- Wedge-local net: a **single** von Neumann algebra $\mathcal{M}$ acted on by spacetime translations

Definition

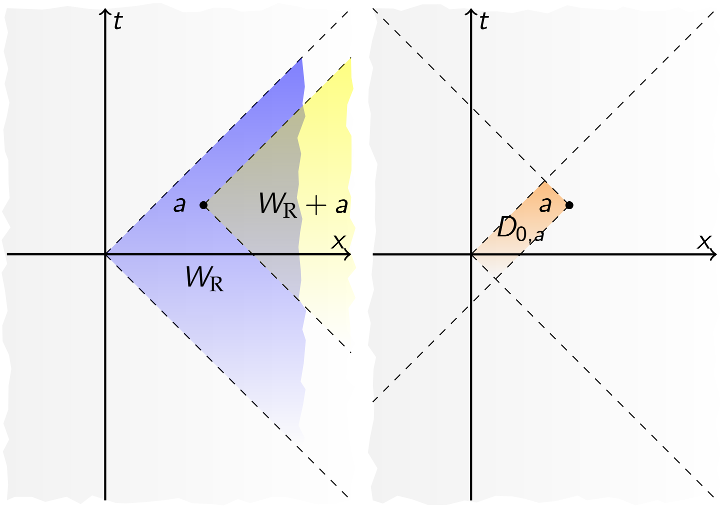
$\mathcal{M}$: vN algebra, T: positive-energy rep of $\mathbb{R}^2$, Ω: vector, is a wedge-local net on 2-dim if Ω is cyclic and separating for $\mathcal{M}$ and

- $\mathrm{Ad}\, T(a)(\mathcal{M}) \subset \mathcal{M}$ for $a \in W_{\mathrm{R}}$, $T(a)\Omega = \Omega$

Correspondence: $\mathcal{A}(W_{\mathrm{R}}) \Leftrightarrow \mathcal{M}$, where $W_{\mathrm{R}} := \{a = (a_0, a_1) : |a_0| < a_1\}$.

examples

- Factorizing S-matrix models (Lechner '06)
- Deformations (Buchholz-Lechner-Summers '10, Dybalski-T. '11, Lechner '11, etc.)



t
a
$W_{\mathrm{R}} + a$
x
W_{R}
t
a
$D_{0,a}$
x

Scattering theory (Buchholz '75, Dybalski-T. '11)

Let $(\mathcal{M}, T, \Omega)$ be a wedge-local net. We define **asymptotic fields**

$$\Phi_+^{\mathrm{out}}(x) := \operatorname*{s-lim}_{\mathcal{T}\to\infty} \int dt\, h_{\mathcal{T}}(t)\mathrm{Ad}\,U(t,t)(x),$$

where $h_{\mathcal{T}}$ has support around $\mathcal{T}$ and $\int dt\, h_{\mathcal{T}}(t) = 1$. Similarly we define $\Phi_\pm^{\mathrm{in}}$. Note that if $x \in \mathcal{M}$ then, $\Phi_+^{\mathrm{out}}(x)$ and $\Phi_-^{\mathrm{in}}(x)$ stays in $\mathcal{M}$.

- The net is **asymptotically complete** if $\Phi_+^{\mathrm{out}}(x)\Phi_-^{\mathrm{out}}(y)\Omega$ spans $\mathcal{H}$.
- The net is **interacting** if $S \neq \mathbb{1}$ where

$$S \cdot \Phi_+^{\mathrm{out}}(x)\Phi_-^{\mathrm{out}}(y)\Omega = \Phi_+^{\mathrm{in}}(x)\Phi_-^{\mathrm{in}}(y)\Omega.$$

Main problem

Construct interacting wedge-local nets.

Chiral conformal net

Definition

A **conformal net** on S^1 is a map $\mathcal{A}$ from the set of intervals in S^1 into the set of von Neumann algebras on $\mathcal{H}$ which satisfies

- Isotony: $I \subset J \Rightarrow \mathcal{A}(I) \subset \mathcal{A}(J)$.
- Locality: $I \cap J \Rightarrow [\mathcal{A}(I), \mathcal{A}(J)] = 0$.
- Möbius covariance: $\exists U$: positive energy rep of $\mathrm{PSL}(2, \mathbb{R})$ such that $\mathrm{Ad}U(g)\mathcal{A}(I) = \mathcal{A}(gI)$.
- Vacuum: $\exists \Omega$ such that $U(g)\Omega = \Omega$ and cyclic for $\mathcal{A}(I)$.

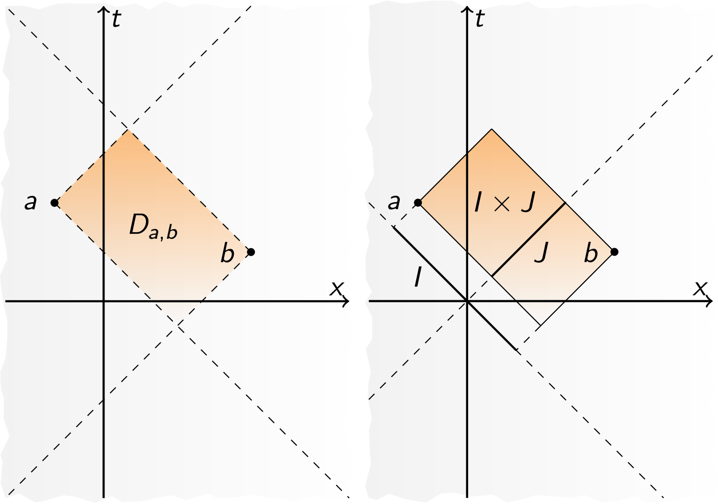
Many examples: $U(1)$*-current* (free massless boson), *Free massless fermion*, Virasoro nets (stress energy tensor), Loop group nets (noncommutative currents).
In the present work, important are the $U(1)$-current net and the free massless fermion which admit the **Fock space structure**.

Chiral CFT in 2 dimensions

Although there exist a plenty of conformal nets on S^1, there is no notion of *interaction* for one-dimensional theory.
However, it is easy to construct a (noninteracting) two-dimensional net from a *pair* of nets on S^1.
For two nets $\mathcal{A}_+, \mathcal{A}_-$ on S^1, we define

- a **chiral** net on $\mathbb{R}^2$: $\mathcal{A}(I \times J) := \mathcal{A}_+(I) \otimes \mathcal{A}_-(J)$
- a representation $U = U_+ \otimes U_-$ of $\mathrm{PSL}(2,\mathbb{R}) \otimes \mathrm{PSL}(2,\mathbb{R}) \supset \mathcal{P}_+^\uparrow$,
- the vacuum $\Omega = \Omega_+ \otimes \Omega_-$

A chiral net $\mathcal{A}$ is *asymptotically complete*, but *not* interacting (Dybalski-T. '11). Such nets with a simple tensor product structure can be considered as free theory in 2 dimensions.


t
a
$D_{a,b}$
b
x
t
a
$I \times J$
J
b
I
x

General structure of asymptotically complete nets (T. '11)

If $(\mathcal{M}, T, \Omega)$ is an *asymptotically complete* wedge-local net with S-matrix S, then

- $\mathcal{H} = \mathcal{H}_+ \otimes \mathcal{H}_-$
- $\Phi_+^{\mathrm{out}}(x)$ and $\Phi_-^{\mathrm{out}}(y)$ act like operators respectively on $\mathcal{H}_+$ and $\mathcal{H}_-$, hence they generate two nets $\mathcal{A}_\pm^{\mathrm{out}}$ on $\mathbb{R}$ (if the net is *strictly local*, they are indeed conformal nets)
- $\mathcal{M} = \{x \otimes \mathbb{1}, S(\mathbb{1} \otimes y)S^* : x \in \mathcal{A}_+^{\mathrm{out}}(\mathbb{R}_-), y \in \mathcal{A}_-^{\mathrm{out}}(\mathbb{R}_+)\}''$

Furthermore, for the modular objects we have

- $\Delta = \Delta^{\mathrm{out}} = \Delta_+^{\mathrm{out}} \otimes \Delta_-^{\mathrm{out}}$
- $J = SJ^{\mathrm{out}} = S \cdot J_+^{\mathrm{out}} \otimes J_-^{\mathrm{out}}$

In other words, any interacting asymptotically complete net is a *twisting* of a chiral CFT by an operator S.
The search for interacting nets is reduced to the search for appropriate S.

Example: the $U(1)$-current net

The Weyl algebra

$$W(f)W(g) = \exp\left(-\frac{i}{2}\int f(t)g'(t)dt\right) W(f+g),$$

where $f, g \in C^\infty(\mathbb{R}, \mathbb{R})$ admits the vacuum representation π_0 on the Fock space $F(L^2(\mathbb{R}, dp))$ (there is a representation U of $\mathrm{M\ddot{o}b}$ which renders W covariant). One defines the $U(1)$-**current net** by

$$\mathcal{A}^{(0)}(I) = \{\pi_0(W(f)) : \mathrm{supp} f \subset I\}'',$$

where we identified $\mathbb{R}$ with $S^1 \setminus \{-1\}$.

- The free massless bosonic field on 2 dimensions decomposes into the tensor product of two copies of the $U(1)$-current net.
- The Fock space structure allows to consider particle number.
- For a unitary V_1 on $\mathrm{L}^2(\mathbb{R}, dp)$, one defines the second quantization $\Gamma(V_1) = \mathbb{1} \oplus V_1 \oplus (V_1 \otimes V_1) \otimes \cdots$

Endomorphisms of $\mathcal{A}^{(0)}(\mathbb{R}_+)$

Definition

A Longo-Witten endomorphism of a net $\mathcal{A}$ on S^1 is an endomorphism of $\mathcal{A}(\mathbb{R}_+)$ implemented by a unitary V commuting with translation $T(t)$.

Simplest examples: $\mathrm{Ad}\, T(s)$ for $s \geq 0$, inner symmetry (automorphism which preserves each local algebra $\mathcal{A}(I)$ and the vacuum state)

An **inner symmetric** function φ is the boundary value of a bounded analytic function on the upper-half plane with $|\varphi(p)| = 1, \varphi(p) = \overline{\varphi(-p)}$ for $p \in \mathbb{R}$. Example: $\varphi(p) = e^{i\kappa p}$ with $\kappa \geq 0$, $\frac{p-i\kappa}{p+i\kappa}$ with $\kappa > 0$

Theorem (Longo-Witten '11)

$\mathcal{A}^{(0)}$: the $U(1)$-current net
$V_\varphi := \Gamma(\varphi(P_1))$ implements a Longo-Witten endomorphism of $\mathcal{A}^{(0)}$, where P_1 is the generator of the translation on the one-particle space.

Construction of wedge-local nets

As we know the general structure of wedge-local nets, we only have to construct the scattering operator. For an inner symmetric function φ, set

- $\mathcal{H}^n := \mathcal{H}_1^{\otimes n}$
- $P_{i,j}^{m,n} := (\mathbb{1} \otimes \cdots \otimes \underset{i\text{-th}}{P_1} \otimes \cdots \otimes \mathbb{1}) \otimes (\mathbb{1} \otimes \cdots \otimes \underset{j\text{-th}}{P_1} \otimes \cdots \otimes \mathbb{1})$, acting on $\mathcal{H}^m \otimes \mathcal{H}^n$, $1 \leq i \leq m$ and $1 \leq j \leq n$.
- $\varphi_{i,j}^{m,n} := \varphi(P_{i,j}^{m,n})$ (functional calculus on $\mathcal{H}^m \otimes \mathcal{H}^n$).
- $S_\varphi := \bigoplus_{m,n} \prod_{i,j} \varphi_{i,j}^{m,n}$

We can take the spectral decomposition of S_φ only with respect to the right component:

$$S_\varphi = \bigoplus_n \int \prod_j \Gamma(\varphi(p_j P_1)) \otimes dE_1(p_1) \otimes \cdots \otimes dE_1(p_n)$$

Note that the integrand is a unitary operator which implements a Longo-Witten endomorphism for any value of $p_j \geq 0$.

Construction of wedge-local nets

We set

- $\mathcal{M}_\varphi := \{x \otimes \mathbb{1}, S_\varphi(\mathbb{1} \otimes y)S_\varphi^* : x \in \mathcal{A}^{(0)}(\mathbb{R}_-), y \in \mathcal{A}^{(0)}(\mathbb{R}_+)\}''$
- $T := T_0 \otimes T_0$
- $\Omega := \Omega_0 \otimes \Omega_0$

Theorem (T. '11)

$(\mathcal{M}_\varphi, T, \Omega)$ *is an asymptotically complete wedge-local net with the S-matrix* S_φ.

Proof) To see that it is a wedge-local net, what is nontrivial is the separating property of Ω. We set

$$\mathcal{M}_\varphi^1 := \{S_\varphi(x \otimes \mathbb{1})S_\varphi^*, \mathbb{1} \otimes y : x \in \mathcal{A}^{(0)}(\mathbb{R}_+), y \in \mathcal{A}^{(0)}(\mathbb{R}_-)\}''$$

$\mathcal{M}_\varphi$ and $\mathcal{M}_\varphi^1$ commute since

$$S_\varphi(x \otimes \mathbb{1})S_\varphi^* = \bigoplus_n \int \mathrm{Ad}\left(\prod_j \Gamma(\varphi(p_j P_1))\right)(x) \otimes dE_1(p_1) \otimes \cdots dE_1(p_n).$$

Example: the complex free massless fermion

- $\mathcal{H}_1 := L^2(S^1)$ with the complex structure $I \cdot e_n = \mathrm{sign}(n) i e_n$
- P: the projection onto the space spanned by $\{e_n : n \geq 0\}$

On the fermionic Fock space $F(\mathcal{H}_1)$, for $f \in \mathcal{H}_1$, one defines

$$A(f)v = f \wedge v, \ \ \psi(f) = A(Pf) + A(P^{\perp} f)^*,$$

and the **complex free fermionic net**

$$\mathcal{F}(I) := \{\psi(f) : \mathrm{supp} f \subset I\}'',$$

which is a graded-local.
For $j =: \psi^* \psi :$, $W(f) = e^{ij(f)}$ satisfies the Weyl relation, hence $\mathcal{F}$ contains the $U(1)$-current net as a subnet. There is an action of $U(1)$ on $\mathcal{F}$ determined by $\alpha_z(\psi(f)) = \psi(zf)$.

Fact

The $U(1)$-current subnet $\mathcal{A}^{(0)}$ is the fixed point of $\mathcal{F}$ with respect to α.

Endomorphisms on $\mathcal{F}$

The one-particle space $\mathcal{H}_1$ has multiplicity 2 as the (projective) representation space of $\mathrm{M\ddot{o}b}$, $P\mathcal{H}_1 \oplus P^\perp\mathcal{H}_1$.
Let φ be an inner (not necessarily symmetric) function. As an application of Longo-Witten '11, one sees that the unitary operator

$$\tilde{\varphi}(P,+) := \begin{pmatrix} \varphi(P) & 0 \\ 0 & \check{\varphi}(P) \end{pmatrix}, \quad \tilde{\varphi}(P,-) := \begin{pmatrix} \check{\varphi}(P) & 0 \\ 0 & \varphi(P) \end{pmatrix}$$

implements an endomorphism of $L^2(\mathbb{R}_+)$, hence the second quantization $\Gamma(\tilde{\varphi}(P,\pm))$ implements a Longo-Witten endomorphism of $\mathcal{F}$.

Theorem (Bischoff-T., in preparation)

$\Gamma(\tilde{\varphi}(P,\pm))$ *commutes with the gauge action. Hence the endomorphism* $\mathrm{Ad}\Gamma(\tilde{\varphi}(P,\pm))$ *restricts to the fixed point subnet* $\mathcal{A}^{(0)}$ *and it is not implemented by any second quantization operator for a generic* φ.

Wedge-local nets with particle production

We set

$$S_\varphi := \bigoplus_n \int \prod_j \Gamma(\widetilde{\varphi}(p_j P_1, \iota)) \otimes dE_1(p_1, \iota) \otimes \cdots \otimes dE_1(p_n, \iota)$$

- $\mathcal{N}_\varphi := \{x \otimes \mathbb{1}, S_\varphi(\mathbb{1} \otimes y)S_\varphi^* : x \in \mathcal{A}^{(0)}(\mathbb{R}_-), y \in \mathcal{A}^{(0)}(\mathbb{R}_+)\}''$
- $T := T_0 \otimes T_0$
- $\Omega := \Omega_0 \otimes \Omega_0$

Theorem (Bischoff-T., in preparation)

$(\mathcal{N}_\varphi|_{\mathcal{H}^\alpha}, T|_{\mathcal{H}^\alpha}, \Omega)$ *is an asymptotically complete wedge-local net with the S-matrix* $S_\varphi|_{\mathcal{H}^\alpha}$*, with the asymptotic algebra* $\mathcal{A}^{(0)} \otimes \mathcal{A}^{(0)}$.
The space $\mathcal{H}_1 \otimes \mathcal{H}_1$ *is* not *preserved by* S_φ *for a generic* φ*, hence it represents* particle production.

Conclusion

Summary

- General structure of two-dimensional massless asymptotically complete nets
- New Longo-Witten type endomorphisms on the $U(1)$-current net
- Interacting wedge-local nets, some with particle production

Open problems

- Further examples with different asymptotic algebra
- Strict locality
- Massive analogue