Deformation of chiral conformal field theory

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**Introduction: Quantum field theory**

- Quantum Field Theory treats infinite particles.
- Physicists calculate observable quantities by perturbation.

**Main problem**

No **interacting** consistent model in 4 spacetime dimensions.

- Different approaches: Operator-valued distributions, \( n \)-point functions, **von Neumann algebraic**.
- Recently, examples in 2 dimensions have been constructed by algebraic approach (Lechner ‘08).

**Main results**

- New candidate of interacting examples in 2 dimensions
- First steps towards classification
We consider a **family of von Neumann algebras** \( \{ \mathcal{A}(O) \} \) parametrized by bounded regions \( \{ O \} \): “physical quantities measured in \( O \”).

**Definition**

A **local net** \( \mathcal{A} \) of observables is a map \( \mathbb{R}^2 \ni O \mapsto \mathcal{A}(O) \subset B(\mathcal{H}) \) such that

- \( O_1 \subset O_2 \Rightarrow \mathcal{A}(O_1) \subset \mathcal{A}(O_2) \)
- \( O_1 \) and \( O_2 \) are causally disjoint \( \Rightarrow [\mathcal{A}(O_1), \mathcal{A}(O_2)] = 0 \)
- \( \exists \) rep \( U \) of Poincaré group \( \mathcal{P}_+ \) such that \( U(g)\mathcal{A}(O)U(g)^* = \mathcal{A}(gO) \)
- \( U|_{\mathbb{R}^2} \) has joint spectrum contained in \( V_+ \)
- \( \exists \Omega \in \mathcal{H} \) such that \( U(g)\Omega = \Omega \), and cyclic for \( \mathcal{A}(O) \)

- Possible to define **interaction**
- Difficult to **construct** interacting examples
The standard right wedge \( W_R := \{ a = (a_0, a_1) : |a_0| < a_1 \} \).

**Definition**

A **wedge-local net** (Borchers triple) is a triple of a single von Neumann algebra \( \mathcal{M} \), a rep \( U \) of \( \mathbb{R}^2 \) with spectrum condition and \( \Omega \) such that

- \( \mathcal{M}(a) := U(a)\mathcal{M}U(a)^* \subset \mathcal{M} \) for \( a \in W_R \)
- \( U(a)\Omega = \Omega \)

**Example**

Let \( (\mathcal{A}, U, \Omega) \) be a local net. Then \( (\mathcal{A}(W_R), U, \Omega) \) is a wedge-local net.

If \( (\mathcal{M}, U, \Omega) \) is a wedge-local net, then \( U \) extends to \( \mathcal{P}^+_\uparrow \) and \( (\mathcal{A}, U, \Omega) \) where \( \mathcal{A}(D_{a,b}) := \mathcal{M}(a) \cap \mathcal{M}(b) \) satisfies the condition of local net **except the cyclicity of** \( \Omega \).
We define **asymptotic fields**

\[
\Phi_{\pm}^{\text{out}}(x) := \text{s-lim}_{T \to \infty} \int dt \, h_T(t) \text{Ad}U(t, \pm t)(x),
\]

where \( h_T \) has support around \( T \) and \( \int dt \, h_T(t) = 1 \). Similarly we define \( \Phi_{\pm}^{\text{in}} \).

The **S-matrix** is defined by:

\[
S \cdot \Phi_{+}^{\text{out}}(x) \Phi_{-}^{\text{out}}(y) = \Phi_{+}^{\text{in}}(x) \Phi_{-}^{\text{in}}(y).
\]

If \( S \) is not a multiple of \( \mathbb{I} \), then we say the net \( \mathcal{A} \) is **interacting**.
Problems

- To construct interacting local net
- To show that the intersection is “sufficiently large”

Results

- Construct several families of interacting wedge-local nets
- Explicit calculation of intersection of some of such families
One considers a net $\mathcal{A}_0$ on $S^1 = \mathbb{R} \cap \{\infty\}$, \textbf{without} the notion of interaction, with Möbius $= \text{PSL}(2, \mathbb{R})$ symmetry $U_0$ and $\Omega_0$.  

$\mathcal{A}_0$ assigns to each interval $I \subset \mathbb{R}$ a von Neumann algebra $\mathcal{A}_0(I)$ satisfying several conditions.

With $\mathcal{A}_0$, one can construct a two-dimensional net. 

A \textbf{chiral conformal net} with a component $\mathcal{A}_0$ is defined as follows $\mathcal{A}(I \times J) = \mathcal{A}_0(I) \otimes \mathcal{A}_0(J)$, a representation $U = U_0 \otimes U_0$ of $\text{PSL}(2, \mathbb{R}) \otimes \text{PSL}(2, \mathbb{R}) \supset \mathcal{P}^+_+$, $\Omega = \Omega_0 \otimes \Omega_0$. A chiral net $\mathcal{A}$ is \textbf{not} interacting.

It is possible to construct from $\mathcal{A}$ an \textbf{interacting} wedge-local net (Buchholz-Lechner-Summers ‘10).
Deformation w.r.t. translation

- \((\mathcal{A}_0, U_0, \Omega_0)\): a net on \(S^1\), \(U_0(\tau(t)) = e^{itP_0}\): translation

**Theorem**

For \(\kappa \geq 0\), the triple

- \(\mathcal{M} := \{\text{Ad} e^{i\kappa P_0 \otimes P_0} (x \otimes 1), 1 \otimes y : x \in \mathcal{A}(\mathbb{R}_+), y \in \mathcal{A}(\mathbb{R}_-)\}\)"
- \(U := U_0 \otimes U_0,\)
- \(\Omega = \Omega_0 \otimes \Omega_0\)

is an **interacting** wedge-local net with S-matrix \(S = e^{i2\kappa P_0 \otimes P_0}\). It coincides with the BLS-deformation.

**Proof:** Use \(e^{i\kappa P_0 \otimes P_0} = \int e^{i\kappa tP_0} \otimes dE_0(t)\) to show that \(\Omega\) is separating for \(\mathcal{M}\). Note that \(\text{Ad} e^{i\kappa tP_0}\) is an **endomorphism** of \(\mathcal{A}_0(\mathbb{R}_+)\). We have \(\text{Ad} e^{i\kappa P_0 \otimes P_0} (x \otimes 1) = \int \text{Ad} e^{i\kappa tP_0} (x) \otimes dE_0(t)\), which commutes with \(z \otimes 1, z \in \mathcal{A}_0(\mathbb{R}_-)\).
Deformation w.r.t. inner symmetry

- \((A_0, U_0, \Omega_0)\): a net on \(S^1\).
- Let \(e^{isQ_0}, s \in \mathbb{R}\) be a periodic inner symmetry:
  \[Ade^{isQ_0}(A_0(I)) = A_0(I), \quad e^{isQ_0}\Omega_0 = \Omega_0.\]

**Theorem**

**The triple**

- \(M := \{Ade^{ikQ_0}Q_0(x \otimes 1), 1 \otimes y : x \in \mathcal{A}(\mathbb{R}_+), y \in \mathcal{A}(\mathbb{R}_-)\}''\),
- \(U := U_0 \otimes U_0\),
- \(\Omega = \Omega_0 \otimes \Omega_0\)

is a **interacting** wedge-local net with \(S\)-matrix \(S = e^{i2kQ_0}Q_0\).

It is possible to determine the strictly local elements.
Theorem

If $\alpha_s = A \text{e}^{isQ_0}$ is periodic and $A_0$ satisfies a technical condition, then it holds that

$$A(D_{a,b}) := \mathcal{M}_{Q_0,\kappa}(a) \cap \mathcal{M}_{Q_0,\kappa}(b)' = A_0(I)^{\alpha_k} \otimes A_0(J)^{\alpha_k}.$$ 

If we let $A_0(I)^{\alpha_k} \otimes A_0(J)^{\alpha_k} \Omega =: \mathcal{H}_{\mathcal{A}}$, we have $S|_{\mathcal{H}_{\mathcal{A}}} = 1|_{\mathcal{H}_{\mathcal{A}}}$.
Constructions through Longo-Witten endomorphisms

- $\mathcal{A}_0$: $U(1)$-current net (Weyl algebra on the symmetric Fock space).
- $\varphi(p) = e^{-i/p}$: an inner symmetric function (the boundary value of an analytic function on the upper-half plain).
- $\Gamma(\varphi(P_0)) := 1 \oplus \varphi(P_0) \oplus (\varphi(P_0) \otimes \varphi(P_0)) \oplus \cdots$: the second quantization. Put $e^{isQ_0} = \Gamma(\varphi(P_0/s))$.

**Theorem (Longo-Witten)**

\[ e^{i\kappa Q_0} \mathcal{A}_0(\mathbb{R}_+) e^{-i\kappa Q_0} \subset \mathcal{A}_0(\mathbb{R}_+) \text{ for } \kappa \geq 0. \]

**Theorem**

For $\kappa \geq 0$, the triple $\mathcal{M} := \{ \text{Ad} e^{i\kappa Q_0 \otimes Q_0} (x \otimes 1) \cdots \}'''$, $U$, $\Omega$ is an interacting wedge-local net with $S$-matrix $S = e^{i2\kappa Q_0 \otimes Q_0}$.
Constructions through Longo-Witten endomorphisms

For a more general inner function $\varphi$, $(\mathcal{M} := \{\text{AdS}(x \otimes 1) \cdots \}'', \ U, \Omega)$ is a wedge-local net, where ...

- $\mathcal{H}^n := \mathcal{H}_0^{\otimes n}, \mathcal{H}^{m,n} := \mathcal{H}^m \otimes \mathcal{H}^n$
- $\mathcal{H}^{\Sigma, \Sigma} = \bigoplus_{m,n=0}^{\infty} \mathcal{H}^{m,n}$.
- $P_{i,j}^{m,n} := (1 \otimes \cdots \otimes P_0 \otimes \cdots \otimes 1) \otimes (1 \otimes \cdots \otimes P_0 \otimes \cdots \otimes 1)$, acting on $\mathcal{H}^{m,n}$, $1 \leq i \leq m$ and $1 \leq j \leq n$.
- $\varphi_{i,j}^{m,n} := \varphi(P_{i,j}^{m,n})$ (functional calculus on $\mathcal{H}^{m,n}$).
- $S^{m,n} := \prod_{i,j} \varphi_{i,j}^{m,n}$ (component of the S-matrix)
- $S^{\Sigma,n} := \bigoplus_m S^{m,n}$
- $S := \bigoplus_{m,n} S^{m,n} = \bigoplus_m \prod_{i,j} \varphi(P_{i,j}^{m,n})$
Towards classification

- S-matrix $S$ is an invariant of a net $\mathcal{A}$
- Asymptotic fields $\Phi_{\pm}^{\text{out}}$ generate a chiral CFT $\mathcal{A}_{0}^{\text{out}} \otimes \mathcal{A}_{0}^{\text{out}}$ (another invariant).

Theorem

If the net $\mathcal{A}$ is asymptotically complete and satisfies standard assumptions (Bisognano-Wichmann property), then

$$\mathcal{A}(\mathcal{W}_R) = \mathcal{A}_{0}^{\text{out}}(\mathbb{R}_+) \lor S\mathcal{A}_{0}^{\text{out}}(\mathbb{R}_-) S^*.$$

In other words, S-matrix and asymptotic fields consist a complete invariant.
Summary

Results:
- New procedure to produce wedge-local QFT
- S-matrix and chiral component as complete invariants

Open problems:
- To find examples with large intersection
- Classification of two-dimensional massless theories