Deformation of chiral conformal field theory

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Introduction: Quantum field theory

- Quantum Field Theory treats infinite particles.
- Physicists calculate observable quantities by perturbation.

Main problem

No **interacting** consistent model in 4 spacetime dimensions.

- Different approaches: Operator-valued distributions, *n*-point functions, **von Neumann algebraic**.
- Recently, examples in 2 dimensions have been constructed by algebraic approach (Lechner '08).

Main results

- New candidate of interacting examples in 2 dimensions
- First steps towards classification

Introduction: Algebraic QFT

We consider a **family of von Neumann algebras** $\{\mathcal{A}(O)\}$ parametrized by bounded regions $\{O\}$: "physical quantities measured in O".

Definition

A **local net** $\mathcal{A}$ **of observables** is a map $\mathbb{R}^2 \supset O \longmapsto \mathcal{A}(O) \subset B(\mathcal{H})$ such that

- $O_1 \subset O_2 \Rightarrow \mathcal{A}(O_1) \subset \mathcal{A}(O_2)$
- O_1 and O_2 are causally disjoint $\Rightarrow [\mathcal{A}(O_1), \mathcal{A}(O_2)] = 0$
- $\exists$ rep U of Poincaré group $\mathcal{P}_+^\uparrow$ such that $U(g)\mathcal{A}(O)U(g)^* = \mathcal{A}(gO)$
- $U|_{\mathbb{R}^2}$ has joint spectrum contained in V_+
- $\exists \Omega \in \mathcal{H}$ such that $U(g)\Omega = \Omega$, and cyclic for $\mathcal{A}(O)$

- Possible to define **interaction**
- Difficult to **construct** interacting examples

Introduction: Wedge-local QFT

The standard right wedge $W_{\mathrm{R}} := \{a = (a_0, a_1) : |a_0| < a_1\}$.

Definition

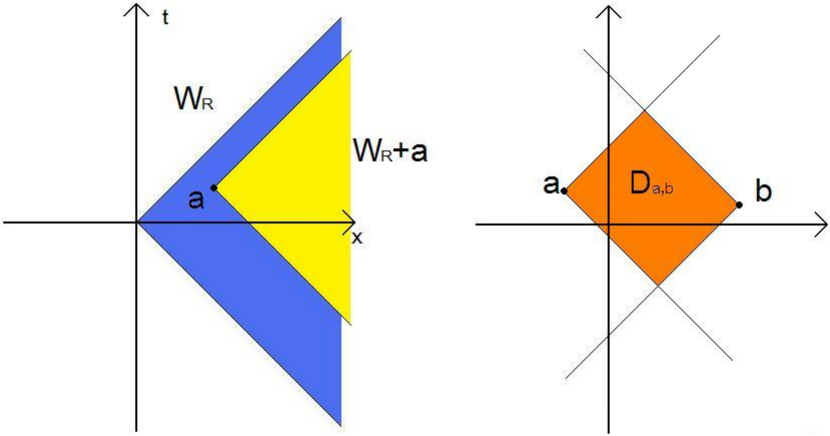
A **wedge-local net** (Borchers triple) is a triple of a **single von Neumann algebra** $\mathcal{M}$, a rep U of $\mathbb{R}^2$ with spectrum condition and Ω such that

- $\mathcal{M}(a) := U(a)\mathcal{M}U(a)^* \subset \mathcal{M}$ for $a \in W_{\mathrm{R}}$
- $U(a)\Omega = \Omega$

Example

Let $(\mathcal{A}, U, \Omega)$ be a local net. Then $(\mathcal{A}(W_{\mathrm{R}}), U, \Omega)$ is a wedge-local net.

If $(\mathcal{M}, U, \Omega)$ is a wedge-local net, then U extends to $\mathcal{P}_+^\uparrow$ and $(\mathcal{A}, U, \Omega)$ where $\mathcal{A}(D_{a,b}) := \mathcal{M}(a) \cap \mathcal{M}(b)$ satisfies the condition of local net **except the cyclicity of** Ω.


t
WR
WR+a
a
x
a
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b

Scattering theory

We define **asymptotic fields**

$$\Phi_\pm^{\mathrm{out}}(x) := \mathrm{s}\text{-}\lim_{T\to\infty} \int dt\, h_T(t) \mathrm{Ad}\, U(t, \pm t)(x),$$

where h_T has support around T and $\int dt\, h_T(t) = 1$. Similarly we define $\Phi_\pm^{\mathrm{in}}$.
The **S-matrix** is defined by:

$$S \cdot \Phi_+^{\mathrm{out}}(x)\Phi_-^{\mathrm{out}}(y) = \Phi_+^{\mathrm{in}}(x)\Phi_-^{\mathrm{in}}(y).$$

If S is not a multiple of $\mathbb{1}$, then we say the net $\mathcal{A}$ is **interacting**.

Problems

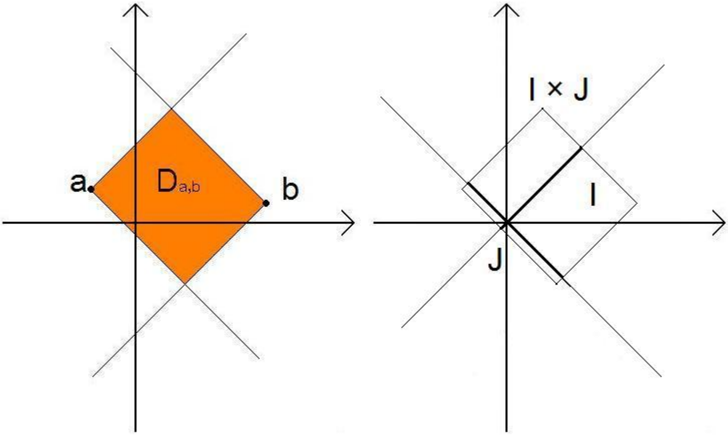
- To construct interacting local net
- To show that the intersection is "sufficiently large"

Results

- Construct several families of interacting wedge-local nets
- Explicit calculation of intersection of some of such families

Introduction: Chiral CFT

- One considers a net $\mathcal{A}_0$ on $S^1 = \mathbb{R} \cap \{\infty\}$, **without** the notion of interaction, with Möbius $= \mathrm{PSL}(2, \mathbb{R})$ symmetry U_0 and Ω_0.
- $\mathcal{A}_0$ assigns to each interval $I \subset \mathbb{R}$ a von Neumann algebra $\mathcal{A}_0(I)$ satisfying several conditions.
- With $\mathcal{A}_0$, one can construct a two-dimensional net.
- A **chiral conformal net** with a component $\mathcal{A}_0$ is defined as follows $\mathcal{A}(I \times J) = \mathcal{A}_0(I) \otimes \mathcal{A}_0(J)$, a representation $U = U_0 \otimes U_0$ of $\mathrm{PSL}(2, \mathbb{R}) \otimes \mathrm{PSL}(2, \mathbb{R}) \supset \mathcal{P}_+^\uparrow$, $\Omega = \Omega_0 \otimes \Omega_0$. A chiral net $\mathcal{A}$ is **not** interacting.
- It is possible to construct from $\mathcal{A}$ an **interacting** wedge-local net (Buchholz-Lechner-Summers '10).


a
b
$D_{a,b}$
$I \times J$
I
J

Deformation w.r.t. translation

- $(\mathcal{A}_0, U_0, \Omega_0)$: a net on S^1, $U_0(\tau(t)) = e^{itP_0}$: translation

Theorem

For $\kappa \geq 0$, the triple

- $\mathcal{M} := \{\mathrm{Ad}e^{i\kappa P_0 \otimes P_0}(x \otimes \mathbb{1}), \mathbb{1} \otimes y : x \in \mathcal{A}(\mathbb{R}_+), y \in \mathcal{A}(\mathbb{R}_-)\}''$,
- $U := U_0 \otimes U_0$,
- $\Omega = \Omega_0 \otimes \Omega_0$

is an **interacting** *wedge-local net with S-matrix $S = e^{i2\kappa P_0 \otimes P_0}$. It coincides with the BLS-deformation.*

Proof: Use $e^{i\kappa P_0 \otimes P_0} = \int e^{i\kappa t P_0} \otimes dE_0(t)$ to show that Ω is separating for $\mathcal{M}$. Note that $\mathrm{Ad}e^{i\kappa t P_0}$ is an **endomorphism** of $\mathcal{A}_0(\mathbb{R}_+)$. We have $\mathrm{Ad}e^{i\kappa P_0 \otimes P_0}(x \otimes \mathbb{1}) = \int \mathrm{Ad}e^{i\kappa t P_0}(x) \otimes dE_0(t)$, which commutes with $z \otimes \mathbb{1}, z \in \mathcal{A}_0(\mathbb{R}_-)$.

Deformation w.r.t. inner symmetry

- $(\mathcal{A}_0, U_0, \Omega_0)$: a net on S^1.
- Let $e^{isQ_0}, s \in \mathbb{R}$ be a periodic inner symmetry: $\mathrm{Ad}e^{isQ_0}(\mathcal{A}_0(I)) = \mathcal{A}_0(I)$, $e^{isQ_0}\Omega_0 = \Omega_0$.

Theorem

The triple

- $\mathcal{M} := \{\mathrm{Ad}e^{i\kappa Q_0 \otimes Q_0}(x \otimes \mathbb{1}), \mathbb{1} \otimes y : x \in \mathcal{A}(\mathbb{R}_+), y \in \mathcal{A}(\mathbb{R}_-)\}''$,
- $U := U_0 \otimes U_0$,
- $\Omega = \Omega_0 \otimes \Omega_0$

is a **interacting** *wedge-local net with S-matrix* $S = e^{i2\kappa Q_0 \otimes Q_0}$.

It is possible to determine the strictly local elements.

Intersection of deformed net

Theorem

If $\alpha_s = \mathrm{Ad}\, e^{isQ_0}$ *is periodic and* $\mathcal{A}_0$ *satisfies a technical condition, then it holds that*

$$\mathcal{A}(D_{a,b}) := \mathcal{M}_{Q_0,\kappa}(a) \cap \mathcal{M}_{Q_0,\kappa}(b)' = \mathcal{A}_0(I)^{\alpha_\kappa} \otimes \mathcal{A}_0(J)^{\alpha_\kappa}.$$

If we let $\overline{\mathcal{A}_0(I)^{\alpha_\kappa} \otimes \mathcal{A}_0(J)^{\alpha_\kappa}\Omega} =: \mathcal{H}_\mathcal{A}$*, we have* $S|_{\mathcal{H}_\mathcal{A}} = \mathbb{1}|_{\mathcal{H}_\mathcal{A}}$.

Constructions through Longo-Witten endomorphisms

- $\mathcal{A}_0$: $U(1)$-current net (Weyl algebra on the symmetric Fock space).
- $\varphi(p) = e^{-i/p}$: an inner symmetric function (the boundary value of an analytic function on the upper-half plain).
- $\Gamma(\varphi(P_0)) := \mathbb{1} \oplus \varphi(P_0) \oplus (\varphi(P_0) \otimes \varphi(P_0)) \oplus \cdots$: the second quantization. Put $e^{isQ_0} = \Gamma(\varphi(P_0/s))$.

Theorem (Longo-Witten)

$e^{i\kappa Q_0}\mathcal{A}_0(\mathbb{R}_+)e^{-i\kappa Q_0} \subset \mathcal{A}_0(\mathbb{R}_+)$ *for* $\kappa \geq 0$.

Theorem

For $\kappa \geq 0$*, the triple* $\mathcal{M} := \{\mathrm{Ad}e^{i\kappa Q_0 \otimes Q_0}(x \otimes \mathbb{1}) \cdots\}''$, U, Ω *is an* **interacting** *wedge-local net with S-matrix* $S = e^{i2\kappa Q_0 \otimes Q_0}$.

Constructions through Longo-Witten endomorphisms

For a more general inner function φ, $(\mathcal{M} := \{\mathrm{Ad}S(x \otimes \mathbb{1}) \cdots\}'', U, \Omega)$ is a wedge-local net, where ...

- $\mathcal{H}^n := \mathcal{H}_0^{\otimes n}, \mathcal{H}^{m,n} := \mathcal{H}^m \otimes \mathcal{H}^n$
- $\mathcal{H}^{\Sigma,\Sigma} = \bigoplus_{m,n=0}^{\infty} \mathcal{H}^{m,n}$.
- $P_{i,j}^{m,n} := (\mathbb{1} \otimes \cdots \otimes \underset{i\text{-th}}{P_0} \otimes \cdots \otimes \mathbb{1}) \otimes (\mathbb{1} \otimes \cdots \otimes \underset{j\text{-th}}{P_0} \otimes \cdots \otimes \mathbb{1})$, acting on $\mathcal{H}^{m,n}$, $1 \leq i \leq m$ and $1 \leq j \leq n$.
- $\varphi_{i,j}^{m,n} := \varphi(P_{i,j}^{m,n})$ (functional calculus on $\mathcal{H}^{m,n}$).
- $S^{m,n} := \prod_{i,j} \varphi_{i,j}^{m,n}$ (component of the S-matrix)
- $S^{\Sigma,n} := \bigoplus_m S^{m,n}$
- $S := \bigoplus_{m,n} S^{m,n} = \bigoplus_m \prod_{i,j} \varphi(P_{i,j}^{m,n})$

Towards classification

- S-matrix S is an invariant of a net $\mathcal{A}$
- Asymptotic fields $\Phi_{\pm}^{\rm out}$ generate a chiral CFT $\mathcal{A}_0^{\rm out} \otimes \mathcal{A}_0^{\rm out}$ (another invariant).

Theorem

If the net $\mathcal{A}$ is asymptotically complete and satisfies standard asumptions (Bisognano-Wichmann property), then
$\mathcal{A}(W_{\rm R}) = \mathcal{A}_0^{\rm out}(\mathbb{R}_+) \vee S\mathcal{A}_0^{\rm out}(\mathbb{R}_-)S^*$.

In other words, S-matrix and asymptotic fields consist a complete invariant.

Summary

Results:

- New procedure to produce wedge-local QFT
- S-matrix and chiral component as complete invariants

Open problems:

- To find examples with large intersection
- Classification of two-dimensional massless theories