Unitarity and reflection positivity in 2d CFT
(Work in progress with M.S. Adamo and Y. Moriwaki)

Let $\phi(z) = Y(v, z) = \sum_n \phi_n z^{-n-d}$ be a primary field in a VOA V (d: conformal dimension). $\phi_n \in \mathrm{End}(V)$

<u>Examples</u> · $J(z) = \sum_n J_n z^{-n-1}$, $[J_m, J_n] = m\delta_{m+n}$ the $U(1)$-current in the vacuum representation V.
· Virasoro algebra · Affine Lie algebras

<u>Unitarity</u>: V admits a scalar product, $\langle \cdot, \cdot \rangle$, and $\langle \Psi_1, \phi_n \Psi_2 \rangle = \langle \phi_{-n} \Psi_1, \Psi_2 \rangle$, or $(\phi_n)^* = \phi_{-n}$.

$\widehat{\phi}(z) = \sum_n \phi_n z^{-n}$. $\widehat{\phi}(z)^* = \widehat{\phi}(z)$ with $z^* = z^{-1}$!

Carpi-Kawahigashi-Longo-Weiner: V satisfying unitarity and "polynomial energy bound" $\Rightarrow$ Wightman field on S^1!
$C^\infty(S^1) \ni f(e^{i\theta}) = \sum_n \hat{f}_n e^{in\theta} \longmapsto \phi(f) = \sum_n \hat{f}_n \phi_n$.
$\Rightarrow$ Wightman field on $\mathbb{R}^{1+1}$.

In general, Osterwalder-Schrader axioms + linear growth $\Rightarrow$ Wightman axioms. Schwinger function for ϕ?

<u>OS axioms</u> for Schwinger functions $\{S_n(z_1, \dots z_n)\}$, $z_k \in \mathbb{R}^2 = \mathbb{C}$.
Locality, Euclidean invariance,

<u>reflection positivity</u>, clustering, linear growth.

$r z = \bar{z}^{-1}$!

$$0 \le \sum_{j,k} \int \overline{f_j(rz_j, \dots, rz_1)}\, f_k(z_{j+1}, \dots z_{j+k})$$
$$S_{j+k}(rz_1, \dots rz_j, z_{j+1}, \dots z_{j+k})\, dz\, J(z).$$

where $J(z)dz$ is r-invariant, f_j have supports $1 > |z_1| > |z_2| > \dots > |z_n|$.

<u>OS axioms from unitary VOA</u>

Recall $\widehat{\phi}(z)^* = \widehat{\phi}(z)$ with $z^* = z^{-1}$, or $z \in S^1$!
$\widehat{\phi}(z)^* = \widehat{\phi}(rz)$ weakly.

Define

$$S_n(z_1,\cdots,z_n)=\langle\Omega,\hat\phi(z_1)\cdots\hat\phi(z_n)\Omega\rangle.$$

$$=\sum_{k_1,\cdots,k_n}\langle\Omega,\phi_{k_1}\cdots\phi_{k_n}\Omega\rangle z_1^{-k_1}\cdots z_n^{-k_n}$$

$$=\sum_{k_2,\cdots,k_n}\langle\Omega,\phi_{-k_2-\cdots-k_n}\phi_{k_2}\cdots\phi_{k_n}\Omega\rangle z_1^{k_2+\cdots+k_n}\cdot z_2^{-k_2}\cdots z_n^{-k_n}$$

$$=\sum_{k_2,\cdots,k_n}\langle\Omega,\phi_{-k_2-\cdots-k_n}\phi_{k_2}\cdots\phi_{k_n}\Omega\rangle\left(\frac{z_1}{z_2}\right)^{k_2+\cdots+k_n}\cdots\left(\frac{z_{n-1}}{z_n}\right)^{k_n}$$

Assume "polynomial energy growth" for $\langle\Omega,\phi\cdots\phi\Omega\rangle$,

then it is convergent for $|z_1|>|z_2|>\cdots>|z_n|$.
($k_n<0, k_n+k_{n-1}<0\cdots$)

Similar in other regions.

Put $\Psi=\sum_k\int_{1>|z_1|>\cdots>|z_n|} f_k(z_1,\cdots,z_k)\hat\phi(z_1)\cdots\hat\phi(z_k)\Omega\,dz\,J(z)$

Unitarity: $0\le\|\Psi\|^2$

$$=\sum_{k,j}\int\overline{f_j(z_1,\cdots,z_j)}f_k(z_{j+1},\cdots,z_{j+k})\langle\hat\phi(z_1)\cdots\hat\phi(z_j)\Omega,\hat\phi(z_{j+1})\cdots\hat\phi(z_{j+k})\Omega\rangle\,dz\,J(z)$$

$$=\sum_{k,j}\int\overline{f_j(z_1,\cdots,z_j)}f_k(z_{j+1},\cdots,z_{j+k})\langle\Omega,\hat\phi(rz_j)\cdots\hat\phi(rz_1)\hat\phi(z_{j+1})\cdots\hat\phi(z_{j+k})\Omega\rangle\,dz\,J(z)$$

$$=\sum_{k,j}\int\overline{f_j(rz_j,\cdots,rz_1)}f_k(z_{j+1},\cdots,z_{j+k})\langle\Omega,\hat\phi(z_1)\cdots\hat\phi(z_{j+k})\Omega\rangle\,dz\,J(z).$$

$\Rightarrow$ Reflection positivity ✓

Linear growth, other axioms ✓

Work in progress: full 2d CFT.

Question Reflection positive (Euclidean) Hilbert space?