Half-sided modular inclusions (and free products in AQFT)

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1. Conformal nets on the circle

In two-dimensional conformal field theory on the Minkowski space, important observables such as currents and stress-energy tensor decompose into the so-called chiral components, which are quantum fields living on one of the lightrays. These chiral components extends further to the circle (the one-point compactification of the lightray) (the Lüscher-Mack theorem, see [FST89, Section 3.2]). *It turns out that operator algebras are useful in constructing such chiral components.*

In the operator-algebraic approach, such a chiral component is realized as a **Möbius covariant net** on S^1: a family $\{\mathcal{A}(I)\}_{I\subset S^1}$ of von Neumann algebras parametrized by open, proper, connected and non empty intervals in S^1, a unitary representation U of the Möbius group $\mathrm{PSL}(2,\mathbb{R})$ and the "vacuum" vector Ω invariant for U, which satisfy the Haag-Kastler axioms, namely, isotony, locality, Möbius covariance, positivity of energy and cyclicity of the vacuum [GF93, Definition 2.5].

It follows from these axioms that Ω is cyclic and separating for each algebra $\mathcal{A}(I)$ (Reeh-Schlieder property), hence we can define the modular objects:

$$S_I : \mathcal{A}(I)\Omega \ni x\Omega \longmapsto x^*\Omega,$$

whose closure is still denoted by S_I and $S_I = J_I\Delta_I^{\frac{1}{2}}$ is the polar decomposition.

Let us recall that the lightray $\mathbb{R}$ is identified as a subset of S^1 by the stereographic projection. With this identification, the positive half-line $\mathbb{R}_+$ is an interval in S^1. For a Möbius covariant net, the Bisognano-Wichmann property holds [GF93, Theorem 2.19]: $\Delta^{it}_{\mathbb{R}_+} = U(D_{\mathbb{R}_+}(2\pi t))$, where $D_{\mathbb{R}_+}$ is the one-parameter subgroup of dilations of $\mathrm{PSL}(2,\mathbb{R})$, which clearly preserves $\mathbb{R}_+$. Now, observe that, by covariance, it holds that $\mathrm{Ad}\,\Delta^{it}(\mathcal{A}(\mathbb{R}_+ + 1)) \subset \mathcal{A}(\mathbb{R}_+ + 1)$ for $t \geq 0$. This inclusion of algebras, called a half-sided modular inclusion, turns out to contain much information of the given net.

2. Half-sided modular inclusions

Let $\mathcal{N} \subset \mathcal{M}$ be von Neumann algebras and Ω be a vector cyclic and separating for both $\mathcal{N}$ and $\mathcal{M}$. We can then define the modular group $\Delta^{it}_{\mathcal{M}}$ of $\mathcal{M}$ with respect to Ω. This triple $(\mathcal{N} \subset \mathcal{M}, \Omega)$ is said to be a **half-sided modular inclusion** (HSMI) if $\mathrm{Ad}\,\Delta^{it}_{\mathcal{M}}(\mathcal{N}) \subset \mathcal{N}$ for $t \geq 0$. From this simple object, many interesting properties follow, among which is that $\Delta^{it}_{\mathcal{M}}$ and $\Delta^{is}_{\mathcal{N}}$ generate a positive-energy representation of the translation-dilation group [Wie93] [AZ05, Theorem 2.1]. Moreover, if Ω is cyclic for $\mathcal{N}' \cap \mathcal{M}$ ($\mathcal{N}'$ is the set of all bounded operators commuting with $\mathcal{N}$), this inclusion is said to be **standard**.

Let $(\mathcal{A}, U, \Omega)$ be a Möbius covariant net. It is said to be **strongly additive** if $\mathcal{A}(I_1)\vee\mathcal{A}(I_2) = \mathcal{A}(I)$ holds, where I_1 and I_2 are two intervals made from an interval

I by removing one point, and $\mathcal{A}(I_1) \vee \mathcal{A}(I_2)$ denotes the von Neumann algebra generated by $\mathcal{A}(I_1)$ and $\mathcal{A}(I_2)$. There is a one-to-one correspondence between

- Standard half-sided modular inclusions $(\mathcal{N} \subset \mathcal{M}, \Omega)$
- Strongly additive Möbius covariant nets $(\mathcal{A}, U, \Omega)$

and the correspondence is given by $\mathcal{N} = \mathcal{A}(\mathbb{R}_+ + 1), \mathcal{M} = \mathcal{A}(\mathbb{R}_+)$ [GLW98, Corollary 1.9].

This correspondence allows one to construct new Möbius covariant nets from standard HSMIs. For example, consider the Virasoro nets Vir_c with $c > 1$ (the nets generated by the stress-energy tensor only). They are known not to be strongly additive [BSM90, Section 4], therefore, from the standard HSMI $(\mathcal{N} = \mathrm{Vir_c}(\mathbb{R}_+ + 1), \mathcal{M} = \mathrm{Vir_c}(\mathbb{R}_+), \Omega)$ one obtains a Möbius covariant net $\mathcal{A}$ which is different from Vir_c. These nets have not been identified with any known Möbius covariant net.

Another construction goes as follows: for a given Möbius covariant net $\mathcal{A}$, one considers its restriction $\mathcal{A}|_\mathbb{R}$ to the real line $\mathbb{R}$. If one takes a KMS state φ on $\mathcal{A}|_\mathbb{R}$ and makes the GNS representation π_φ with the GNS vector Ω_φ, with respect to the translation group, then the inclusion $(\pi_\varphi(\bigcup_{I\Subset\mathbb{R}_+} \mathcal{A}(I)) \subset \pi_\varphi(\bigcup_{I\Subset\mathbb{R}} \mathcal{A}(I))'', \Omega_\varphi)$ is a standard HSMI, and hence one can construct a Möbius covariant net [Lon01, Proposition 3.2]. We have constructed continuously many differerent KMS states on $\mathrm{Vir}_c, c \geq 1$ [CLTW12, Section 5], therefore, from these standard HSMIs one can construct (possibly new) Möbius covariant nets.

Problem 2.1. *Determine whether these Möbius covariant nets are isomorphic to any known net.*

3. Non standard HSMI from free products

If $\mathcal{A}$ is a Haag-Kastler net in $1 + 1$ or more dimensions with the Bisognano-Wichmann property (meaning that the modular group is the Lorentz boost), then $(\mathcal{A}(W + a) \subset \mathcal{A}(W), \Omega)$ is a half-sided modular inclusion, where W is a wedge region and a is a past lightlike vector in W. It is in general a difficult problem to determine whether such a given HSMI is standard or not, if $\mathcal{A}$ is not the free field net. Some successful cases are those coming from two-dimensional interacting nets [Tan14], and they are some fixed point subnets of the tensor product of the so-called U(1)-current net [BT15, Section 5.3]. Many other cases are open.

Problem 3.1. *Determine whether the HSMIs coming from the interacting nets of* [Lec08] *are standard.*

Now it is a natural question whether there is a HSMI $(\mathcal{N} \subset \mathcal{M}, \Omega)$ where $\mathcal{N}' \cap \mathcal{M}$ is trivial, i.e. containing only the multiples of the identity operator. We answer this positively by constructing an example from free products [LTU17].

A **free product** of a family of von Neumann algebras $\{\mathcal{M}_k\}_{k\in K}$ with respect to cyclic and separating vectors $\{\Omega_k\}_{k\in K}$ is a large von Neumann algebra containing isomorphic images of all $\mathcal{M}_k$'s which are highly non commutative, and equipped with a cyclic separating vector Ω [Voi85]. One can determine the modular objects of $(\mathcal{M}, \Omega)$ in terms of those of $(\mathcal{M}_k, \Omega_k)$ [Bar95, Lemma 1].

Let $(\mathcal{N}_0 \subset \mathcal{M}_0, \Omega_0)$ be a standard HSMI and $\{(\mathcal{N}_k \subset \mathcal{M}_k, \Omega_k)\}_{k\in K}$ isomorphic copies of $(\mathcal{N}_0 \subset \mathcal{M}_0, \Omega_0)$. We can then construct an inclusion of the free product von Neumann algebras $(\mathcal{N} \subset \mathcal{M}, \Omega)$. As the modular objects are known, it is immediate to see that this is a HSMI.

Theorem 3.1. *If $|K| = \infty$, then for the free product HSMI $(\mathcal{N} \subset \mathcal{M}, \Omega)$, $\mathcal{N}' \cap \mathcal{M}$ is trivial.*

Problem 3.2. *Determine $\mathcal{N}' \cap \mathcal{M}$ when $|K| < \infty$, and if it is nontrivial, study the corresponding Möbius covariant net.*

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