Operator-algebraic construction of integrable QFT YOH TANIMOTO

1. Half-sided modular inclusions

Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of von Neumann algebras on a Hilbert space \mathcal{H} . Assume that Ω is cyclic for \mathcal{M} and \mathcal{N} , separating for \mathcal{M} . Let σ^t be the modular automorphisms group for \mathcal{M} with respect to Ω . We say that this inclusion $\mathcal{N} \subset \mathcal{M}$ is a half-sided modular inclusion if $\sigma^t(\mathcal{N}) \subset \mathcal{N}$ holds for $t \geq 0$.

Differently from the study of subfactors, the situation where the relative commutant $\mathcal{M} \cap \mathcal{N}'$ is large is of a great interest. We say that a half-sided modular inclusion is **standard** if Ω is cyclic for $\mathcal{M} \cap \mathcal{N}'$.

A remarkable result is that there is a one-to-one correspondence between standard half-sided modular inclusions and a class of quantum field theories, namely strongly additive Möbius covariant nets on S^1 [8] as we explain below.

This is the recurring theme of this article. In many cases, a few operator algebras with some additional information are enough to construct the full quantum field theory.

2. QUANTUM FIELD THEORY AND HAAG-KASTLER NETS

Quantum field theory is a theoretical framework for elementary particle physics. While classical field theory is concerned with functions (possibly in a extended sense) on the spacetime \mathbb{R}^d , Streater and Wightman pointed out that quantum fields should be operator-valued distributions $\phi(x), x \in \mathbb{R}^d$ [17]. Haag proposed then to consider the net of von Neumann algebras $\mathcal{A}(O) = \{e^{i\phi(f)} : \operatorname{supp} f \subset O\}''$ generated by quantum field ϕ , where $O \subset \mathbb{R}^d$ is a spacetime region [9]. Such a net $\{\mathcal{A}(O)\}$, together with the representation U of the spacetime symmetry and the vacuum vector Ω , turns out to contain most of information of physical interest, e.g. the scattering amplitudes of particles, therefore, can be regarded as a mathematical framework of quantum field theory, and called **Haag-Kastler net**.

A Haag-Kastler net (\mathcal{A}, U, Ω) by definition should satisfy various axioms (see a recent review on conformal field theories [15]). An important axiom is **locality**, namely, if O_1 and O_2 are spacelike separated, then the corresponding von Neumann algebras $\mathcal{A}(O_1)$ and $\mathcal{A}(O_2)$ should commute. The action of U should be consistent with \mathcal{A} , namely $U(g)\mathcal{A}(O)U(g)^* = \mathcal{A}(gO)$ (covariance). When one considers one-dimensional spacetime d = 1, spacetime regions are intervals and spacelike separation of I_1, I_2 is simply replaced by disjointness $I_1 \cap I_2 = \emptyset$.

If one has a standard half-sided modular inclusion ($\mathcal{N} \subset \mathcal{M}, \Omega$), one can construct a net on \mathbb{R} by setting $\mathcal{A}(\mathbb{R}_+) = \mathcal{M}, \mathcal{A}(\mathbb{R}_++1) = \mathcal{N}, \mathcal{A}(0,1) = \mathcal{M} \cap \mathcal{N}'$. The crucial point is that one can obtain the spacetime symmetry group, in this case the Möbius group, from the modular operators for these von Neumann algebras, and indeed, one can extend the net to the circle S^1 [8].

3. Examples of Haag-Kastler nets

The most important problem in mathematical quantum field theory is the scarcity of examples in higher dimensions. Although there are many conformal field theories for d = 1, 2 [6] and even certain classification results are available [10], and under the program of Constructive Quantum Field Theory several interacting models have been obtained for d = 2, 3 [7], currently the only examples of Haag-Kastler nets in $d \ge 4$ are the (generalized) free fields [16].

Therefore, it is significant to study methods and techniques to produce examples of nets. We explain below some recent results for d = 2. We also mention several attempts to higher dimensions [4, 12] and new ideas which go through de Sitter spacetime [2].

4. Longo-Witten endomorphisms

Let us see that small additional information to a one-dimensional net is sufficient in order to construct a class of nets on the two-dimensional half-plane $\mathbb{R}^2_+ = \{(t,x) \in \mathbb{R}^2 : x > 0\}$. Let (\mathcal{A}, U, Ω) be a net on \mathbb{R} , where U is a unitary representation of the translation group \mathbb{R} (we are not considering Möbius or conformal covariance here). We say that a unitary operator V implements a **Longo-Witten endomorphism** of the net (\mathcal{A}, U, Ω) if V commutes with U and $\mathrm{Ad}V(\mathcal{A}(\mathbb{R}_+)) \subset \mathcal{A}(\mathbb{R}_+)$. For two intervals $I_1 < I_2$ on the time axis x = 0, one associates a diamond $D = \{(x,t) \in \mathbb{R}^2_+ : t - x \in I_1, t + x \in I_2\}$. Conversely, any diamond in \mathbb{R}^2_+ is of this form. If we define $\mathcal{A}_V(D) = \mathcal{A}(I_1) \lor \mathrm{Ad}V(\mathcal{A}(I_2))$, then \mathcal{A}_V is a local net on \mathbb{R}^2_+ , time-translation covariant (with respect to U) [14].

Examples of Longo-Witten endomorphisms are given on the U(1)-current net by the second quantization operators [14]. Further examples which are not second quantization were obtained by the so-called boson-fermion correspondence [3].

5. Integrable QFT

By considering more specific examples, one can do more. Namely, we can construct a two-dimensional net on the full spacetime \mathbb{R}^2 . Let (\mathcal{A}, U, Ω) be the massive complex free field net on \mathbb{R}^2 . This can be identified with the the tensor product of two copies of the massive real free field net, the simplest two-dimensional net. For the wedge-shaped region $W_{\mathbb{R}} := \{(t, x) \in \mathbb{R}^2 : x > |t|\}$, the inclusion $\mathcal{A}(W_{\mathbb{R}} + (1, 1)) \subset \mathcal{A}(W_{\mathbb{R}})$ is a standard half-sided modular inclusion with respect to Ω . The associated one-dimensional net is the tensor product of two copies of the U(1)-current net.

There is a one-parameter 2π -periodic Longo-Witten automorphisms acting on this one-dimensional net. Let us denote the unitary operators by V(s). We can take its generator Q and write them as $V(s) = e^{isQ}, s \in \mathbb{R}$. Now the new net is defined on the tensor product Hilbert space. Let us introduce $\widetilde{V}(s) = e^{isQ\otimes Q}$. Define the new net first for the wedge by $\widetilde{\mathcal{A}}_s(W_R) := \mathcal{A}(W_R) \otimes \mathbb{C}I \vee \operatorname{Ad}\widetilde{V}(s)(\mathbb{C}I \otimes \mathcal{A}(W_R))$. The other elements are simply the tensor products $\widetilde{U} = U \otimes U, \widetilde{\Omega} = \Omega \otimes \Omega$. The von Neumann algebras $\widetilde{\mathcal{A}}(O)$ for general region O is defined by locality (one assigns the commutant $\mathcal{A}(W_{\mathrm{R}})'$ to the reflected wedge W_{L}) and covariance with respect to \widetilde{U} . Then the triple $(\widetilde{\mathcal{A}}, \widetilde{U}, \widetilde{\Omega})$ is a two-dimensional Haag-Kastler net [18]. One can compute the so-called S-matrix, an invariant of a net, which is different from the identity operator and hence one says that this net is interacting.

With more complicated Longo-Witten endomorphisms, one can construct more examples. The resulting nets have S-matrices with a particularly simple structure, called factorizing, and some of them are believed to be related to the quantization of classical integrable Lagrangians. Another class of nets with factorizing S-matrices has been constructed by a different technique [11]. Further examples are currently under investigation [13, 1, 5].

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