Noninteraction of waves in two-dimensional conformal field theory

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Abstract

In higher dimensional quantum field theory, irreducible representations of the Poincaré group are associated with particles. Their counterpart in two-dimensional massless models are "waves" introduced by Buchholz. In this paper we show that waves do not interact in two-dimensional Möbius covariant theories and in- and out-asymptotic fields coincide. We identify the set of the collision states of waves with the subspace generated by the chiral components of the Möbius covariant net from the vacuum. It is also shown that Bisognano-Wichmann property, dilation covariance and asymptotic completeness (with respect to waves) imply Möbius symmetry.

Under natural assumptions, we observe that the maps which give asymptotic fields in Poincaré covariant theory are conditional expectations between appropriate algebras. We show that a two-dimensional massless theory is asymptotically complete and noninteracting if and only if it is a chiral Möbius covariant theory.

1 Introduction

Quantum field theory (QFT) is designed to describe interactions between elementary particles and can successfully account for a wide range of physical phenomena. However, its mathematical foundations are still unsettled and constitute an active area of research in mathematical physics. While the most important open problem in QFT is the existence of interacting models in physical four-dimensional spacetime, theories in lower dimensional spacetime have also attracted considerable interest. For instance, two-dimensional conformal field theories (CFT), whose infinite dimensional symmetry group is a powerful tool for structural analysis, have been thoroughly investigated. Superselection structure of such theories has been clarified and deep classification results have been obtained [19, 20]. On

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the other hand, particle aspects of two-dimensional CFT have only recently attracted attention [13, 14], although a general framework for scattering of massless excitations ("waves") in two-dimensional theories dates back to [6]. In particular, it was shown in [13] that "waves" do not interact in *chiral* conformal field theories. In this paper we generalize this result to any CFT[1], relying on ideas from [8]. Moreover, we show that a conformal field theory is asymptotically complete (i.e. collision states of waves span the entire Hilbert space) if and and only if it is chiral. This latter result is obtained by a careful analysis of the chiral components [28] of the theory.

In view of a large body of highly non-trivial results concerning two-dimensional CFT, both on the sides of physics and mathematics [12, 16, 17], our assertion that these theories have trivial scattering theory may seem surprising. In this connection we emphasise that the presence of interaction in scattering theory *cannot* be inferred solely from the fact that a particular expression for the Hamiltonian or the correlation functions differ from those familiar from free field theory. In fact, the Ising model, the most fundamental "interacting" model, can be considered as a subtheory of "free" fermionic field [26], hence the conventional term of "interaction" seems ill-defined. Instead, a conclusive argument should rely on a scattering theory which implements, in the theoretical setting, the quantum mechanical procedure of state preparation at asymptotic times. Such an intrinsic scattering theory was developed by Buchholz [6] in the framework of algebraic quantum field theory [18], which we also adopt in this work. The elementary excitations of this collision theory, called "waves" in [6], are eigenstates of the relativistic mass operator. However, they are not necessarily particles in the conventional sense of Wigner i.e. states in an irreducible representation space of the Poincaré group. This less restrictive concept of the particle is natural in two-dimensional massless theories, where irreducible representations of the Poincaré group typically have infinite multiplicity (cf. Section 3.1).

As the classical results on the absence of interaction in dilation-covariant theories in physical spacetime require the existence of irreducible representations of the Poincaré group with finite multiplicity [10, 8], they cannot be applied to two-dimensional CFT directly. We combine essential ideas from [8] with the representation theory of the Möbius group to overcome this difficulty and obtain triviality of the scattering matrix. Under asymptotic completeness with respect to waves, one can even prove that dilation covariance implies Möbius covariance, hence also noninteraction. Exploiting again the Möbius symmetry, we construct chiral observables following Rehren [28] which live on the positive or negative lightrays and show that they generate all the collision states of waves from the vacuum. In examples of non-chiral two-dimensional CFT, the profile of chiral observables is well-known [20], hence this result gives an explicit description of the subspace of collision states. As a by-product we obtain an alternative proof of the noninteraction of waves and the insight that asymptotic completeness with respect to waves of a conformal field theory (in the sense of waves) is equivalent to chirality. This suggest that chiral Möbius covariant theories are generic examples of noninteracting massless theories in two-dimensional spacetime (cf. [31]). Indeed, it turns out that Poincaré covariant theory satisfying Bisognano-Wichmann

[1] The terms "conformal" and "Möbius covariant" will be clarified in Section 2.

property and Haag duality is noninteracting and asymptotically complete (with respect to waves) if and only if it is chiral. This is a strengthened converse of the sufficient condition for noninteraction by Buchholz [6]. We prove this based on an observation that the maps which give asymptotic fields are conditional expectations between appropriate algebras. Another consequence of this observation is that the maps which give in- and out-asymptotic fields are the conditional expectations onto the chiral components in Möbius covariant theory, hence they coincide.

This paper is organized as follows. In Section 2 we recall the basic notions of Poincaré covariant nets with various higher symmetries in two-dimensional spacetime and the scattering theory of massless "waves" studied in [6]. In Section 3 we demonstrate that these waves have always trivial scattering matrix in Möbius covariant nets. This proof is based on the representation theory of Möbius group, and one derives Möbius symmetry from Bisognano-Wichmann property, dilation covariance and asymptotic completeness with respect to waves. In Section 4 the chiral components are defined following [28]. They turn out to generate all the waves from the vacuum. In Section 5.1, under Bisognano-Wichmann property and Haag duality, we show that asymptotic fields are given by conditional expectations and that a Poincaré covariant net is asymptotically complete (with respect to waves) and noninteracting if and only if it is isomorphic to a chiral Möbius net. In Section 5.2 we show that in- and out-asymptotic fields coincide in Möbius covariant nets. In Section 6 we discuss open problems and perspectives. In Appendix A we collect fundamental facts about conditional expectations and in Appendix B remarks about various definitions of chiral component are given.

2 Preliminaries

2.1 Conformal nets

In algebraic QFT, we consider nets of observables. Let us briefly recall the definitions. The two-dimensional Minkowski space $\mathbb{R}^2$ is represented as a product of two lightlines $\mathbb{R}^2 = \mathrm{L}_+ \times \mathrm{L}_-$, where $\mathrm{L}_\pm := \{(a_0, a_1) \in \mathbb{R}^2 : a_0 \pm a_1 = 0\}$ are the positive and the negative lightlines. The fundamental group of spacetime symmetry is the (proper orthochronous) Poincaré group $\mathcal{P}_+^\uparrow$, which is generated by translations and Lorentz boosts.

Let $\mathcal{O}$ be the family of open bounded regions in $\mathbb{R}^2$. A **(local) Poincaré covariant net** $\mathcal{A}$ assigns to $O \in \mathcal{O}$ a von Neumann algebra $\mathcal{A}(O)$ on a common separable Hilbert space $\mathcal{H}$ satisfying the following conditions:

(1) **Isotony.** If $O_1 \subset O_2$, then $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$.

(2) **Locality.** If O_1 and O_2 are spacelike separated, then $[\mathcal{A}(O_1), \mathcal{A}(O_2)] = 0$.

(3) **Additivity.** If $O = \bigcup_i O_i$, then $\mathcal{A}(O) = \bigvee_i \mathcal{A}(O_i)$.

(4) **Poincaré covariance.** There exists a strongly continuous unitary representation U of the Poincaré group $\mathcal{P}_+^\uparrow$ such that

$$U(g)\mathcal{A}(O)U(g)^* = \mathcal{A}(gO), \text{ for } g \in \mathcal{P}_+^\uparrow.$$

(5) **Positivity of energy.** The joint spectrum of the translation subgroup in $\mathcal{P}_+^\uparrow$ in the representation U is contained in the forward lightcone $V_+ := \{(p_0, p_1) \in \mathbb{R}^2 : p_0 + p_1 \geq 0, p_0 - p_1 \geq 0\}$.

(6) **Existence of the vacuum.** There is a unique (up to a phase) unit vector Ω in $\mathcal{H}$ which is invariant under the action of U, and cyclic for $\bigvee_{O\in\mathcal{O}} \mathcal{A}(O)$.

From these assumptions, the following properties automatically follow [3].

(7) **Reeh-Schlieder property.** The vector Ω is cyclic and separating for each $\mathcal{A}(O)$.

(8) **Irreducibility.** The von Neumann algebra $\bigvee_{O\in\mathcal{O}} \mathcal{A}(O)$ is equal to $B(\mathcal{H})$.

We identify the circle S^1 as the one-point compactification of the real line $\mathbb{R}$ by the Cayley transform:

$$t = i\frac{z-1}{z+1} \Longleftrightarrow z = -\frac{t-i}{t+i}, \quad t \in \mathbb{R}, \quad z \in S^1 \subset \mathbb{C}.$$

The Möbius group $\mathrm{PSL}(2,\mathbb{R})$ acts on $\mathbb{R} \cup \{\infty\}$ by the linear fractional transformations, hence it acts on $\mathbb{R}$ locally (see [4] for local actions). Then the group $\overline{\mathrm{PSL}(2,\mathbb{R})} \times \overline{\mathrm{PSL}(2,\mathbb{R})}$ acts locally on $\mathbb{R}^2$, where $\overline{\mathrm{PSL}(2,\mathbb{R})}$ is the universal covering group of $\mathrm{PSL}(2,\mathbb{R})$. Note that the group $\overline{\mathrm{PSL}(2,\mathbb{R})} \times \overline{\mathrm{PSL}(2,\mathbb{R})}$ contains translations, Lorentz boosts and dilations, so in particular it includes the Poincaré group $\mathcal{P}_+^\uparrow$. We refer to [20] for details.

Let $\mathcal{A}$ be a Poincaré covariant net. If the representation U of $\mathcal{P}_+^\uparrow$ (associated to the net $\mathcal{A}$) extends to $\overline{\mathrm{PSL}(2,\mathbb{R})} \times \overline{\mathrm{PSL}(2,\mathbb{R})}$ such that for any open region O there is a small neighborhood $\mathcal{U}$ of the unit element in $\overline{\mathrm{PSL}(2,\mathbb{R})} \times \overline{\mathrm{PSL}(2,\mathbb{R})}$ such that $gO \subset \mathbb{R}^2$ and it holds that

$$U(g)\mathcal{A}(O)U(g)^* = \mathcal{A}(gO), \text{ for } g \in \mathcal{U},$$

then we say that $\mathcal{A}$ is a **Möbius covariant net**.

If the net $\mathcal{A}$ is Möbius covariant, then it extends to a net on the Einstein cylinder $\mathcal{E} := \mathbb{R} \times S^1$ [20]. On $\mathcal{E}$ one can define a natural causal structure which extends the one on $\mathbb{R}^2$ (see [25]). We take a coordinate system on $\mathcal{E}$ used in [28]: Let $\mathbb{R} \times \mathbb{R}$ be the universal cover of $S^1 \times S^1$. The cylinder $\mathcal{E}$ is obtained from $\mathbb{R} \times \mathbb{R}$ by identifying points (a, b) and $(a + 2\pi, b - 2\pi) \in \mathbb{R} \times \mathbb{R}$. Any double cone of the form $(a, a + 2\pi) \times (b, b + 2\pi) \subset \mathbb{R} \times \mathbb{R}$ represents a copy of the Minkowski space. The causal complement of a double cone $(a, c) \times (b, d)$, where $0 < c - a < 2\pi, 0 < d - b < 2\pi$, is $(c, a + 2\pi) \times (d - 2\pi, b)$ or equivalently $(c - 2\pi, a) \times (d, b + 2\pi)$. If O is a double cone, we denote the causal complement by O'. For an interval $I = (a, b)$, we denote by I^+ the interval $(b, a + 2\pi) \subset \mathbb{R}$ and by I^- the interval $(b - 2\pi, a) \subset \mathbb{R}$.

Furthermore, it is well-known that, from Möbius covariance, the following properties automatically follow (see [4]):

(9M) **Haag duality in $\mathcal{E}$.** For a double cone O in $\mathcal{E}$ it holds that $\mathcal{A}(O)' = \mathcal{A}(O')$, where O' is defined in $\mathcal{E}$ as above.

(10M) **Bisognano-Wichmann property in $\mathcal{E}$** For a double cone O in $\mathcal{E}$, the modular automorphism group Δ_O^{it} of $\mathcal{A}(O)$ with respect to the vacuum state $\omega := \langle\Omega, \cdot\Omega\rangle$ equals to $U(\Lambda_t^O)$ where Λ_t^O is a one-parameter group in $\overline{\mathrm{PSL}(2,\mathbb{R})} \times \overline{\mathrm{PSL}(2,\mathbb{R})}$ which preserves O (see [4] for concrete expressions).

We denote by $\mathrm{Diff}(\mathbb{R})$ the group of diffeomorphisms of S^1 which preserve the point -1. If we identify $S^1 \setminus \{-1\}$ with $\mathbb{R}$, this can be considered as a group of diffeomorphisms of $\mathbb{R}$ [2]. The Minkowski space $\mathbb{R}^2$ can be identified with a double cone in $\mathcal{E}$. The group $\mathrm{Diff}(\mathbb{R}) \times \mathrm{Diff}(\mathbb{R})$ acts on $\mathbb{R}^2$ and this action extends to $\mathcal{E}$ by periodicity. The group generated by this action of $\mathrm{Diff}(\mathbb{R}) \times \mathrm{Diff}(\mathbb{R})$ and the action of $\overline{\mathrm{PSL}(2,\mathbb{R})} \times \overline{\mathrm{PSL}(2,\mathbb{R})}$ (which acts on $\mathcal{E}$ through quotient by the relation $(r_{2\pi}, r_{-2\pi}) = (\mathrm{id}, \mathrm{id})$ [20]) is denoted by $\mathrm{Conf}(\mathcal{E})$. Explicitly, $\mathrm{Conf}(\mathcal{E})$ is isomorphic to the quotient group of $\overline{\mathrm{Diff}(S^1)} \times \overline{\mathrm{Diff}(S^1)}$ by the normal subgroup generated by $(r_{2\pi}, r_{-2\pi})$, where $\overline{\mathrm{Diff}(S^1)}$ is the universal covering group of $\mathrm{Diff}(S^1)$ (note that $r_{2\pi}$ is an element in the center of $\overline{\mathrm{Diff}(S^1)}$).

A Möbius covariant net is said to be **conformal** if the representation U further extends to a projective representation of $\mathrm{Conf}(\mathcal{E})$ such that

$$U(g)\mathcal{A}(O)U(g)^* = \mathcal{A}(gO), \text{ for } g \in \mathrm{Diff}(\mathbb{R}) \times \mathrm{Diff}(\mathbb{R}),$$

and if it holds that

$$U(g)xU(g)^* = x$$

for $x \in \mathcal{A}(O)$, where O is a double cone and $g \in \mathrm{Diff}(\mathbb{R}) \times \mathrm{Diff}(\mathbb{R})$ has a support disjoint from $O \subset \mathbb{R}^2$.

Proposition 2.1. *If the net $\mathcal{A}$ is conformal, the intersection $\bigcap_J \mathcal{A}(I \times J)$ contains representatives of diffeomorphisms of the form $g_+ \times \mathrm{id}$ where $\mathrm{supp}(g_+) \subset I$,*

Proof. If g is a diffeomorphism of the form $g_+ \times \mathrm{id}$ and $\mathrm{supp}(g_+) \subset I$, then $U(g)$ commutes with $\mathcal{A}(I^+ \times J)$ for arbitrary J, thus Proposition follows by the Haag duality in $\mathcal{E}$. □

In the rest of the present paper, conformal covariance will not be assumed except Appendix B, although a major part of examples of Möbius covariant nets is in fact conformal.

If it holds that $\mathcal{A}(O_1) \vee \mathcal{A}(O_2) = \mathcal{A}(O)$ where O_1 and O_2 are the two components of the causal complement (in O) of an interior point of a double cone O, we say that $\mathcal{A}$ is **strongly additive**. This implies the **chiral additivity** [28], namely that $\mathcal{A}(I \times J_1) \vee \mathcal{A}(I \times J_2) = \mathcal{A}(I \times J)$ if J_1 and J_2 are obtained from J by removing an interior point.

We recall that $\mathcal{A}(O)$ is interpreted as the algebra of observables measured in a spacetime region O. A typical example of a conformal net is constructed in the following way: If

[2]Note that not all diffeomorphisms of $\mathbb{R}$ extend to diffeomorphisms of S^1, hence the group $\mathrm{Diff}(\mathbb{R})$ is *not* the group of all the diffeomorphisms of $\mathbb{R}$. However, this notation is common in the context of conformal field theory.

we have a local conformal field Ψ, namely an operator-valued distribution, then we define $\mathcal{A}(O)$ as the von Neumann algebra generated by $e^{i\Psi(f)}$ where the support of f is included in O. But our framework does not assume the existence of any field. Indeed, there are examples of nets for which no local field description is at hand [22]. Thus the algebraic approach is more general than the conventional one. It also provides a natural scattering theory, as we recall in the next section.

2.2 Scattering theory of waves

Here we summarize the scattering theory of massless two-dimensional models established in [6]. This theory is stated in terms of Poincaré covariant nets of observables.

Let us denote by $T(a) := U(\tau(a))$ the representative of spacetime translation $\tau(a)$ by $a \in \mathbb{R}^2$. Furthermore, we denote the lightlike translations by $T_\pm(t) := T(t, \pm t)$. Let $\mathbf{P}$ denote the subgroup of $\mathrm{PSL}(2,\mathbb{R})$ generated by (one-dimensional) translations and dilations. Note that $\mathbf{P}$ is simply connected, hence it can be considered as a subgroup of $\overline{\mathrm{PSL}(2,\mathbb{R})}$. As will be seen in the following, the representation U of $\overline{\mathrm{PSL}(2,\mathbb{R})} \times \overline{\mathrm{PSL}(2,\mathbb{R})}$ restricted to $\mathbf{P} \times \mathbf{P}$ has typically a big multiplicity in Möbius covariant theories. The subspaces $\mathcal{H}_+ = \{\xi \in \mathcal{H} : T_+(t)\xi = \xi \text{ for all } t\}$ and $\mathcal{H}_- = \{\xi \in \mathcal{H} : T_-(t)\xi = \xi \text{ for all } t\}$ are referred to as the spaces of waves with positive and negative momentum, respectively. Let $P_\pm$ be the orthogonal projections onto $\mathcal{H}_\pm$, respectively.

Let x be a local operator, i.e., an element of $\mathcal{A}(O)$ for some O. We set $x(a) := T(a)xT(a)^*$ for $a \in \mathbb{R}^2$ and consider a family of operators parametrized by $\mathcal{T}$:

$$x_\pm(h_\mathcal{T}) := \int dt\, h_\mathcal{T}(t) x(t, \pm t),$$

where $h_\mathcal{T}(t) = |\mathcal{T}|^{-\varepsilon} h(|\mathcal{T}|^{-\varepsilon}(t - \mathcal{T}))$, $0 < \varepsilon < 1$ is a constant, $\mathcal{T} \in \mathbb{R}$ and h is a nonnegative symmetric smooth function on $\mathbb{R}$ such that $\int dt\, h(t) = 1$.

Lemma 2.2 ([6] Lemma 1,2,3). *Let x be a local operator. Then the limit $\Phi^{\mathrm{in}}_\pm(x) := \operatorname*{s-lim}_{\mathcal{T}\to-\infty} x_\pm(h_\mathcal{T})$ exists and it holds that*

- $\Phi^{\mathrm{in}}_\pm(x)\Omega = P_\pm x\Omega$.
- $\Phi^{\mathrm{in}}_\pm(x)\mathcal{H}_\pm \subset \mathcal{H}_\pm$.
- $\mathrm{Ad}\, U(g)\Phi^{\mathrm{in}}_\pm(x) = \Phi^{\mathrm{in}}_\pm(\mathrm{Ad}\, U(g)(x))$, *where* $g \in \mathcal{P}^\uparrow_+$.

Furthermore, the limit $\Phi^{\mathrm{in}}_\pm(x)$ depends only on $P_\pm x\Omega$, respectively. We call these limit operators the "incoming asymptotic fields". It holds that $[\Phi^{\mathrm{in}}_+(x), \Phi^{\mathrm{in}}_-(y)] = 0$ for arbitrary local x and y.

Similarly one defines the "outgoing asymptotic fields" by $\Phi^{\mathrm{out}}_\pm(x) := \operatorname*{s-lim}_{\mathcal{T}\to\infty} x_\pm(h_\mathcal{T})$

Remark 2.3. As the asymptotic field is defined as the limit of local operators, it still has certain local properties. For example, let O_+ and O_0 be two regions such that O_+ stays in the future of O_0 and $x \in O_+, y \in O_0$. Then it holds that $[\Phi_\pm^{\mathrm{in}}(x), y] = 0$, since for a negative $\mathcal{T}$ with sufficiently large absolute value, $x_\pm(h_\mathcal{T})$ lies in the spacelike complement of y. Similar observations apply also to $\Phi_\pm^{\mathrm{out}}$.

Lemma 2.2 captures the dispersionless kinematics of elementary excitations in two-dimensional massless theories: since $\Phi_\pm^{\mathrm{in}}(x)\mathcal{H}_\pm \subset \mathcal{H}_\pm$, by composing two waves travelling to the right we obtain again a wave travelling to the right. Thus waves are, in general, composite objects, associated with reducible representations of the Poincaré group. Moreover, it follows that collision states of waves may contain at most two excitations: One wave with positive momentum and the other with negative momentum.

Let us now construct these collision states: For $\xi_\pm \in \mathcal{H}_\pm$, there are sequences of local operators $\{x_{\pm,n}\}$ such that $\operatorname*{s-lim}_{n\to\infty} P_\pm x_{\pm,n}\Omega = \xi_\pm$ and Using these sequences let us define collision states following [6] (see also [13]):

$$\begin{aligned}
\xi_+ \overset{\mathrm{in}}{\times} \xi_- &= \operatorname*{s-lim}_{n\to\infty} \Phi_+^{\mathrm{in}}(x_{+,n})\Phi_-^{\mathrm{in}}(x_{-,n})\Omega \\
\xi_+ \overset{\mathrm{out}}{\times} \xi_- &= \operatorname*{s-lim}_{n\to\infty} \Phi_+^{\mathrm{out}}(x_{+,n})\Phi_-^{\mathrm{out}}(x_{-,n})\Omega
\end{aligned}$$

We interpret $\xi_+ \overset{\mathrm{in}}{\times} \xi_-$ (respectively $\xi_+ \overset{\mathrm{out}}{\times} \xi_-$) as the incoming (respectively outgoing) state which describes two non-interacting waves ξ_+ and ξ_-. These asymptotic states have the following natural properties.

Lemma 2.4 ([6] Lemma 4)**.** *For the collision states $\xi_+ \overset{\mathrm{in}}{\times} \xi_-$ and $\eta_+ \overset{\mathrm{in}}{\times} \eta_-$ it holds that*

1. $\langle \xi_+ \overset{\mathrm{in}}{\times} \xi_-, \eta_+ \overset{\mathrm{in}}{\times} \eta_- \rangle = \langle \xi_+, \eta_+ \rangle \cdot \langle \xi_-, \eta_- \rangle$.
2. $U(g)(\xi_+ \overset{\mathrm{in}}{\times} \xi_-) = (U(g)\xi_+) \overset{\mathrm{in}}{\times} (U(g)\xi_-)$ *for all* $g \in \mathcal{P}_+^\uparrow$.

And analogous formulae hold for outgoing collision states.

Furthermore, we define the spaces of collision states: Namely, we let $\mathcal{H}^{\mathrm{in}}$ (respectively $\mathcal{H}^{\mathrm{out}}$) be the subspace generated by $\xi_+ \overset{\mathrm{in}}{\times} \xi_-$ (respectively $\xi_+ \overset{\mathrm{out}}{\times} \xi_-$). From the Lemma above, we see that the following map

$$S : \xi_+ \overset{\mathrm{out}}{\times} \xi_- \longmapsto \xi_+ \overset{\mathrm{in}}{\times} \xi_-$$

is an isometry. The operator $S : \mathcal{H}^{\mathrm{out}} \to \mathcal{H}^{\mathrm{in}}$ is called the **scattering operator** or the **S-matrix**. We say the waves in $\mathcal{A}$ are **interacting** if S is not a constant multiple of the identity operator on $\mathcal{H}^{\mathrm{out}}$. The purpose of this paper is to show that $S = \mathbb{1}$ on $\mathcal{H}^{\mathrm{out}}$ for Möbius covariant nets and to determine $\mathcal{H}^{\mathrm{out}} = \mathcal{H}^{\mathrm{in}}$ in terms of chiral observables (see

Section 4). As a corollary one observes that a Möbius covariant net is chiral if and only if it is **asymptotically complete (with respect to waves)**, i.e. $\mathcal{H}^{\mathrm{out}} = \mathcal{H}^{\mathrm{in}} = \mathcal{H}$. We remark that this notion of asymptotic completeness refers only to massless excitations. If one considers the massive free field, all the asymptotic fields considered here reduce to multiples of the identity. Throughout this article, we are concerned only with waves.

Moreover, we show in Section 5.1 that if a net is Poincaré covariant and asymptotically complete, then it is noninteracting if and only if it is a chiral Möbius covariant net (see Section 4.1) and in Section 5.2 that in- and out- asymptotic fields coincide in a (possibly non-chiral) Möbius covariant net.

3 Noninteraction of waves

3.1 Representations of the spacetime symmetry group

As a preliminary for the proof of the main result, we need to examine the structure of representations of the group generated by translations and dilations.

Recall that we denote by $\mathbf{P}$ the subgroup of $\mathrm{PSL}(2,\mathbb{R})$ generated by (one-dimensional) translations and dilations. The group $\mathbf{P}$ is simply connected, hence it can be considered as a subgroup of $\overline{\mathrm{PSL}(2,\mathbb{R})}$. The direct product $\mathbf{P}\times\mathbf{P} \subset \mathrm{PSL}(2,\mathbb{R})\times\mathrm{PSL}(2,\mathbb{R})$ is the group of (two-dimensional) translations, Lorentz boosts and dilations. For the later use, we only have to consider representations of $\mathbf{P}\times\mathbf{P}$ which extend to positive-energy representations of $\overline{\mathrm{PSL}(2,\mathbb{R})}\times\overline{\mathrm{PSL}(2,\mathbb{R})}$.

Recall further that irreducible positive-energy representations of $\overline{\mathrm{PSL}(2,\mathbb{R})}$ are classified by a nonnegative number l, which is the lowest eigenvalue of the generator of (the universal covering of) the group of rotations (see [24]). We claim that irreducible representations of $\overline{\mathrm{PSL}(2,\mathbb{R})}\times\overline{\mathrm{PSL}(2,\mathbb{R})}$ are classified by pairs of nonnegative numbers $l_{\mathrm{L}}, l_{\mathrm{R}}$. Indeed, we can take the Gårding domain $\mathcal{D}$ since $\overline{\mathrm{PSL}(2,\mathbb{R})}\times\overline{\mathrm{PSL}(2,\mathbb{R})}$ is a finite dimensional Lie group. Furthermore, if a representation is irreducible, then the center of the group must act as scalars. From this it follows that the joint spectrum of generators of left and right rotations is discrete and each point must have positive components by the assumed positivity of energy. The same argument as in [24] shows that an eigenvector with minimal eigenvalues of rotations generates an irreducible representation, hence irreducible representations are classified by this pair of minimal eigenvalues. Conversely, all of these representations are realized by product representations. Let us sum up these observations:

Proposition 3.1. *All the irreducible representations of* $\overline{\mathrm{PSL}(2,\mathbb{R})}\times\overline{\mathrm{PSL}(2,\mathbb{R})}$ *are completely classified by pairs of nonnegative numbers* $(l_{\mathrm{L}}, l_{\mathrm{R}})$*. A representation with a given* $(l_{\mathrm{L}}, l_{\mathrm{R}})$ *is unitarily equivalent to the product of representations of* $\overline{\mathrm{PSL}(2,\mathbb{R})}$ *with lowest weights* $l_{\mathrm{L}}, l_{\mathrm{R}}$ *(*$l_{\mathrm{L}} = 0$ *or* $l_{\mathrm{R}} = 0$ *correspond to the trivial representation). A vector in any of these irreducible representations is invariant under the subgroup* $\overline{\mathrm{PSL}(2,\mathbb{R})}\times\mathrm{id}$ *if and only if it is invariant under* $\tau_0\times\mathrm{id}$*, where* τ_0 *is the translation subgroup of* $\overline{\mathrm{PSL}(2,\mathbb{R})}$ *(and the same holds for the right component).*

We know that if $l \neq 0$ then the restriction of the representation to $\mathbf{P}$ is the unique strictly positive-energy representation [24] (here "positive-energy" means that the generator of translations is positive). As a consequence of Proposition 3.1, we can classify positive-energy irreducible representations of $\mathbf{P} \times \mathbf{P}$ which appear in Möbius covariant nets.

Corollary 3.2. *Let ι and ρ be the trivial and the unique strictly positive-energy representation of $\mathbf{P}$ respectively. Any irreducible positive-energy representation of $\mathbf{P} \times \mathbf{P}$ which extends to $\overline{\mathrm{PSL}(2,\mathbb{R})} \times \overline{\mathrm{PSL}(2,\mathbb{R})}$ is one of the following four representations.*

- $\iota \otimes \iota$,
- $\rho \otimes \iota$,
- $\iota \otimes \rho$,
- $\rho \otimes \rho$.

Any (possibly reducible) representation of $\mathbf{P} \times \mathbf{P}$ extending to $\overline{\mathrm{PSL}(2,\mathbb{R})} \times \overline{\mathrm{PSL}(2,\mathbb{R})}$ is a direct sum of copies of the above four representations.

Proof. The first part of the statement follows directly from Proposition 3.1. The second part is a consequence of the general result (for example, see [11, Sections 8.5 and 18.7]) that any continuous unitary representation (on a separable Hilbert space) of a (separable) locally compact group is unitarily equivalent to a direct integral of irreducible representations. Since by assumption the given representation extends to $\overline{\mathrm{PSL}(2,\mathbb{R})} \times \overline{\mathrm{PSL}(2,\mathbb{R})}$, it decomposes into a direct integral, and the components have positive-energy almost everywhere. Hence they are classified by $(l_{\mathrm{L}}, l_{\mathrm{R}})$ and when restricted to $\mathbf{P} \times \mathbf{P}$ they fall into irreducible representations listed above. Since the integrand takes only four different values (up to unitary equivalence), the direct integral reduces to a direct sum. □

3.2 Proof of noninteraction

As waves are defined in terms of representations of translations, we need to analyse the representation U. We continue to use notations from the previous section. A net $\mathcal{A}$ in this section is always assumed to be Möbius covariant.

The representation ρ of $\mathbf{P}$ does not admit any nontrivial invariant vector with respect to (one-dimensional) translations. The subgroup of dilations is noncompact (isomorphic to $\mathbb{R}$) and for any vector ξ in the representation space of ρ it holds that $\rho(\delta_s)\xi$ tends weakly 0 as $s \to \pm\infty$, where δ_s represents the group element of dilation by e^s.

Remark 3.3. At this point we use the assumed covariance under the action of the two dimensional Möbius group $\overline{\mathrm{PSL}(2,\mathbb{R})} \times \overline{\mathrm{PSL}(2,\mathbb{R})}$. If we assume only the dilation covariance (as in [8]), the present author is not able to exclude the possibility of occurrence of a representation of $\mathbf{P}$ which is trivial only on translations in general. As we will see, the absence of such representations is essential to identify all the waves in the relevant representation space.

But if one assumes Bisognano-Wichmann property and asymptotic completeness in addition, it is possible to show that the representation of the spacetime symmetry extends to the Möbius group: As observed in [31], for an asymptotically complete Poincaré covariant net with Bisognano-Wichmann property, one can define the asymptotic net which is chiral Möbius covariant. The representation of the Möbius group is a natural extension of the given representation of the Poincaré group given through the Bisognano-Wichmann property. Their actions on asymptotic fields are determined by the boosts, hence the representation extends also the given representation of dilation. Summing up, under Bisognano-Wichmann property and asymptotic completeness, the representation of the Poincaré group and dilation extends to the Möbius group.

Among the four irreducible positive-energy representations of $\mathbf{P}\times\mathbf{P}$ (see Corollary 3.2), only $\iota\otimes\iota$ contains a nonzero invariant vector with respect to two-dimensional translations. The representation space of $\iota\otimes\rho$ consists of invariant vectors with respect to positive-lightlike translations but contains no nonzero invariant vectors with respect to negative-lightlike translations. An analogous statement holds for $\rho\otimes\iota$. The representation $\rho\otimes\rho$ contains no nonzero invariant vectors, neither with respect to negative- nor positive-lightlike translations.

Let us consider the representation U of $\overline{\mathrm{PSL}(2,\mathbb{R})}\times\overline{\mathrm{PSL}(2,\mathbb{R})}$ associated with a Möbius covariant net $\mathcal{A}$. The restriction of U to $\mathbf{P}\times\mathbf{P}$ is a direct sum of copies of representations which appeared in Corollary 3.2. By the uniqueness of the vacuum, the representation $\iota\otimes\iota$ appears only once. Waves of positive (respectively negative) momentum correspond precisely to $\rho\otimes\iota$ (respectively $\iota\otimes\rho$). From these observations, it is straightforward to see the following.

Lemma 3.4. *Let us denote by P the spectral measure of the representation $T = U|_{\mathbb{R}^2}$ of translations. Each of the following spectral subspaces of T carries the multiple of one of the irreducible representations in Corollary 3.2 (the correspondence is the order of appearance)*

- $Q_0 := P(\{(0,0)\})$,
- $Q_{\mathrm{L}} := P(\{(a_0,a_1) : a_0 = a_1, a_0 > 0\})$,
- $Q_{\mathrm{R}} := P(\{(a_0,a_1) : a_0 = -a_1, a_0 > 0\})$,
- $Q_{\mathrm{L,R}} := P(\{(a_0,a_1) : a_0 > a_1, a_0 > -a_1\})$.

Let δ^{L} be the dilation in the left-component of $\overline{\mathrm{PSL}(2,\mathbb{R})}\times\overline{\mathrm{PSL}(2,\mathbb{R})}$. Then for any vector $\xi\in\mathcal{H}$, $\operatorname{w-}\lim_{s\to 0} U(\delta^{\mathrm{L}}_s)\xi = (Q_{\mathrm{R}}+Q_0)\xi$. Similarly for the dilation in the right component δ^{R} we have $\operatorname{w-}\lim_{s\to 0} U(\delta^{\mathrm{R}}_s)\xi = (Q_{\mathrm{L}}+Q_0)\xi$. Furthermore, it holds that $Q_{\mathrm{L}}+Q_0 = P_+$, $Q_{\mathrm{R}}+Q_0 = P_-$ (see Section 2.2 for definitions)

After this preparation we proceed to our main result:

Theorem 3.5. *Let $\mathcal{A}$ be a Möbius covariant net. We have the equality $\xi_+ \overset{\mathrm{in}}{\times} \xi_- = \xi_+ \overset{\mathrm{out}}{\times} \xi_-$ for any pair $\xi_+ \in \mathcal{H}_+$ and $\xi_- \in \mathcal{H}_-$. In particular, such waves do not interact and we have $\mathcal{H}^{\mathrm{out}} = \mathcal{H}^{\mathrm{in}}$.*

Proof. We show the equality $\langle \xi_+ \overset{\mathrm{in}}{\times} \xi_-, \eta_+ \overset{\mathrm{out}}{\times} \eta_- \rangle = \langle \xi_+ \overset{\mathrm{in}}{\times} \xi_-, \eta_+ \overset{\mathrm{in}}{\times} \eta_- \rangle$ for any $\xi_+, \eta_+ \in \mathcal{H}_-$ and $\xi_-, \eta_- \in \mathcal{H}_-$. This is in fact enough for the first statement, since we know that $\|\eta_+ \overset{\mathrm{out}}{\times} \eta_-\| = \|\eta_+ \overset{\mathrm{in}}{\times} \eta_-\|$. As a particular case we have $\langle \eta_+ \overset{\mathrm{in}}{\times} \eta_-, \eta_+ \overset{\mathrm{out}}{\times} \eta_- \rangle = \langle \eta_+ \overset{\mathrm{in}}{\times} \eta_-, \eta_+ \overset{\mathrm{in}}{\times} \eta_- \rangle$, which is possible only if $\eta_+ \overset{\mathrm{out}}{\times} \eta_- = \eta_+ \overset{\mathrm{in}}{\times} \eta_-$.

Obviously it suffices to show the equality for a dense set of vectors in $\mathcal{H}_+$ and $\mathcal{H}_-$. Let us take three double cones O_+, O_0, O_- which are timelike separated in this order, more precisely O_0 stays in the future of O_- and in the past of O_+, and assume that O_0 is a neighborhood of the origin. We choose elements $x_+ \in \mathcal{A}(O_+)$ and $y_+, y_- \in \mathcal{A}(O_-)$. We take a self-adjoint element $b \in \mathcal{A}(O_0)$ and set $b_s := \mathrm{Ad}\,(U(\delta_s^{\mathrm{L}}))(b)$ for $s < 0$. Then $\{b_s\}$ are still contained in $\mathcal{A}(O_0)$. We set:

$$
\begin{aligned}
\xi_+ &:= \Phi_+^{\mathrm{in}}(x_+)\Omega, & \xi_- &:= \operatorname*{w-lim}_{s\to 0} b_s\Omega = \operatorname*{w-lim}_{s\to 0} U(\delta_s^{\mathrm{L}})b\Omega,\\
\eta_+ &:= \Phi_+^{\mathrm{out}}(y_+)\Omega, & \eta_- &:= \Phi_-^{\mathrm{out}}(y_-)\Omega,\\
\zeta_- &:= \Phi_-^{\mathrm{out}}(y_-^*)\Omega = \Phi_-^{\mathrm{out}}(y_-)^*\Omega.
\end{aligned}
$$

Note that b_s commutes with $\Phi_+^{\mathrm{in}}(x_+)$, $\Phi_+^{\mathrm{out}}(y_+)$ and $\Phi_-^{\mathrm{out}}(y_-)$ since Φ^{in} and Φ^{out} are defined as strong limits of local operators and from some point they are spacelike separated (see Remark 2.3). Note also that $\Phi_+^{\mathrm{in}}(x_+)\Omega = P_+x_+\Omega, \Phi_+^{\mathrm{out}}(y_+)\Omega = P_+y_+\Omega, \Phi_-^{\mathrm{out}}(y_-)\Omega = P_-y_-\Omega$ and we have $\lim_s b_s\Omega = P_-b\Omega$ by Lemma 3.4.

We see that

$$
\begin{aligned}
\langle \xi_+ \overset{\mathrm{in}}{\times} \xi_-, \eta_+ \overset{\mathrm{out}}{\times} \eta_- \rangle &= \langle \Phi_+^{\mathrm{in}}(x_+)(\operatorname*{w-lim}_{s\to 0} b_s\Omega), \Phi_+^{\mathrm{out}}(y_+)\Phi_-^{\mathrm{out}}(y_-)\Omega \rangle\\
&= \lim_s \langle \Phi_+^{\mathrm{in}}(x_+)b_s\Omega, \Phi_+^{\mathrm{out}}(y_+)\Phi_-^{\mathrm{out}}(y_-)\Omega \rangle\\
&= \lim_s \langle \Phi_-^{\mathrm{out}}(y_-^*)\Phi_+^{\mathrm{in}}(x_+)\Omega, \Phi_+^{\mathrm{out}}(y_+)b_s\Omega \rangle,
\end{aligned}
$$

where we used Remark 2.3 in the 3rd line. Continuing the calculation, with the help of the definition of asymptotic fields, this can be transformed as

$$
\begin{aligned}
\langle \xi_+ \overset{\mathrm{in}}{\times} \xi_-, \eta_+ \overset{\mathrm{out}}{\times} \eta_- \rangle &= \langle \Phi_-^{\mathrm{out}}(y_-^*)\Phi_+^{\mathrm{in}}(x_+)\Omega, \Phi_+^{\mathrm{out}}(y_+)(\operatorname*{w-lim}_{s\to 0} b_s\Omega) \rangle\\
&= \langle \Phi_-^{\mathrm{out}}(y_-^*)\xi_+, \Phi_+^{\mathrm{out}}(y_+)\xi_- \rangle\\
&= \langle \xi_+ \overset{\mathrm{out}}{\times} \zeta_-, \eta_+ \overset{\mathrm{out}}{\times} \xi_- \rangle\\
&= \langle \xi_+, \eta_+ \rangle \cdot \langle \zeta_-, \xi_- \rangle\\
&= \langle \xi_+, \eta_+ \rangle \cdot \langle \Phi_-^{\mathrm{out}}(y_-^*)\Omega, (\operatorname*{w-lim}_{s\to 0} b_s\Omega) \rangle\\
&= \langle \xi_+, \eta_+ \rangle \cdot \langle (\operatorname*{w-lim}_{s\to 0} b_s\Omega), \Phi_-^{\mathrm{out}}(y_-)\Omega \rangle\\
&= \langle \xi_+, \eta_+ \rangle \cdot \langle \xi_-, \eta_- \rangle\\
&= \langle \xi_+ \overset{\mathrm{in}}{\times} \xi_-, \eta_+ \overset{\mathrm{in}}{\times} \eta_- \rangle,
\end{aligned}
$$

where the 6th equality follows from Remark 2.3 and the self-adjointness of b, the 4th and 8th equalities follow from Lemma 2.4. This equation is linear with respect to b (which is implicitly contained in ξ_-), hence it holds for any $b \in \mathcal{A}(O_0)$.

By the Reeh-Schlieder property, each set of vectors of the forms above is dense in $\mathcal{H}_+$ and $\mathcal{H}_-$, respectively. Thus the required equality is obtained for dense subspaces and this concludes the proof. □

The proof of this Theorem uses only the fact that $\mathcal{A}$ is Poincaré-dilation covariant and that the representation of the Poincaré-dilation group extends to the Möbius group. Putting together with Remark 3.3, we obtain

Corollary 3.6. *If a dilation-covariant net $\mathcal{A}$ satisfies Bisognano-Wichmann property and asymptotic completeness, then the waves in $\mathcal{A}$ are not interacting.*

4 Subspace of collision states of waves

It has been shown by Rehren that any Möbius covariant net contains the maximal chiral subnet, consisting of observables localized on the lightrays [28]. Here we show that the vectors generated by such observables from the vacuum exhaust the subspace of collision states. With this information at hand, we provide an alternative proof of noninteraction of waves and show that a Möbius covariant field theory is asymptotically complete if and only if it is chiral.

4.1 Preliminaries on chiral nets

In this section we discuss a fundamental class of examples of two-dimensional Möbius covariant nets, namely chiral theories. A chiral theory is obtained by a tensor product construction from two nets of von Neumann algebras on a circle S^1, defined below, and each of these nets is referred to as **the chiral component** of the theory.

An open nonempty connected nondense subset I of the circle S^1 is called an interval. A **(local) Möbius covariant net $\mathcal{A}_0$ on S^1** assigns to each interval a von Neumann algebra $\mathcal{A}_0(I)$ on a fixed separable Hilbert space $\mathcal{H}_0$ satisfying the following conditions:

(1) **Isotony.** If $I_1 \subset I_2$, then $\mathcal{A}_0(I_1) \subset \mathcal{A}_0(I_2)$.

(2) **Locality.** If $I_1 \cap I_2 = \emptyset$, then $[\mathcal{A}_0(I_1), \mathcal{A}_0(I_2)] = 0$.

(3) **Möbius covariance.** There exists a strongly continuous unitary representation U_0 of the Möbius group $\mathrm{PSL}(2,\mathbb{R})$ such that for any interval I it holds that

$$U_0(g)\mathcal{A}_0(I)U_0(g)^* = \mathcal{A}_0(gI), \text{ for } g \in \mathrm{PSL}(2,\mathbb{R}).$$

(4) **Positivity of energy.** The generator of the one-parameter subgroup of rotations in the representation U_0 is positive.

(5) **Existence of the vacuum.** There is a unique (up to a phase) unit vector Ω_0 in $\mathcal{H}_0$ which is invariant under the action of U_0, and cyclic for $\bigvee_{I \Subset S^1} \mathcal{A}_0(I)$.

Among consequences of these axioms are (see [16])

(6) **Reeh-Schlieder property.** The vector Ω_0 is cyclic and separating for each $\mathcal{A}_0(I)$.

(7) **Additivity.** If $I = \bigcup_i I_i$, then $\mathcal{A}_0(I) = \bigvee_i \mathcal{A}_0(I_i)$.

(8) **Haag duality in S^1.** For an interval I it holds that $\mathcal{A}_0(I)' = \mathcal{A}_0(I')$, where I' is the interior of the complement of I in S^1.

(9) **Bisognano-Wichmann property.** The modular group Δ_0^{it} of $\mathcal{A}_0(\mathbb{R}_+)$ with respect to Ω_0 is equal to $U_0(\delta(-2\pi t))$, where δ is the one-parameter group of dilations.

It is known that the positivity of energy is equivalent to the positivity of the generator of translations [24].

We say that $\mathcal{A}_0$ is **strongly additive** if it holds that $\mathcal{A}_0(I) = \mathcal{A}_0(I_1) \vee \mathcal{A}_0(I_2)$, where I_1 and I_2 are intervals obtained by removing an interior point of I.

Let $\mathrm{Diff}(S^1)$ be the group of orientation-preserving diffeomorphisms of the circle S^1. This group naturally includes $\mathrm{PSL}(2,\mathbb{R})$. If the representation U_0 associated to a Möbius covariant net $\mathcal{A}_0$ extends to a projective unitary representation of $\mathrm{Diff}(S^1)$ such that for any interval I and $x \in \mathcal{A}_0(I)$ it holds that

$$U_0(g)\mathcal{A}_0(I)U_0(g)^* = \mathcal{A}_0(gI), \text{ for } g \in \mathrm{Diff}(S^1),$$
$$U_0(g)xU_0(g)^* = x, \text{ if } \mathrm{supp}(g) \subset I',$$

then $\mathcal{A}_0$ is said to be a **conformal net on** S^1 or to be **diffeomorphism covariant** ($\mathrm{supp}(g) \subset I'$ means that g acts identically on I).

Let $\mathcal{A}_0$ be a Möbius covariant net on S^1. As in Section 2.1, we identify S^1 and $\mathbb{R} \cup \{\infty\}$ by the Cayley transform. Under this identification, for an interval $I \Subset \mathbb{R}$ we write $\mathcal{A}_0(I)$.

Let $\mathcal{A}_\pm$ be two Möbius covariant nets on S^1 defined on the Hilbert spaces $\mathcal{H}_\pm$ with the vacuum vectors $\Omega_\pm$ and the representations of $U_\pm$. We define a two-dimensional net $\mathcal{A}$ as follows: Let $\mathrm{L}_\pm := \{(t_0, t_1) \in \mathbb{R}^2 : t_0 \pm t_1 = 0\}$ be two lightrays. For a double cone O of the form $I \times J$ where $I \subset \mathrm{L}_+$ and $J \subset \mathrm{L}_-$, we set $\mathcal{A}(O) = \mathcal{A}_+(I) \otimes \mathcal{A}_-(J)$. For a general open region $O \subset \mathbb{R}^2$, we set $\mathcal{A}(O) := \bigvee_{I\times J} \mathcal{A}(I \times J)$ where the union is taken among intervals such that $I \times J \subset O$. If we take the vacuum vector as $\Omega := \Omega_+ \otimes \Omega_-$ and define the representation U of $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ by $U(g_+ \times g_-) := U_+(g_1) \times U_-(g_2)$, it is easy to see that all the conditions for Möbius covariant net follow from the corresponding properties of nets on S^1. We say that such $\mathcal{A}$ is **chiral**. If $\mathcal{A}_\pm$ are conformal, then the representation U naturally extends to a projective representation of $\mathrm{Diff}(S^1) \times \mathrm{Diff}(S^1)$. Hence $\mathcal{A}$ is a two-dimensional conformal net.

4.2 The maximal chiral subnet and collision states

As we have seen in Section 4.1, from a pair of Möbius covariant nets on S^1 we can construct a two-dimensional Möbius covariant net. In this section we explain a converse procedure: Namely, starting with a two-dimensional Möbius covariant net $\mathcal{A}$, we find a pair of Möbius covariant nets $\mathcal{A}_\pm$ on S^1 which are maximally contained in $\mathcal{A}$. In general, such a chiral part is just a subnet of the original net. Moreover, we show that the subspace generated by this subnet from the vacuum coincides with the subspace of collision states of waves. It follows that a Möbius covariant net is asymptotically complete if and only if it is chiral.

It is possible to define chiral components in several ways. We follow the definition by Rehren [28]. Recall that the two-dimensional Möbius group $\overline{\mathrm{PSL}(2,\mathbb{R})} \times \overline{\mathrm{PSL}(2,\mathbb{R})}$ is a direct product of two copies of the universal covering group of $\mathrm{PSL}(2,\mathbb{R})$. We write this as $\widetilde{G}_\mathrm{L} \times \widetilde{G}_\mathrm{R}$, where $\widetilde{G}_\mathrm{L}$ and $\widetilde{G}_\mathrm{R}$ are copies of $\overline{\mathrm{PSL}(2,\mathbb{R})}$ [3].

Definition 4.1. *For a two-dimensional Möbius net $\mathcal{A}$ we define nets of von Neumann algebras on $\mathbb{R}$ by the following: For an interval $I \subset \mathbb{R}$ we set the von Neumann algebras*

$$\mathcal{A}_\mathrm{L}(I) := \mathcal{A}(I \times J) \cap U(\widetilde{G}_\mathrm{R})',$$
$$\mathcal{A}_\mathrm{R}(J) := \mathcal{A}(I \times J) \cap U(\widetilde{G}_\mathrm{L})'.$$

The definition of $\mathcal{A}_\mathrm{L}$ (respectively $\mathcal{A}_\mathrm{R}$) does not depend on the choice of J (respectively of I) since $\widetilde{G}_\mathrm{R}$ (respectively $\widetilde{G}_\mathrm{L}$) acts transitively on the family of intervals.

If the net $\mathcal{A}$ is conformal, then the components $\mathcal{A}_\mathrm{L}$ and $\mathcal{A}_\mathrm{R}$ are nontrivial (see Remark B.3)

Lemma 4.2 ([28])**.** *The nets $\mathcal{A}_\mathrm{L}, \mathcal{A}_\mathrm{R}$ extend to Möbius nets on S^1. For a fixed double cone $I \times J$, there holds*

$$\mathcal{A}_\mathrm{L}(I) \vee \mathcal{A}_\mathrm{R}(J) \simeq \mathcal{A}_\mathrm{L}(I) \otimes \mathcal{A}_\mathrm{R}(J).$$

Then we determine $\mathcal{H}^\mathrm{out} = \mathcal{H}^\mathrm{in}$ in terms of chiral components. The key is the following lemma.

Lemma 4.3 ([28], Lemma 2.3)**.** *Let $\mathcal{A}$ be a Möbius covariant net. The subspace $\overline{\mathcal{A}_\mathrm{L}(I)\Omega}$ coincides with the subspace of $\widetilde{G}_\mathrm{R}$-invariant vectors. A corresponding statement holds for $\mathcal{A}_\mathrm{R}(J)$.*

Remark 4.4. The proof of this lemma requires Möbius covariance of the net. On the other hand, in Section 3.2, where we utilized the fact that the representation U of $\mathbf{P} \times \mathbf{P}$ extends to $\overline{\mathrm{PSL}(2,\mathbb{R})} \times \overline{\mathrm{PSL}(2,\mathbb{R})}$, what was really needed is that U decomposes into a direct sum of copies of the four irreducible representations in Corollary 3.2.

Theorem 4.5. *It holds that $\mathcal{H}^\mathrm{out} = \mathcal{H}^\mathrm{in} = \overline{\mathcal{A}_\mathrm{L}(I) \vee \mathcal{A}_\mathrm{R}(J)\Omega}$.*

[3] Generally, the symbol $\widetilde{G}$ is used to indicate the universal covering group for a group G, but for $\mathrm{PSL}(2,\mathbb{R})$ it is customary to use the notation $\overline{\mathrm{PSL}(2,\mathbb{R})}$ for its universal cover.

Proof. As we have seen in Proposition 3.1, the spaces of invariant vectors with respect to $\widetilde{G}_{\mathrm{R}}, \widetilde{G}_{\mathrm{L}}$ and to positive/negative lightlike translations coincide. Lemma 4.3 tells us that $\overline{\mathcal{A}_{\mathrm{L}}(I)\Omega} = \mathcal{H}_+$ and $\overline{\mathcal{A}_{\mathrm{R}}(J)\Omega} = \mathcal{H}_-$.

As elements in $\mathcal{A}_{\mathrm{L}}$ are fixed under the action of $\widetilde{G}_{\mathrm{R}}$, for $x \in \mathcal{A}_{\mathrm{L}}(I)$ it holds that $\Phi_+^{\mathrm{in}}(x) = x$. Similarly we have $\Phi_-^{\mathrm{in}}(y) = y$ for $y \in \mathcal{A}_{\mathrm{R}}(J)$. Thus we see that

$$x\Omega \overset{\mathrm{in}}{\times} y\Omega = \Phi_+^{\mathrm{in}}(x)\Phi_-^{\mathrm{in}}(y)\Omega = xy\Omega \in \overline{\mathcal{A}_{\mathrm{L}}(I) \vee \mathcal{A}_{\mathrm{R}}(J)\Omega}$$

Conversely, since $\mathcal{A}_{\mathrm{L}}(I)$ and $\mathcal{A}_{\mathrm{R}}(J)$ commute, any element in $\mathcal{A}_{\mathrm{L}}(I) \vee \mathcal{A}_{\mathrm{R}}(J)$ can be approximated strongly by linear combinations of elements of product form xy. This implies the required equality of subspaces. □

As a simple corollary, we have another proof of noninteraction of waves and a relation between asymptotic completeness and chirality:

Corollary 4.6. *Let $\mathcal{A}$ be a Möbius covariant net.*

(a) (same as Theorem 3.5) We have the equality $\xi_+ \overset{\mathrm{in}}{\times} \xi_- = \xi_+ \overset{\mathrm{out}}{\times} \xi_-$ for any pair $\xi_+ \in \mathcal{H}_+$ and $\xi_- \in \mathcal{H}_-$. In particular, such waves do not interact.

(b) $\mathcal{H}^{\mathrm{out}} = \mathcal{H}^{\mathrm{in}} = \mathcal{H}$ if and only if $\mathcal{A}$ coincides with its maximal chiral subnet.

Proof. Theorem 4.5 tells us that the space of collision states of waves is generated by chiral observables $\mathcal{A}_{\mathrm{L}}(I) \vee \mathcal{A}_{\mathrm{R}}(J)$. Lemma 2.2 assures that to investigate the S-matrix it is enough to consider observables which generate the collision states. Then, on the space of waves $\mathcal{H}_0 = \overline{\mathcal{A}_{\mathrm{L}}(I) \vee \mathcal{A}_{\mathrm{R}}(J)\Omega}$ and regarding the chiral observables, it has been shown that a chiral net is asymptotically complete ($\mathcal{H}^{\mathrm{out}} = \mathcal{H}^{\mathrm{in}} = \mathcal{H}_0$) and the S-matrix is trivial [13, 14].

If $\mathcal{H}_0 \neq \mathcal{H}$, then by the Reeh-Schlieder property, the full net $\mathcal{A}$ must contain non-chiral observables, and $\mathcal{A}_{\mathrm{L}} \otimes \mathcal{A}_{\mathrm{R}} \neq \mathcal{A}$. If $\mathcal{H}_0 = \mathcal{H}$, since both $\mathcal{A}_{\mathrm{L}} \otimes \mathcal{A}_{\mathrm{R}}$ and $\mathcal{A}$ are Möbius covariant, there is a conditional expectation $E_O : \mathcal{A}(O) \to \mathcal{A}_{\mathrm{L}} \otimes \mathcal{A}_{\mathrm{R}}(O)$ which preserves $\langle \cdot\Omega, \Omega\rangle$, but E_O is in fact the identity map since Ω is cyclic for $\mathcal{A}_{\mathrm{L}} \otimes \mathcal{A}_{\mathrm{R}}(O)$ (see Theorem A.1). □

4.3 How large is the space of collision states?

We have seen that a part $\overline{\mathcal{A}_{\mathrm{L}}(I) \vee \mathcal{A}_{\mathrm{R}}(J)\Omega}$ of the Hilbert space $\mathcal{H}$ can be interpreted as the space of collision states of waves and that these waves do not interact. Then of course it is natural to investigate the particle aspects of the orthogonal complement of this space. We do not go into the detail of this problem here, but restrict ourselves to a few comments.

The algebra of chiral observables $\mathcal{A}_{\mathrm{L}} \otimes \mathcal{A}_{\mathrm{R}}$ is represented on the the full Hilbert space $\mathcal{H}$ in a reducible way. One can decomposes $\mathcal{H}$ into a direct sum of the irreducible components with respect to $\mathcal{A}_{\mathrm{L}} \otimes \mathcal{A}_{\mathrm{R}}$:

$$\mathcal{H} = \bigoplus_i \mathcal{H}_{\rho_i},$$

where $\{\rho_i\}$ are irreducible representations (see [23]) of $\mathcal{A}_{\mathrm{L}} \otimes \mathcal{A}_{\mathrm{R}}$. When $\mathcal{A}_{\mathrm{L}}$ and $\mathcal{A}_{\mathrm{R}}$ are completely rational [21], then the representations ρ_i are tensor products $\rho_i^{\mathrm{L}} \otimes \rho_i^{\mathrm{R}}$ of representations ρ_i^{L} of $\mathcal{A}_{\mathrm{L}}$ and ρ_i^{R} of $\mathcal{A}_{\mathrm{R}}$. As we consider the maximal chiral subnet introduced by Rehren, the vacuum representations $\rho_0^{\mathrm{L}}, \rho_0^{\mathrm{R}}$ appear only once, in the form $\rho_0^{\mathrm{L}} \otimes \rho_0^{\mathrm{R}}$ [28, Corollary 3.5]. This representation $\rho_0^{\mathrm{L}} \otimes \rho_0^{\mathrm{R}}$ is realized on the subspace $\mathcal{H}_0 = \overline{\mathcal{A}_{\mathrm{L}}(I) \vee \mathcal{A}_{\mathrm{R}}(J)\Omega}$. Theorem 4.5 says that the waves are contained only in $\mathcal{H}_0$.

Hence, when $\mathcal{A}$ is not chiral, the space of collision states is at most a half of the full Hilbert space, if we simply count the number of representations which appear in the decomposition. A conceptually more satisfactory measure is the index of the inclusion $[\mathcal{A} : \mathcal{A}_{\mathrm{L}} \otimes \mathcal{A}_{\mathrm{R}}]$. The minimal value of the index of a nontrivial inclusion is 2, which would mean again that waves occupy half of the available space. This case indeed happens: Let $\mathcal{A}_0$ be a Möbius covariant net on S^1 with $\mathbb{Z}_2$ symmetry. If we define $\mathcal{A} = (\mathcal{A}_0 \otimes \mathcal{A}_0)^{\mathbb{Z}_2}$, where $\mathbb{Z}_2$ acts on $\mathcal{A}_0 \otimes \mathcal{A}_0$ by the diagonal action and $(\mathcal{A}_0 \otimes \mathcal{A}_0)^{\mathbb{Z}_2}$ is the fixed point subnet of this action, then $\mathcal{A}$ has $\mathcal{A}_0^{\mathbb{Z}_2} \otimes \mathcal{A}_0^{\mathbb{Z}_2}$ as the maximal chiral subnet and the index $[\mathcal{A} : \mathcal{A}_0^{\mathbb{Z}_2} \otimes \mathcal{A}_0^{\mathbb{Z}_2}]$ is 2. But in this case it is natural to say that the orthogonal complement can be interpreted as collision states in a bigger net $\mathcal{A}_0 \otimes \mathcal{A}_0$ which do not interact. In general, if a given net is not the fixed point, such a reinterpretation of the orthogonal complement as waves is impossible and the index is typically larger than 2. New ideas are needed to clarify this general case.

5 Asymptotic fields given through conditional expectations

5.1 Characterization of noninteracting nets

In [6], in the general setting of Poincaré covariant nets, Buchholz has proved that timelike commutativity implies the absence of interaction. The purpose of this subsection is to show a strengthened converse, namely that if a two-dimensional Poincaré covariant net is asymptotically complete and noninteracting, then under natural assumptions it is (unitarily equivalent to) a chiral Möbius covariant net.

For this purpose, it is appropriate to extend the definition of a net also to unbounded regions. Let $\mathcal{A}$ be a Poincaré covariant net. For an arbitrary open region O, we define $\mathcal{A}(O) := \bigvee_{D \subset O} \mathcal{A}(D)$, where D runs over all bounded regions included in O (this definition coincides with the original net if O is bounded). Among important unbounded regions are **wedges**. The standard left and right wedges are defined as follows:

$$\begin{aligned} W_{\mathrm{L}} &:= \{(t_0, t_1) : t_0 > t_1, t_0 < -t_1\} \\ W_{\mathrm{R}} &:= \{(t_0, t_1) : t_0 < t_1, t_0 > -t_1\} \end{aligned}$$

The regions W_{L} and W_{R} are invariant under Lorentz boosts. The causal complement of W_{L} is W_{R} (and vice versa). All the regions obtained by translations of standard wedges are still called left- and right- wedges, respectively. Moreover, any double cone is obtained as the intersection of a left wedge and a right wedge. Let O' denote the causal complement of O in $\mathbb{R}^2$ (not in $\mathcal{E}$). It holds that $W_{\mathrm{L}}' = W_{\mathrm{R}}$, and if $D = (W_{\mathrm{R}} + a) \cap (W_{\mathrm{L}} + b)$ is a double cone, $a, b \in \mathbb{R}^2$, then $D' = (W_{\mathrm{L}} + a) \cup (W_{\mathrm{R}} + b)$. It is easy to see that Ω is still cyclic and separating for $\mathcal{A}(W_{\mathrm{R}})$ and $\mathcal{A}(W_{\mathrm{L}})$.

Let us introduce some additional assumptions on the structure of nets.

- **Haag duality.** If O is a wedge or a double cone, then it holds that $\mathcal{A}(O)' = \mathcal{A}(O')$.
- **Bisognano-Wichmann property.** The modular group Δ^{it} of $\mathcal{A}(W_{\mathrm{R}})$ with respect to Ω is equal to $U(\Lambda(-2\pi t))$, where $\Lambda(t) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$ denotes the Lorentz boost.

Duality for wedges (namely $\mathcal{A}(W_{\mathrm{L}})' = \mathcal{A}(W_{\mathrm{R}})$) follows from the Bisognano-Wichmann property (see Proposition A.2). If $\mathcal{A}$ is Möbius covariant, then the Bisognano-Wichmann property is automatic [4], and Haag duality is equivalent to strong additivity [28]. Apart from Möbius nets, these properties are common even in massive interacting models [22]. Furthermore, starting with $\mathcal{A}(W_{\mathrm{L}})$, it is possible to construct a net which satisfies both properties [5, 22]. Hence we believe that these additional assumptions are natural and throughout this section we assume that the net $\mathcal{A}$ satisfies them.

Let $\mathcal{A}$ be a Poincaré covariant net satisfying the Bisognano-Wichmann property. We start with general remarks on asymptotic fields. Let $\mathcal{N}_+^{\mathrm{out}}$ be the von Neumann algebra generated by $\Phi_+^{\mathrm{out}}(x)$ where $x \in \mathcal{A}(O)$, $O \subset W_{\mathrm{R}}$ and O is bounded [4].

Lemma 5.1. *The the map Φ_+^{out} which gives the asymptotic field is a conditional expectation (cf. Appendix A) from $\mathcal{A}(W_{\mathrm{R}})$ onto $\mathcal{N}_+^{\mathrm{out}}$ which preserves the vacuum state $\omega := \langle \Omega, \cdot\Omega\rangle$.*

Proof. By construction, $\Phi_+^{\mathrm{out}}(x) \in \mathcal{A}(W_{\mathrm{R}})$ for such $x \in \mathcal{A}(O), O \subset W_{\mathrm{R}}$ as above. Recall that if g is a Poincaré transformation, it holds that $\mathrm{Ad}\, U(g)\Phi_+^{\mathrm{out}}(x) = \Phi_+^{\mathrm{out}}(\mathrm{Ad}\, U(g)(x))$ (see Lemma 2.2). Hence $\mathcal{N}_+^{\mathrm{out}}$ is invariant under Lorentz boosts $\mathrm{Ad}\, U(\Lambda(-2\pi t)), t \in \mathbb{R}$. Since we assume the Bisognano-Wichmann property, $\mathcal{N}_+^{\mathrm{out}}$ is invariant under the modular group of $\mathcal{A}(W_{\mathrm{R}})$ with respect to ω.

By Takesaki's Theorem A.1, there is a conditional expectation E from $\mathcal{A}(W_{\mathrm{R}})$ onto $\mathcal{N}_+^{\mathrm{out}}$ and this is implemented by the projection P_+^{out} onto $\overline{\mathcal{N}_+^{\mathrm{out}}\Omega}$. By Lemma 2.2, we know that $P_+^{\mathrm{out}} = P_+$. Two operators $E(x)$ and $\Phi_+^{\mathrm{out}}(x)$ in $\mathcal{A}(W_{\mathrm{R}})$ satisfy $E(x)\Omega = P_+^{\mathrm{out}}x\Omega = P_+x\Omega = \Phi_+^{\mathrm{out}}(x)\Omega$. The vacuum vector Ω is separating for $\mathcal{A}(W_{\mathrm{R}})$, hence they coincide. $\square$

Analogously, we consider $\mathcal{N}_-^{\mathrm{in}}$ generated by $\{\Phi_-^{\mathrm{in}}(x) : x \in \mathcal{A}(O), O \subset W_{\mathrm{R}}, O \text{ bounded}\}$. The map Φ_-^{in} is the conditional expectation from $\mathcal{A}(W_{\mathrm{R}})$ onto $\mathcal{N}_-^{\mathrm{in}}$.

Proposition 5.2. *Let us assume that $\mathcal{A}$ is asymptotically complete. The wedge algebra $\mathcal{A}(W_{\mathrm{R}})$ is generated by $\mathcal{N}_+^{\mathrm{out}}$ and $\mathcal{N}_-^{\mathrm{in}}$.*

[4]From Lemma 5.1 it is immediate that Φ_+^{out} naturally extends to $\mathcal{A}(W_{\mathrm{R}})$, but it is convenient to define $\mathcal{N}_+^{\mathrm{in}}$ with bounded regions since we see the relation between Φ_+^{in} and Φ_-^{out} in Lemma 5.3.

Proof. As we observed before Lemma 5.1, $\mathcal{N}_+^{\mathrm{out}}$ and $\mathcal{N}_-^{\mathrm{in}}$ are invariant under Lorentz boosts. Hence the same holds for $\mathcal{N}_{\mathrm{R}} := \mathcal{N}_+^{\mathrm{out}} \vee \mathcal{N}_-^{\mathrm{in}}$. Again by Theorem A.1, there is a conditional expectation E from $\mathcal{A}(W_{\mathrm{R}})$ onto $\mathcal{N}_{\mathrm{R}}$. The wedge algebra $\mathcal{A}(W_{\mathrm{R}})$ is already in the GNS representation of the vacuum ω since Ω is cyclic and separating for $\mathcal{A}(W_{\mathrm{R}})$. $\mathcal{N}_R\Omega$ contains all the collision states, since $\mathcal{N}_{\mathrm{R}}\Omega \supset \{\Phi_+^{\mathrm{out}}(x)\Phi_-^{\mathrm{in}}(y)\Omega\}$ and the assumption of asymptotic completeness tells us that $\mathcal{N}_{\mathrm{R}}\Omega$ is dense in $\mathcal{H}$, hence the projection $P_{\mathcal{N}_{\mathrm{R}}}$ onto $\overline{\mathcal{N}_{\mathrm{R}}\Omega}$ is equal to $\mathbb{1}$. Therefore the conditional expectation E is in fact the identity map and $\mathcal{N}_{\mathrm{R}} = \mathcal{A}(W_{\mathrm{R}})$. □

Lemma 5.3. *Let us assume that $\mathcal{A}$ is asymptotically complete and noninteracting. Then it holds that $\Phi_+^{\mathrm{out}}(x) = \Phi_+^{\mathrm{in}}(x)$ and $\Phi_-^{\mathrm{in}}(x) = \Phi_-^{\mathrm{out}}(x)$ for $x \in \mathcal{A}(O)$.*

Proof. We present the proof for "+" objects only, since the other assertion is analogous. By the assumption that $S = \mathbb{1}$, it follows that $\xi_+ \overset{\mathrm{in}}{\times} \xi_- = \xi_+ \overset{\mathrm{out}}{\times} \xi_-$ for any pair $\xi_+ \in \mathcal{H}_+, \xi_- \in \mathcal{H}_-$. Then we have

$$\begin{aligned}
\Phi_+^{\mathrm{out}}(x) \cdot \xi_+ \overset{\mathrm{out}}{\times} \xi_- &= (\Phi_+^{\mathrm{out}}(x)\xi_+) \overset{\mathrm{out}}{\times} \xi_- \\
&= P_+ x \xi_+ \overset{\mathrm{out}}{\times} \xi_- \\
&= P_+ x \xi_+ \overset{\mathrm{in}}{\times} \xi_- \\
&= (\Phi_+^{\mathrm{in}}(x)\xi_+) \overset{\mathrm{in}}{\times} \xi_- \\
&= \Phi_+^{\mathrm{in}}(x) \cdot \xi_+ \overset{\mathrm{in}}{\times} \xi_- \\
&= \Phi_+^{\mathrm{in}}(x) \cdot \xi_+ \overset{\mathrm{out}}{\times} \xi_-,
\end{aligned}$$

where, in the 1st and 5th lines we used the fact that right- and left- moving asymptotic fields commute, the 2nd and 4th equalities come from Lemma 2.2 and the rest is particular cases of the equivalence between "$\overset{\mathrm{in}}{\times}$" and "$\overset{\mathrm{out}}{\times}$". By the assumption of asymptotic completeness, $\xi_+ \overset{\mathrm{in}}{\times} \xi_- = \xi_+ \overset{\mathrm{out}}{\times} \xi_-$ span the whole space, hence we have the equality of operators $\Phi_+^{\mathrm{out}}(x) = \Phi_+^{\mathrm{in}}(x)$. □

Lemma 5.4. *Let us assume that $\mathcal{A}$ is asymptotically complete and noninteracting. The map*

$$W : \xi_+ \otimes \xi_- \mapsto \xi_+ \overset{\mathrm{in}}{\times} \xi_- = \xi_+ \overset{\mathrm{out}}{\times} \xi_-$$

gives a natural unitary equivalence $(P_+\mathcal{N}_+^{\mathrm{out}}) \otimes (P_-\mathcal{N}_-^{\mathrm{in}}) \simeq \mathcal{A}(W_{\mathrm{R}})$, which is elementwise expressed as $P_+\Phi_+^{\mathrm{out}}(x) \otimes P_-\Phi_-^{\mathrm{in}}(y) \mapsto \Phi_+^{\mathrm{out}}(x)\Phi_-^{\mathrm{in}}(y)$. Furthermore, this decomposition is compatible with the action of the Poincaré group $\mathcal{P}_+^{\uparrow}$: $\mathcal{H}_+$ and $\mathcal{H}_-$ are invariant under $\mathcal{P}_+^{\uparrow}$, hence there is a tensor product representation on $\mathcal{H}_+ \otimes \mathcal{H}_-$ and it holds that $W \cdot (U(g)P_+\Phi_+^{\mathrm{out}}(x)\Omega \otimes U(g)P_-\Phi_-^{\mathrm{in}}(y)\Omega) = U(g)W \cdot (P_+\Phi_+^{\mathrm{out}}(x)\Omega \otimes P_-\Phi_-^{\mathrm{in}}(y)\Omega)$.

Proof. The unitarity of the map W in the statement is clear from Lemma 2.4 and it follows that W intertwines the actions of asymptotic fields by Lemma 2.2: Namely, Φ_+^{out} and Φ_-^{out}

act as in a tensor product (Lemma 2.2, 2.4) but we know that $\Phi_-^{\mathrm{out}}(x) = \Phi_-^{\mathrm{in}}(x)$ from noninteraction (Lemma 5.3). As for the action of the Poincaré group, we see from Lemma 2.2, for x and y as in Lemma 5.3, that

$$\begin{aligned} W \cdot U(g)\Phi_+^{\mathrm{out}}(x)\Phi_-^{\mathrm{in}}(y)\Omega &= W \cdot \mathrm{Ad}\, U(g)(\Phi_+^{\mathrm{out}}(x))\mathrm{Ad}\, U(g)(\Phi_-^{\mathrm{in}}(y))\Omega \\ &= W \cdot \Phi_+^{\mathrm{out}}(\mathrm{Ad}\, U(g)(x))\Phi_-^{\mathrm{in}}(\mathrm{Ad}\, U(g)(y))\Omega \\ &= P_+\Phi_+^{\mathrm{out}}(\mathrm{Ad}\, U(g)(x))\Omega \otimes P_-\Phi_-^{\mathrm{in}}(\mathrm{Ad}\, U(g)(y))\Omega \\ &= P_+U(g)\Phi_+^{\mathrm{out}}(x)\Omega \otimes P_-U(g)\Phi_-^{\mathrm{in}}(y)\Omega \\ &= U(g)P_+\Phi_+^{\mathrm{out}}(x)\Omega \otimes U(g)P_-\Phi_-^{\mathrm{in}}(y)\Omega, \end{aligned}$$

where in the last step we used the fact that $\mathcal{H}_+$ and $\mathcal{H}_-$ are invariant under $U(g)$. This completes the proof. $\square$

Let us recall the notion of a half-sided modular inclusion due to Wiesbrock, with which we recover the Möbius symmetry of a given noninteracting net.

Theorem 5.5 ([32, 2])**.** *Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of von Neumann algebras, Ω be a cyclic and separating vector for $\mathcal{N}, \mathcal{M}$ and $\mathcal{M} \cap \mathcal{N}'$. Let us assume that the modular group σ_t^Ω of $\mathcal{M}$ with respect to the state $\langle\Omega, \cdot\Omega\rangle$ preserves $\mathcal{N}$ with $t \geq 0$ (respectively $t \leq 0$). Then there is a Möbius covariant net $\mathcal{A}_0$ on S^1 such that $\mathcal{A}_0(\mathbb{R}_-) = \mathcal{M}$ and $\mathcal{A}_0(\mathbb{R}_- - 1) = \mathcal{N}$ (respectively $\mathcal{A}_0(\mathbb{R}_+) = \mathcal{M}$ and $\mathcal{A}_0(\mathbb{R}_+ + 1) = \mathcal{N}$).*

If a unitary representation T_0 of $\mathbb{R}$ with positive spectrum satisfies $T_0(t)\Omega = \Omega$ for $t \in \mathbb{R}$, $\mathrm{Ad}\, T_0(t)(\mathcal{M}) \subset \mathcal{M}$ for $t \leq 0$ (respectively $t \geq 0$) and $\mathrm{Ad}\, T_0(-1)(\mathcal{M}) = \mathcal{N}$ (respectively $\mathrm{Ad}\, T_0(1)(\mathcal{M}) = \mathcal{N}$), then T_0 is the representation of the translation group of the Möbius covariant net constructed above.

Such an inclusion $\mathcal{N} \subset \mathcal{M}$ is called a **standard $\pm$half-sided modular inclusion** (standardness refers to the condition that Ω is cyclic and separating for $\mathcal{M} \cap \mathcal{N}'$). If T_0 is the representation of the translation group, the Möbius net on S^1 restricted to the real line $\mathbb{R}$ has an explicit form [32, 2]

$$\mathcal{A}_0((s,t)) = T_0(s)\mathcal{M}'T_0(s)^* \cap T_0(t)\mathcal{M}T_0(t)^*$$
$$(\text{respectively } \mathcal{A}_0((s,t)) = T_0(s)\mathcal{M}T_0(s)^* \cap T_0(t)\mathcal{M}'T_0(t)^*).$$

For a von Neumann algebra $\mathcal{N}$ on the Hilbert space $\mathcal{H}$ (on which the net $\mathcal{A}$ is defined), we denote $\mathcal{N}(a) = \mathrm{Ad}\, T(a)(\mathcal{N})$ for $a \in \mathbb{R}^2$, where T is the representation of the translation group for the net $\mathcal{A}$ (see Section 2.2). We put $a_1 := (1,1), a_{-1} := (-1,1) \in \mathbb{R}^2$.

Lemma 5.6. *The inclusion $P_+\mathcal{N}_+^{\mathrm{out}}(a_{-1}) \subset P_+\mathcal{N}_+^{\mathrm{out}}$ is a standard $+$half-sided modular inclusion with respect to Ω on $\mathcal{H}_+$. Analogously, $P_-\mathcal{N}_-^{\mathrm{in}}(a_1) \subset P_-\mathcal{N}_-^{\mathrm{in}}$ is a standard $-$half-sided modular inclusion with respect to Ω on $\mathcal{H}_-$.*

Proof. We prove only the former claim, since the latter is analogous. Recall that the conditional expectation Φ_+^{out} commutes with translations (Lemma 2.2), hence $\mathcal{N}_+^{\mathrm{out}}(a_{-1})$ is generated by $\{\Phi_+^{\mathrm{out}}(x) : x \in \mathcal{A}(O), O \subset W_{\mathrm{R}} + a_{-1}, O \text{ bounded}\}$. The region $W_{\mathrm{R}} + a_{-1}$ is mapped into itself by Lorentz boosts $\Lambda(-t), t \geq 0$. Lemma 5.1 tells us that Φ_+^{out} is a conditional expectation which preserves $\omega := \langle \Omega, \cdot \Omega \rangle$, hence the modular automorphism of $\mathcal{N}_+^{\mathrm{out}}$ with respect to ω is the restriction of the modular automorphism of $\mathcal{A}(W_{\mathrm{R}})$. Thus Bisognano-Wichmann property shows that $\mathcal{N}_+^{\mathrm{out}}(a_{-1})$ is invariant under the modular automorphism σ_t^Ω of $\mathcal{N}_+^{\mathrm{out}}$ for $t \geq 0$. The projection P_+ commutes with both of $\mathcal{N}_+^{\mathrm{out}}$ and $\mathcal{N}_+^{\mathrm{out}}(a_{-1})$, hence it is a +half-sided modular inclusion.

As for standardness, note that $\mathcal{A}(W_{\mathrm{R}}) \cap \mathcal{A}(W_{\mathrm{L}} + a_{-1} + a_1)$ contains $\mathcal{A}(D)$ where $D = W_{\mathrm{R}} \cap (W_{\mathrm{L}} + a_{-1} + a_1)$ is a double cone. Recall that $\mathcal{A}(W_{\mathrm{R}}) \simeq P_+\mathcal{N}_+^{\mathrm{out}} \otimes P_-\mathcal{N}_-^{\mathrm{in}}$ and the action of the Poincaré group splits as well (Lemma 5.4). According to this unitary equivalence we have $\mathcal{A}(W_{\mathrm{L}} + a_{-1} + a_1) \simeq P_+\mathcal{N}_+^{\mathrm{out}}(a_{-1})' \otimes P_-\mathcal{N}_-^{\mathrm{in}}(a_1)'$ and $\mathcal{A}(W_{\mathrm{R}}) \cap \mathcal{A}(W_{\mathrm{L}} + a_{-1} + a_1) \simeq P_+(\mathcal{N}_+^{\mathrm{out}} \cap \mathcal{N}_+^{\mathrm{out}}(a_{-1})') \otimes P_-(\mathcal{N}_-^{\mathrm{in}} \cap \mathcal{N}_-^{\mathrm{in}}(a_1)')$, since we have wedge duality (Proposition A.2). The vector $\Omega \simeq \Omega \otimes \Omega$ is cyclic for $\mathcal{A}(D)$ (Reeh-Schlieder property) and this is possible only if Ω is cyclic for both $P_+ \left(\mathcal{N}_+^{\mathrm{out}} \cap \mathcal{N}_+^{\mathrm{out}}(a_{-1})'\right)$ and $P_- \left(\mathcal{N}_-^{\mathrm{in}} \cap \mathcal{N}_-^{\mathrm{in}}(a_1)'\right)$. The cyclicity for $P_+ \left(\mathcal{N}_+^{\mathrm{out}} \cap \mathcal{N}_+^{\mathrm{out}}(a_{-1})'\right)$ is the standardness. □

Theorem 5.7. *Let $\mathcal{A}$ be a Poincaré covariant net, asymptotically complete and noninteracting (satisfying Haag duality and Bisognano-Wichmann property). Then it is a chiral Möbius covariant net.*

Proof. First we have to prepare two Möbius covariant nets on S^1: This has been done in Lemma 5.6. Namely, putting $a_{\pm t} = (t, \pm t) \in \mathbb{R}^2$ for $t \in \mathbb{R}$, we have two nets

$$\begin{aligned}\mathcal{A}_{\mathrm{L}}((s,t)) &= P_+ \left(\mathcal{N}_+^{\mathrm{out}}(a_{-s})' \cap \mathcal{N}_+^{\mathrm{out}}(a_{-t})\right), \\ \mathcal{A}_{\mathrm{R}}((s,t)) &= P_- \left(\mathcal{N}_-^{\mathrm{in}}(a_s) \cap \mathcal{N}_-^{\mathrm{in}}(a_t)'\right).\end{aligned}$$

Under the unitary equivalence between $\mathcal{H}$ and $\mathcal{H}_+ \otimes \mathcal{H}_-$ from Lemma 5.4, Haag duality implies that, for the double cone $D = W_{\mathrm{R}} \cap (W_{\mathrm{L}} + a_{-1} + a_1)$, we have

$$\begin{aligned}\mathcal{A}(D) &= \mathcal{A}(W_{\mathrm{R}}) \cap \mathcal{A}(W_{\mathrm{L}} + a_{-1} + a_1) \\ &\simeq P_+(\mathcal{N}_+^{\mathrm{out}} \cap \mathcal{N}_+^{\mathrm{out}}(a_{-1})') \otimes P_-(\mathcal{N}_-^{\mathrm{in}} \cap \mathcal{N}_-^{\mathrm{in}}(a_1)') \\ &= \mathcal{A}_{\mathrm{L}}((-1,0)) \otimes \mathcal{A}_{\mathrm{R}}((0,1)).\end{aligned}$$

The corresponding equality for general intervals $(s_{\mathrm{L}}, t_{\mathrm{L}}), (s_{\mathrm{R}}, t_{\mathrm{R}})$ follows from the above definition of nets $\mathcal{A}_{\mathrm{L}}, \mathcal{A}_{\mathrm{R}}$. In Lemma 5.4 we saw that the actions of the Poincaré group are compatible with this unitary equivalence. □

Remark 5.8. Haag duality is used only in Theorem 5.7. Since a Poincaré covariant net $\mathcal{A}$ with Bisognano-Wichmann property is wedge dual (Propositions A.2, A.3), we can infer that the dual net $\mathcal{A}^{\mathrm{d}}$ (see [3]) is a chiral Möbius net even if we do not assume Haag duality.

Combining this and Corollary 3.6, we see the following:

Corollary 5.9. *An asymptotically complete, Poincaré-dilation covariant net $\mathcal{A}$ (satisfying Haag duality and Bisognano-Wichmann property) is (unitarily equivalent to) a chiral Möbius covariant net.*

5.2 Asymptotic fields in Möbius covariant nets

Finally, as a further consequence of the considerations on conditional expectations, we show that in- and out-asymptotic fields coincide in Möbius covariant nets even without assuming the asymptotic completeness. Lemma 5.1 has been proved for general Poincaré covariant nets with Bisognano-Wichmann property, hence it applies to Möbius covariant nets as well (see (10M) in Section 2.1). We use the same notations as in Section 4.

Let $\mathcal{A}_\mathrm{L} \otimes \mathcal{A}_\mathrm{R}$ be the maximal chiral subnet. Since both nets $\mathcal{A}$ and $\mathcal{A}_\mathrm{L} \otimes \mathcal{A}_\mathrm{R}$ are Möbius covariant, they satisfy Bisognano-Wichmann property in $\mathcal{E}$ ((10M) in Section 2.1). Theorem A.1 of Takesaki implies that there is a conditional expectation E_D from $\mathcal{A}(D)$ onto $\mathcal{A}_\mathrm{L}(I) \otimes \mathcal{A}_\mathrm{R}(J)$, where $D = I \times J$ is a double cone in $\mathcal{E}$, which is implemented by the projection P onto $\mathcal{H}^\mathrm{in} = \mathcal{H}^\mathrm{out} = \overline{\mathcal{A}_\mathrm{L}(I) \vee \mathcal{A}_\mathrm{R}(J)\Omega}$ (see Theorem 4.5). Since the projection P does not depend on D, the conditional expectations $\{E_D\}_{D \subset \mathcal{E}}$ are compatible, namely, if $D_1 \subset D_2$ then it holds that $E_{D_2}|_{\mathcal{A}(D_1)} = E_{D_1}$. Indeed, it holds that $E_{D_1}(x)\Omega = Px\Omega = E_{D_2}(x)\Omega$ and Ω is separating for $\mathcal{A}(D_2)$.

In addition, there is a conditional expectation $\mathrm{id} \otimes \omega$ from $\mathcal{A}_\mathrm{L}(I) \otimes \mathcal{A}_\mathrm{R}(J) \simeq \mathcal{A}_\mathrm{L}(I) \vee \mathcal{A}_\mathrm{R}(J)$ onto $\mathcal{A}_\mathrm{L}(I)$ which obviously preserves ω and is implemented by P_+ (see Theorem A.1). If we take intervals $I_1 \subset I_2$, then the corresponding expectations are obviously compatible. By composing this expectation and E_D, we find an expectation E_L from $\mathcal{A}(D)$ onto $\mathcal{A}_\mathrm{L}(I)$ which preserves ω and is implemented by P_+ (we omit the dependence on D since this family of expectations is compatible). Analogous statements hold for $\mathcal{A}_\mathrm{R}(J)$.

Theorem 5.10. *If $\mathcal{A}$ is a Möbius covariant net, then for $x \in \mathcal{A}(D)$ with some bounded double cone $D = I \times J$, it holds that $\Phi_+^\mathrm{out}(x) = \Phi_+^\mathrm{in}(x)$ and $\Phi_-^\mathrm{out}(x) = \Phi_-^\mathrm{in}(x)$.*

Proof. We exhibit the proof only for "+" objects since the other is analogous. As we have seen in Lemma 5.1, Φ_+^out is a conditional expectation from $\mathcal{A}(W_\mathrm{R})$ onto $\mathcal{N}_+^\mathrm{out}$ which preserves ω.

We claim that $\Phi_+^\mathrm{out}(x) = E_\mathrm{L}(x)$. We may assume that $D \subset W_\mathrm{R}$ since Φ_+^out commutes with translations, and E_L is compatible and the translated expectation $\mathrm{Ad}\, T(a) \circ E_\mathrm{L} \circ \mathrm{Ad}\, T(-a)$ still preserves ω (hence E_L commutes with translation $\mathrm{Ad}\, T(a)$ as well). It holds that $\Phi_+^\mathrm{out}(x) \subset \mathcal{A}(W_\mathrm{R})$ and $E_\mathrm{L}(x) \in \mathcal{A}_\mathrm{L}(I) \subset \mathcal{A}(D) \subset \mathcal{A}(W_\mathrm{R})$. In addition we have $\Phi_+^\mathrm{out}(x)\Omega = P_+x\Omega = E_\mathrm{L}(x)\Omega$, hence by the separating property of Ω for $\mathcal{A}(W_\mathrm{R})$ we obtain the claimed equality.

Similarly one sees $\Phi_+^\mathrm{in}(x) = E_\mathrm{L}(x)$, hence two maps Φ_+^out and Φ_+^in coincide. □

6 Concluding remarks

In the first part of this work we showed that waves in two-dimensional Möbius nets do not interact. This result can be seen as a (non-trivial) adaptation of an argument of Buchholz and Fredenhagen [8] to the two-dimensional case. Moreover, we showed that collision states of waves correspond precisely to states generated from the vacuum by observables from the maximal chiral subnet. This implies the equivalence between asymptotic completeness and chirality of a given Möbius covariant theory. As we observed in Appendix B, chiral observables admit geometric definitions. This is a special feature of two-dimensional Möbius theory, which, to our knowledge, does not have a counterpart in higher-dimensional theories.

The second part of this paper relies on our observation that, in a Poincaré covariant net with the Bisognano-Wichmann property, the maps which give asymptotic fields are conditional expectations. Exploiting this fact, we showed that a Haag dual net is asymptotically complete and noninteracting if and only if it is a chiral Möbius net. We also strengthened our result on noninteraction by showing that in- and out-asymptotic fields in any (possibly non-chiral) Möbius net coincide.

The orthogonal complement of the space of collision states, which may be quite large as we explained in Section 4.3, is a natural subject of future research. Fortunately, we have tools to investigate this orthogonal complement: They include the theory of particle weights [9, 27], developed to study infraparticles. With the help of this theory we have confirmed that infraparticles are present in all states in product representations of the chiral subnet, hence in the orthogonal complement of the space of collision states of waves in any completely rational net [13, 21]. The question of interaction and asymptotic completeness of these excitations remains open to date (for a general account on asymptotic completeness, see [7]). However, the fact that the incoming and outgoing asymptotic fields coincide in Möbius covariant theories on the entire Hilbert space suggests the absence of interaction. These issues are under investigation [15].

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Appendix A Remarks on conditional expectations

The Bisognano-Wichmann property asserts a relation between the dynamics of the net and the Tomita-Takesaki modular theory. In the modular theory, one of the fundamental tools is conditional expectation. We briefly recall here its definition and discuss some immediate consequences. A **conditional expectation** from a von Neumann algebra $\mathcal{M}$ onto a subalgebra $\mathcal{N}$ is a linear map $E : \mathcal{M} \to \mathcal{N}$ satisfying the following properties:

- $E(x) = x$ for $x \in \mathcal{N}$.
- $E(xyz) = xE(y)z$ for $x, z \in \mathcal{N}$, $y \in \mathcal{M}$.
- $E(x)^*E(x) \leq E(x^*x)$ for $x \in \mathcal{M}$.

We see in Section 5.1 that the maps which give asymptotic fields can be considered as conditional expectations between appropriate von Neumann algebras. Let us recall the fundamental theorem of Takesaki [30, Theorem IX.4.2].

Theorem A.1. *Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of von Neumann algebras and φ be a faithful normal state on $\mathcal{M}$. Then the following are equivalent:*

- *$\mathcal{N}$ is invariant under the modular automorphism group σ_t^φ.*
- *There is a normal conditional expectation E from $\mathcal{M}$ onto $\mathcal{N}$ such that $\varphi = \varphi \circ E$.*

Furthermore, if the above conditions hold, then the conditional expectation E is implemented by a projection in the following sense: We consider the GNS representation π_φ and Φ be the GNS vector. Let $P_\mathcal{N}$ be the projection onto the subspace $\overline{\mathcal{N}\Phi}$. Then it holds that $E(x)\Phi = P_\mathcal{N} x\Phi$. In particular, $\mathcal{N} = \mathcal{M}$ if and only if $P_\mathcal{N} = \mathbb{1}$ (hence $E = \mathrm{id}$). The modular automorphism group of $\varphi|_\mathcal{N}$ is equal to $\sigma^\varphi|_\mathcal{N}$.

A Poincaré covariant net $\mathcal{A}$ is said to be **wedge dual** if it holds that $\mathcal{A}(W_\mathrm{L})' = \mathcal{A}(W_\mathrm{L}')(= \mathcal{A}(W_\mathrm{R}))$ (see Section 5.1 for W_L and W_R). With the help of conditional expectation, it is easy to deduce that Bisognano-Wichmann property (see Section 5.1) implies wedge duality, although this implication has been essentially known [5, 29].

Proposition A.2. *If a Poincaré covariant net $\mathcal{A}$ satisfies Bisognano-Wichmann property, then it is wedge dual.*

Proof. The modular automorphism group σ_t^Ω of $\mathcal{A}(W_\mathrm{L})'$ is implemented by Δ_Ω^{-it}, which is equal to $U(\Lambda(-2\pi t))$ by Bisognano-Wichmann property. It is obvious that $\mathcal{A}(W_\mathrm{R}) \subset \mathcal{A}(W_\mathrm{L})'$ and $\mathcal{A}(W_\mathrm{R})$ is invariant under $\mathrm{Ad}\, U(\Lambda(-2\pi t)) = \mathrm{Ad}\, \Delta_\Omega^{-it}$. Hence by Takesaki's Theorem A.1, there is a conditional expectation E from $\mathcal{A}(W_\mathrm{L})'$ onto $\mathcal{A}(W_\mathrm{R})$ which preserves $\omega = \langle \Omega, \cdot \Omega \rangle$ and it is implemented by the projection onto the subspace $\overline{\mathcal{A}(W_\mathrm{R})\Omega}$. But by Reeh-Schlieder property it is the whole space $\mathcal{H}$, hence E is the identity map and we obtain $\mathcal{A}(W_\mathrm{L})' = \mathcal{A}(W_\mathrm{R})$. $\square$

For a given net $\mathcal{A}$, we can associate the **dual net** $\mathcal{A}^\mathrm{d}$ ([3, Section 1.14]), defined by

$$\mathcal{A}^\mathrm{d}(O_0) = \bigcap_{O \perp O_0} \mathcal{A}(O)',$$

where $O \perp O_0$ means that O and O_0 are causally disjoint. $\mathcal{A}^\mathrm{d}$ does not necessarily satisfy locality nor additivity. Additivity is usually necessary only in proving Reeh-Schlieder property, so we do not discuss here. We have the following.

Proposition A.3 ([3])**.** *If a Poincaré covariant net $\mathcal{A}$ is wedge dual, then $\mathcal{A}^\mathrm{d}$ is local and Haag dual.*

Thus, if we consider the dual net $\mathcal{A}^\mathrm{d}$ as a natural extension, Haag duality for a net with Bisognano-Wichmann property is not a strong requirement and the only essential additional assumption in Section 5.1 is Bisognano-Wichmann property.

Appendix B Chiral components of conformal nets

In this appendix we consider various definitions of chiral components when a net $\mathcal{A}$ is conformal. These observations are not needed in the main text at the technical level but justify the interpretation of chiral observables as observables localized on lightlines.

We use the notations from Section 4.

Proposition B.1. *For distant intervals $J_1, J_2 \subset \mathbb{R}$ (i.e. they are disjoint with nonzero distance), it holds that*

$$\mathcal{A}_{\mathrm{L}}(I) = \mathcal{A}(I \times J_1) \cap \mathcal{A}(I \times J_2).$$

Proof. Since the Möbius group $\mathrm{PSL}(2,\mathbb{R}) = G_{\mathrm{R}}$ acts transitively on the family of intervals in $\mathbb{R} \subset S^1$, the inclusion $\mathcal{A}_{\mathrm{L}}(I) \subset \mathcal{A}(I \times J)$ holds for any interval J by the covariance of the net $\mathcal{A}$. Thus inclusions in one direction is proven.

To see the converse inclusion, we consider the commutants. By the Haag duality in $\mathcal{E}$, we have

$$\mathcal{A}_{\mathrm{L}}(I)' = (\mathcal{A}(I \times J) \cap U(\widetilde{G}_{\mathrm{R}})')' = \mathcal{A}(I^+ \times J^-) \vee U(\widetilde{G}_{\mathrm{R}}) \left(= \mathcal{A}(I^- \times J^+) \vee U(\widetilde{G}_{\mathrm{R}})\right),$$

where $I^{\pm}$, $J^{\pm}$ are defined in Section 2.1, and

$$(\mathcal{A}(I \times J_1) \cap \mathcal{A}(I \times J_2))' = \mathcal{A}(I^+ \times J_1^-) \vee \mathcal{A}(I^+ \times J_2^-).$$

Recall that we can choose an arbitrary J. Let J be an interval which includes both J_1 and J_2. In this case we have $J^- \subset J_1^-$ and $J^- \subset J_2^-$, hence

$$\mathcal{A}(I^+ \times J^-) \subset \mathcal{A}(I^+ \times J_1^-) \vee \mathcal{A}(I^+ \times J_2^-).$$

Furthermore, the fact that J_1 and J_2 are distant implies that the family of (two) intervals $\{J_1^-, J_2^-\}$ is an open cover of a closed interval of length 2π. The algebra $\mathcal{A}(I^+ \times J_1^-)$ (respectively $\mathcal{A}(I^+ \times J_2^-)$) contains the representatives of diffeomorphisms of the form $\mathrm{id} \times g_{\mathrm{R}}$ with $\mathrm{supp}(g_{\mathrm{R}}) \subset J_1$ (respectively $\mathrm{supp}(g_{\mathrm{R}}) \subset J_2$) in the sense that $\mathrm{Conf}(\mathcal{E})$ is a quotient group of $\overline{\mathrm{Diff}(S^1)} \times \overline{\mathrm{Diff}(S^1)}$ (see Section 2.1).

We claim that the algebra $\mathcal{A}(I^+ \times J_1^-) \vee \mathcal{A}(I^+ \times J_2^-)$ contains any representative of the form $\mathrm{id} \times g_{\mathrm{R}}$ where g_- is an arbitrary element in $\overline{\mathrm{Diff}(S^1)}$. Note that $\overline{\mathrm{Diff}(S^1)}$ can be identified with the group of diffeomorphisms of $\mathbb{R}$ commuting with the translation by 2π and an element of the form $\mathrm{id} \times g_{\mathrm{R}}$ where $\mathrm{supp}(g_{\mathrm{R}}) \subset J_1^-$ or $\mathrm{supp}(g_{\mathrm{R}}) \subset J_2^-$ can be viewed as a diffeomorphism with a periodic support. The group $\overline{\mathrm{Diff}(S^1)}$ is generated by such elements, hence we obtain the claim. In particular it contains the representatives of the universal cover $\widetilde{G}_{\mathrm{R}}$ of the Möbius group. Summing up, we have shown the inclusion

$$\mathcal{A}(I^+ \times J^-) \vee U(\widetilde{G}_{\mathrm{R}}) \subset \mathcal{A}(I^+ \times J_1^-) \vee \mathcal{A}(I^+ \times J_2^-).$$

The commutant of this relation gives the required inclusion. □

In [20], the intersection $\bigcap_J \mathcal{A}(I \times J)$ is taken as the definition of the chiral component. In fact, this and Definition 4.1 coincide under the diffeomorphism covariance.

Corollary B.2. *We have $\mathcal{A}_{\mathrm{L}}(I) = \bigcap_J \mathcal{A}(I \times J)$. Here, the intersection can be taken over the set of finite length intervals contained in $\mathbb{R} = S^1 \setminus \{-1\}$ or even all intervals in the universal covering space of S^1 by considering $\mathcal{A}$ as a net on $\mathcal{E}$.*

Remark B.3. From Proposition 2.1, it follows that, for a conformal net $\mathcal{A}$, $\mathcal{A}_{\mathrm{L}}(I)$ contains the representatives of diffeomorphisms $g_{\mathrm{L}} \times \mathrm{id}$ with g_{L} supported in I and hence it is nontrivial, although the intersection of regions $\bigcap_J I \times J$ is empty. A similar statement holds for $\mathcal{A}_{\mathrm{R}}$.

If the chiral components $\mathcal{A}_{\mathrm{L}}, \mathcal{A}_{\mathrm{R}}$ satisfy strong additivity, another (potentially useful) definition is possible. This should support an intuitive view that $\mathcal{A}_{\mathrm{L}}, \mathcal{A}_{\mathrm{R}}$ live on lightrays.

Proposition B.4. *Assume that $\mathcal{A}_{\mathrm{R}}$ is strongly additive. If $\{J_n\}$ is a sequence of intervals shrinking to a point, then it holds that $\mathcal{A}_{\mathrm{L}}(I) = \bigcap_n \mathcal{A}(I \times J_n)$.*

Proof. First we claim that $\mathcal{A}_{\mathrm{L}}(I) = \mathcal{A}(I \times J_1) \cap \mathcal{A}(I \times J_2)$ if J_1 and J_2 are obtained from an interval J by removing an interior point. One sees that the proof of Proposition B.1 works except the part concerning the diffeomorphisms. Namely, it holds that $\mathcal{A}_{\mathrm{R}}(J_1 \cup J_2) \subset \mathcal{A}(I^+ \times J_1^-) \vee \mathcal{A}(I^+ \times J_2^-)$

This time, the union $J_1^- \cup J_2^-$ is of length 2π. By the assumed strong additivity of the component $\mathcal{A}_{\mathrm{R}}$, this implies that $\mathcal{A}_{\mathrm{R}}(S^1) \subset \mathcal{A}(I^+ \times J_1^-) \vee \mathcal{A}(I^+ \times J_2^-)$. In fact, if J is an interval with length less than 2π which contains a boundary point of $J_1^- \cup J_2^-$, then $\mathcal{A}_{\mathrm{R}}(J)$ is contained in $\mathcal{A}(I^+ \times J_1^-) \vee \mathcal{A}(I^+ \times J_2^-)$ by strong additivity (note that the restriction of $\mathcal{A}_{\mathrm{R}}(I)$ to its vacuum representation is injective if I is a bounded interval). By the additivity of the chiral component, $\mathcal{A}(I^+ \times J_1^-) \vee \mathcal{A}(I^+ \times J_2^-)$ contains all the representatives of diffeomorphisms of the form $\mathrm{id} \times g$, $g \in \overline{\mathrm{Diff}(S^1)}$, in particular representatives of $\mathrm{id} \times \widetilde{G}_{\mathrm{R}}$. The rest of the argument is the same as Proposition B.1.

Let $\{J_n\}$ be a sequence of intervals shrinking to a point, where $J_n = (a_n, b_n)$. Let $\{J_{1,n}\}$ and $\{J_{2,n}\}$ be sequences of intervals which are obtained by $J_{1,n} = (a_0, b_n)$ and $J_{2,n} = (a_n, b_0)$. Let us denote $J_1 := \mathrm{int}(\bigcap_n J_{1,n}) = (a_0, c), J_2 := \mathrm{int}(\bigcap_n J_{2,n}) = (c, b_0)$, where $c = \lim_n a_n = \lim_n b_n$ and $\mathrm{int}(\cdot)$ means the interior. It is clear that

$$\mathcal{A}_{\mathrm{L}}(I) \subset \mathcal{A}(I \times J_n) \subset \mathcal{A}(I \times J_{1,n}) \cap \mathcal{A}(I \times J_{2,n}),$$

but the last expression tends to

$$\begin{aligned} \bigcap_n \mathcal{A}(I \times J_{1,n}) \cap \mathcal{A}(I \times J_{2,n}) &= \left(\bigcap_n \mathcal{A}(I \times J_{1,n})\right) \cap \left(\bigcap_n \mathcal{A}(I \times J_{2,n})\right) \\ &= \mathcal{A}(I \times J_1) \cap \mathcal{A}(I \times J_2), \end{aligned}$$

where the last equality follows from the Haag duality in $\mathcal{E}$ and additivity. We have seen that this is equal to $\mathcal{A}_{\mathrm{L}}(I)$, hence the intersection $\bigcap_n \mathcal{A}(I \times J_n)$ is equal to this as well. $\square$

Remark B.5. Rehren defined the "generating property" of the net by

$$U(\widetilde{G}_{\mathrm{L}}) \subset \mathcal{A}(I \times J) \vee \mathcal{A}(I' \times J)$$
$$U(\widetilde{G}_{\mathrm{R}}) \subset \mathcal{A}(I \times J) \vee \mathcal{A}(I \times J'),$$

for any I, J. We proved Proposition B.4 by showing the generating property for $\mathcal{A}$ with the strongly additive conformal components. It has been shown in [28] that the generating property implies that $\mathcal{A}_{\mathrm{L}}(I) = \mathcal{A}(I \times J_1) \cap \mathcal{A}(I \times J_2)$ where J_1 and J_2 are obtained by removing an interior point from an interval.

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