Unitary ~~vertex~~ a Möbius vertex algebras and Wightman fields.

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1. Conformal net on S^1

A mathematical approach to 2d Conformal field theory

~~$A(I)$~~: $S^1 \supset I \longmapsto A(I) \subset B(\mathcal{H})$ von Neumann alg.

$U: \text{Möb} = PSL(2,\mathbb{R}) \to U(\mathcal{H})$. unitary. rep. $\Omega \in \mathcal{H}$.

satisfying isotony, locality, covariance, positive energy, vacuum.

$\Rightarrow$ representation theory, subfactors, tensor categories ...

Examples? Vertex algebras, Wightman fields (c.f. Carpi-Kawahigashi-Longo-Weiner)

2. Wightman fields on S^1.

ϕ_j: operator-valued distributions on $\mathcal{D} \subset \mathcal{H}$.

U: rep of Möb. $\Omega \in \mathcal{D}$.

Locality: If $f, g \in C^\infty(S^1)$, $\operatorname{supp} f \cap \operatorname{supp} g = \phi$, then $[\phi(f), \phi(g)] = 0$

Covariance ($\phi(f)\Omega$ gives an irrep of Möb), positive energy, vacuum.

3. Vertex algebras

V: vectorspace, $\{L_1, L_0, L_{-1}\}$: rep of $sl(2,\mathbb{C})$ $[L_m, L_n] = (m-n)L_{m+n}$.

For $v \in V$, there is a formal series $Y(v,z) = \sum_{n \in \mathbb{Z}} v_{(n)} z^{-n-1}$,

$v_{(n)} \in \operatorname{End}(V)$. satisfying covariance,

spectrum condition $V = \bigoplus_{n \in \mathbb{N}} \ker(L_0 - n)$,

vacuum, locality: for $v, u \in V$,

there is N s.t. $(z-w)^N [Y(v,z), Y(u,w)] = 0$

as a formal series.

We say that $Y(v,z)$ is a quasi-primary field if

$Y(v,0)\Omega$ is a lowest-weight vector for $sl(2,\mathbb{C})$.

d_v: the lowest weight (conformal dimension).

Example $U(1)$-current / Heisenberg alg / free massless field.

Consider the Lie alg $\{J_n; n\in\mathbb{Z}, \mathbb{C}\}$, $[J_m, J_n] = m\delta_{m+n}$.

This admits the "vacuum rep" (Fock space / Verma module with trivial weight)

$V \ni \Omega, J_{-1}\Omega, J_{-2}\Omega, \dots J_{-1}J_{-1}J_{-2}\Omega, \dots \quad J_n\Omega = 0, \ n\geq 0.$

~~$J_0\Omega = 0$~~ Wightman field: Take $f\in C^\infty(S^1)$. $\hat{f}_n = \int_{S^1} f(e^{i\theta})e^{-in\theta}d\theta$

$J(f) = \sum_{n\in\mathbb{Z}} J_n \hat{f}_n$, converges on $V. \subset \mathcal{H} = \mathrm{span}\{J(f_1)\cdots J(f_k)\}$

$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} :J_n J_{m-n}:$, where $:J_k J_l: = \begin{cases} J_k J_l & \text{if } k>l \\ J_l J_k & \text{if } k\leq l. \end{cases}$

$\Rightarrow V.$

(J, V, Ω) is a Wightman field on S^1.

Vertex alg. We have $\{L_n\}, \Omega$, First, $J(z) = \sum_{n\in\mathbb{Z}} J_n z^{-n-1}$.

Then $Y(J_{-1}\Omega, z) = J(z)$, $Y(L_{-1}J_{-1}\Omega, z) = J(J_{-2}\Omega, z) = \partial J(z)$.

$Y(J_{-1}J_{-1}, z) = :J(z)^2: = J_+(z)J(z) + J(z)J_-(z)$.

$J_+(z) = \sum_{n<0} J_n z^{-n-1}$, $J_-(z) = \sum_{n\geq 0} J_n z^{-n-1}$. ...

Wightman fields: collection of good (quasi-primary) fields

Vertex alg : collection of all (non-primary / descendant) fields.

Thm There is a one-to-one correspondence between

$\{\phi_j\}$ ~~: collection of (all)~~ ~~to~~ a collection of Wightman fields + (UBO).

V: vertex alg.

UBO: For $v, u \in V = \bigoplus_{n\in\mathbb{N}} \ker(L_0 - n)$, $\phi_1, \dots, \phi_k$, there is a polynomial $P_{u,v,\{\phi_j\}}$, s.t. $|\langle \phi_{1,n_1}\cdots\phi_{k,n_k} u, v\rangle| \leq P_{u,v,\{\phi_j\}}(n_1, \dots, n_k)$

$\phi_{j,n_j} = \int e^{in_j\theta}\phi_j(e^{i\theta})d\theta$, $P_{u,v,\{\phi_j\}}$'s degree does not depend on u, v.

proof (of locality) Wightman + UBO $\Rightarrow$ VA.

$Y(u,z)$ can be constructed from $\{\phi_j\}$ as before.

By Wightman locality, $\Delta_{v,u}(f,g) = \langle u, [\phi_1(f), \phi_2(g)] u\rangle$ is a distribution, supported on $z=w$. By UBO, the order of the distribution does not depend on u, v.

$\Rightarrow (z-w)^N \langle u, [\phi_1(f), \phi_2(g)] v\rangle = 0$ for all $u, v \in V$

$\Rightarrow (z-w)^N [\phi_1(f), \phi_2(g)] = 0$.

VA $\Rightarrow$ UBO + Wightman.

Lemma (commutator formula). Let $a, b \in V$, $Y(a,z), Y(b,w)$ are quasi-primary

$$[a_{(m)}, b_{(n)}] = \sum_{j\geq 0} \binom{m}{j} (a_{(j)} b)_{(m+n-j)}.$$

Estimate $\langle u, \phi_{1,n_1} \cdots \phi_{k,n_k} v\rangle$ by induction in k:

$k=1$. $\langle u, \phi_{1,n_1} v\rangle$ has only finitely many terms. $\leq C_{u,v}$.

k. $\langle u, \phi_{1,n_1} \cdots \phi_{k,n_k} v\rangle$

$= \sum_{j=1}^{k} \langle u, \phi_{2,n_2} \cdots [\phi_{1,n_1}, \phi_{j,n_j}] \cdots \phi_{k,n_k} v\rangle + \langle u, \phi_{2,n_2} \cdots \phi_{k,n_k} \phi_{1,n_1} v\rangle$

Terms without $[,]$ are finite in n_1, so by induction $\leq P_{u,v,\{\phi_2 \cdots \phi_k\}}$.

Terms with $[,]$ have $k-1$ fields. $\leq P_{u,v,\{\phi_2, \cdots \phi_k, (a_{(j)} b)\}} \cdot |n_1|^{m+1}$

We get UBO. $\Rightarrow \langle u, \phi_1(f_1) \cdots \phi_k(f_k) v\rangle$ converge.

$\Rightarrow$ Wightman locality. //.

Remark. UBO may be automatic (Carpi).