Self-adjointness of bound state operators in integrable QFT

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Introduction

Goal

Construct Haag-Kastler net $\{\mathcal{A}(O)\}$ (local observables in O) for an integrable model with bound states (factorizing S-matrices with **poles**).
Non-perturbative, non-trivial quantum field theories in $d = 2$.

- Problem of **strong commutativity, domains** of candidates $\widetilde{\phi}(f), \widetilde{\phi}'(g)$ for observables $\mathcal{A}(W_{\mathrm{L}}), \mathcal{A}(W_{\mathrm{R}})$ (bounded operators) in wedges $W_{\mathrm{L}}, W_{\mathrm{R}}$

Why self-adjointness important?

Example: differential operator

$\mathcal{H} = L^2((0,1), dt) \supset \mathcal{D} = \{\xi \in \mathrm{AC}(0,1) : \xi(0) = \xi(1) = 0\}, A = i\frac{d}{dt}$.
Deficiency indices $n_{\pm}(A) = \dim \mathrm{Ker}(A^* \mp i) = 1$. For $\alpha \in \mathbb{C}, |\alpha| = 1$, $\mathcal{D}_\alpha = \{\xi \in \mathrm{AC}(0,1) : \xi(0) = \alpha\xi(1)\}, A_\alpha = i\frac{d}{dt}$, them A_α's are different self-adjoint extensions.

different extensions $\Leftrightarrow$ different boundary conditions

Self-adjointness criteria

Many criteria were proved by mathematical physicists / have applications in mathematical physics.

- Analytic vector theorem (Nelson): If A is symmetric and has a dense set of analytic vectors in its domain, then it is essentially self-adjoint. $\Longrightarrow$ (Lie groups,) free fields.
- Commutator theorem (Glimm-Jaffe, Nelson): If H is positive self-adjoint and A and $[H, A]$ are symmetric, "can be bounded" by H, then A is essentially self-adjoint. $\Longrightarrow$ $P(\phi)_2$-models.
- Perturbation arguments (Kato-Rellich,...): If H is positive self-adjoint and A is "smaller" than H, then $H + A$ is self-adjoint. $\Longrightarrow$ atomic Hamiltonian in QM.

Domain problem often appears when one considers new models...

Integrable models with bound states: current status

- Goal: Construct local observables in integrable models with bound states.
- Strategy: Construct first observables in wedge-regions.
- Partial results: $\widetilde{\phi}(f)$ and $\widetilde{\phi}'(g)$ **weakly** commute on $\mathrm{Dom}(\widetilde{\phi}(f)) \cap \mathrm{Dom}(\widetilde{\phi}'(g))$ if $\mathrm{supp}\, f \subset W_{\mathrm{L}}, \mathrm{supp}\, g \subset W_{\mathrm{R}}$.
- **Next step**: $\widetilde{\phi}(f)$ and $\widetilde{\phi}'(g)$ have self-adjoint extensions and they **strongly** commute.
- Define $\mathcal{M} := \{e^{i\widetilde{\phi}'(g)} : \mathrm{supp}\, g \subset W_{\mathrm{R}}\}'$.
- Consequence of "next step":
    - Ω is cyclic and **separating** for $\mathcal{M}$.
    - The modular group Δ^{it} of $\mathcal{M}$ with respect to Ω is the Lorentz boosts.
    - Modular nuclearity: $\mathcal{M} \ni x \longmapsto \Delta^{\frac{1}{4}} U(a) x\Omega$ is nuclear if $a \in W_{\mathrm{R}}$ is sufficiently large.
    - Existence of local observables for sufficiently large double cone (Haag-Kastler net with minimal radius).
    - Two-particle S-matrix S (work in progress).

Integrable models with bound states: next step

- Prove that $\widetilde{\phi}(f)$ and $\widetilde{\phi}'(g)$ have self-adjoint extensions.

Recall: $\widetilde{\phi}(f) = \phi(f) + \chi(f)$, where $\phi(f)$ is very similar to the free field (under control), while $\chi(f) = \bigoplus_n \chi_n(f)$ is unbounded at each n. $\chi_1(f)$ was first defined on a Hardy space $H^2(-\frac{\pi}{3}, 0)$, which is **not** self-adjoint. $\chi_n(f) = P_n(\chi_1(f) \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1})P_n$.

  - need to find an extension **maintaining** weak commutativity.
  - find a nice extension of $\chi_1(f)$ and extend $\chi_n(f)$ accordingly.

- Prove that $\widetilde{\phi}(f)$ and $\widetilde{\phi}'(g)$ strongly commute.

  - follows if $\chi(f) + \chi'(g) + cN$ is self-adjoint.

Self-adjointness

A linear operator A on a domain $\mathrm{Dom}(A) \subset \mathcal{H}$ is **symmetric** if

$$\langle \xi, A\eta \rangle = \langle A\xi, \eta \rangle, \xi, \eta \in \mathrm{Dom}(A).$$

Its **adjoint** A^* is defined on $\mathrm{Dom}(A^*) = \{\xi \mid \eta \mapsto \langle \xi, A\eta \rangle \text{is continuous}\}$, by

$$\langle \xi, A\eta \rangle = \langle A^*\xi, \eta \rangle, \eta \in \mathrm{Dom}(A).$$

If A is symmetric, then $A \subset A^*$. A is **self-adjoint** if and only if $A = A^*$.
A has self-adjoint extension(s) if and only if deficiency indices $n_\pm(A) = \dim \mathrm{Ker}(A^* \mp i)$ coincide.

Example: differential operator

$\mathrm{Dom}(A) = \{\xi \in L^2(0,1) \mid \xi \in \mathrm{AC}(0,1), \xi(0) = \xi(1) = 0\}, (A\xi)(t) = i\frac{d}{dt}\xi(t)$. Then $A \neq A^*$,
$\mathrm{Dom}(A^*) = \{\xi \in L^2(0,1) \mid \xi \in \mathrm{AC}(0,1)\}, (A^*\xi)(t) = i\frac{d}{dt}\xi(t)$.
$\xi_\pm(t) = e^{\pm\frac{\pi i}{2}t} \in \mathrm{Ker}(A^* \mp i)$. $n_\pm(A) = 1$.

The one-particle bound state operator

$\mathcal{H}_1 = L^2(\mathbb{R}), \mathrm{Dom}(\chi_1(f)) := H^2\left(-\frac{\pi}{3}, 0\right)$, where $H^2(\alpha, \beta)$ is the space of analytic functions in $\mathbb{R} + i(\alpha, \beta)$ such that $\xi(\cdot + \gamma i)$ is uniformly bounded in L^2-norm, $\gamma \in (\alpha, \beta)$, and f^+ is analytic.

$$(\chi_1(f))\xi(\theta) := \sqrt{2\pi|R|} f^+\left(\theta + \frac{\pi i}{3}\right)\xi\left(\theta - \frac{\pi i}{3}\right),$$

Problem

- Compute deficiency indices $n_\pm(\chi_1(f))$.
- **Classify** self-adjoint extensions of $\chi_1(f)$.

One-particle bound state operator

- $\mathcal{H}_1 = L^2(\mathbb{R})$
- $h(\zeta)$: analytic in $\mathbb{R} + i(-\pi, 0)$, $\overline{h(\theta)} = h(\theta - \pi i)$.
- $\mathrm{Dom}(\chi_1(h)) = H^2(-\pi, 0)$: the space of analytic functions in $\mathbb{R} + i(-\pi, 0)$ such that $\xi(\cdot + \gamma i)$ is uniformly bounded in L^2-norm, $\gamma \in (-\pi, 0)$
- $(\chi_1(h))\xi(\theta) := h(\theta - \pi i)\,\xi(\theta - \pi i) \quad (= \overline{h(\theta)}\xi(\theta - \pi i))$

The main problem

What are self-adjoint extensions of $\chi_1(h)$?

- Write $\chi_1(h) = M_{\overline{h}}\Delta^{\frac{1}{2}}$, $(\Delta^{\frac{1}{2}}\xi)(\theta) = \xi(\theta - \pi i)$
- Classify extensions: **compute** $\mathrm{Ker}(\chi_1(h)^* \pm i)$, $\chi_1(h)^* = \Delta^{\frac{1}{2}} M_h$

Case: Blaschke factors

Consider

$$h(\zeta) = \frac{e^\zeta + \frac{\pi i}{2}}{e^\zeta - \frac{\pi i}{2}}, \quad h(\zeta - \pi i) = \frac{e^\zeta - \frac{\pi i}{2}}{e^\zeta + \frac{\pi i}{2}}$$

How many solutions ξ of $h(\zeta - \pi i)\xi(\zeta - \pi i) = \pm i\xi(\zeta)$, where $h\xi \in H^2(-\pi, 0)$?

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How many solutions ξ of $h(\zeta - \pi i)\xi(\zeta - \pi i) = \pm i\xi(\zeta)$, where $h\xi \in H^2(-\pi, 0)$?

$$\xi(\zeta) = \frac{e^{\frac{1}{2}\zeta}}{e^\zeta + \frac{\pi i}{2}},$$
$$h(\zeta)\xi(\zeta) = \frac{e^{\frac{1}{2}\zeta}}{e^\zeta - \frac{\pi i}{2}}, \quad h(\zeta - \pi i)\xi(\zeta - \pi i) = \frac{-ie^{\frac{1}{2}\zeta}}{e^\zeta + \frac{\pi i}{2}} = -i\xi(\zeta).$$

The deficiency indices are $(1, 0)$. $\chi_1(h)$ has **no** self-adjoint extension.

If h has $n = 2m$ or $2m + 1$ factors, then the deficiency indices are (m, m) or $(m + 1, m)$, respectively.

Case: singular inner functions

Consider

$$h(\zeta) = \exp\left(-i\alpha_+ e^{\zeta} + i\alpha_- e^{-\zeta}\right), \quad h(\zeta - \pi i) = \exp\left(i\alpha_+ e^{\zeta} - i\alpha_- e^{-\zeta}\right),$$

$\alpha_\pm > 0$. How many solutions ξ of $h(\zeta - \pi i)\xi(\zeta - \pi i) = \pm i\xi(\zeta)$, where $h\xi \in H^2(-\pi, 0)$?

Case: singular inner functions

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$\alpha_\pm > 0$. How many solutions ξ of $h(\zeta - \pi i)\xi(\zeta - \pi i) = \pm i\xi(\zeta)$, where $h\xi \in H^2(-\pi, 0)$?

Outside H^2, there is $g(\zeta) = h(\zeta - \pi i)g(\zeta - \pi i)$, $\frac{\xi(\zeta)e^{\pm\frac{1}{2}\zeta}}{g(\zeta)}$ is **periodic**:
$g(\zeta) = \exp\left(\frac{1}{2}\left(i\alpha_+ e^\zeta - i\alpha_- e^{-\zeta}\right)\right)$.

$$\xi_{g_+,g_-}(\zeta) = \int dk_+ \cos\left(\frac{k_+}{2}e^\zeta\right) g_+(k_1) \cdot \int dk_- \cos\left(\frac{k_-}{2}e^{-\zeta}\right) g_-(k_-),$$

where $g_\pm$ are smooth, $g_\pm(k) = g_\pm(-k)$, $\mathrm{supp}\, g_\pm \subset (-\alpha_\pm, \alpha_\pm)$. Then the **product** $g(\zeta)\xi_{g_+,g_-}(\zeta)e^{\left(n+\frac{1}{2}\right)\zeta}$ is a solution.
The deficiency indices are: (∞, ∞).
The operator $\chi_1(h)$ has **infinitely many** self-adjoint extensions.

Case: outer functions

Consider

$$h(\zeta) = \exp\left(-i\int_{-\infty}^{\infty} \frac{ds}{1+s^2}\frac{1+e^{\zeta}s}{e^{\zeta}-s}\log\phi(s)\right),$$

$$|\log\phi(s)| < B|s|^{\alpha} + A, 0 < \alpha < 1.$$

How many solutions ξ of $h(\zeta-\pi i)\xi(\zeta-\pi i) = \pm i\xi(\zeta)$, where $h\xi \in H^2(-\pi, 0)$?

Case: outer functions

Consider

$$h(\zeta) = \exp\left(-i\int_{-\infty}^{\infty} \frac{ds}{1+s^2}\frac{1+e^{\zeta}s}{e^{\zeta}-s}\log\phi(s)\right),$$
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How many solutions ξ of $h(\zeta - \pi i)\xi(\zeta - \pi i) = \pm i\xi(\zeta)$, where $h\xi \in H^2(-\pi, 0)$?

Outside H^2, there is $g(\zeta) = h(\zeta - \pi i)g(\zeta - \pi i)$, $\frac{\xi(\zeta)e^{\pm\frac{1}{2}\zeta}}{g(\zeta)}$ is **periodic**:

$$g(\zeta) = \exp\left(-i\int_{-\infty}^{0} \frac{ds}{1+s^2}\frac{1+e^{\zeta}s}{e^{\zeta}-s}\log\phi(s)\right),$$

Divide an arbitrary solution ξ by g and $\frac{\xi}{g}$ is **periodic**, with an estimate of the form $\frac{\xi(\zeta)}{g(\zeta)} < Ae^{Be^{\alpha|\mathrm{Re}\,\zeta|}} \Rightarrow \xi(\zeta) = Cg(\zeta)$ (but ξ had to be H^2, $\lightning$).
The operator $\chi_1(h)$ is **essentially self-adjoint**.

Summary of $M_{\bar{f}}\Delta^{\frac{1}{2}}$

- Deficiency indices depend very much on f.
- There is not always self-adjoint extensions.
- Even if there are, there is no particular choice.

$\Longrightarrow$ consider a particular class of functions.

- What if $h = f^2$?

Self-adjoint extensions for squares, spectral calculus

Any H^∞-function h admits the decomposition $h = h_B h_{\mathrm{in}} h_{\mathrm{out}}$, where

- h_B is a Blaschke factor
- h_{in} is singular inner
- h_{out} is outer:

$$h(\zeta) = \exp\left(-i\int_{-\infty}^{\infty} \frac{ds}{1+s^2}\frac{1+e^\zeta s}{e^\zeta - s}\log\phi(s)\right) = h_+(\zeta)h_-(\zeta),$$

$$h_+(\zeta - \pi i) = \exp\left(-i\int_{0}^{\infty} \frac{ds}{1+s^2}\frac{1-e^\zeta s}{-e^\zeta - s}\log\phi(s)\right),$$

$$\theta\in\mathbb{R}, h_-(\theta) = \exp\left(-i\int_{-\infty}^{0} \frac{ds}{1+s^2}\frac{1+e^\theta s}{e^\theta - s}\log\phi(s)\right) = \overline{h_+(\theta - \pi i)}$$

If $h = f^2, f = f_B f_{\mathrm{in}} f_{\mathrm{out}}$, then f_B is a Blaschke factor, h_{in} is singular inner, $M_{f_B}, M_{f_{\mathrm{in}}}, M_{h_-}$ are unitary and

$$M_{\overline{h}}\Delta^{\frac{1}{2}} \subset M_{\overline{f_B}}M_{\overline{f_{\mathrm{in}}}}M_{\overline{h_-}}\Delta^{\frac{1}{2}}M_{f_B}M_{f_{\mathrm{in}}}M_{h_-}$$

n-particle bound state operator

S: S-matrix, D_n: the S-representation of $\mathfrak{S}_n$, $P_{S,n}$: S-symmetrization, $\mathcal{H} = \bigoplus P_{S,n}\mathcal{H}_1^{\otimes n}$, $\mathcal{H}_1 = L^2(\mathbb{R})$,

$$\chi_n(f) = nP_{S,n}\left(\chi_1(f) \otimes I \otimes \cdots \otimes I\right)P_{S,n},$$
$$\chi(f) := \bigoplus \chi_n(f).$$

As $\chi_1(f)$ has an extension of the form $u_f \Delta^{\frac{1}{6}} u_f^*$, $\chi_n(f)$ has an extension

$$(u_f \otimes \cdots \otimes u_f)P_n(\Delta^{\frac{1}{6}} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1})P_n(u_f \otimes \cdots \otimes u_f)^*.$$

Problem: is $P_n(\Delta^{\frac{1}{6}} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1})P_n$ self-adjoint?

Note: for $\rho_k = (k, 1, 2, \cdots, k-1, \cdots, n)$,
$D_n(\rho_k)(\Delta^{\frac{1}{6}} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1})D_n(\rho_k)^{-1} =$
$(M_S)_{1,k} \cdots (M_S)_{k-1,k}(\mathbb{1} \otimes \cdots \underset{k\text{-th}}{\Delta}^{\frac{1}{6}} \cdots \otimes \mathbb{1})(M_S)_{1,k}^* \cdots (M_S)_{k-1,k}^* \supset$
$(M_S)_{1,k} \cdots (M_S)_{k-1,k}(M_{S(\cdot+\frac{\pi i}{3})})_{1,k}^* \cdots (M_{S(\cdot+\frac{\pi i}{3})})_{k-1,k}^*(\mathbb{1}\otimes \cdots \underset{k\text{-th}}{\Delta}^{\frac{1}{6}} \cdots \otimes \mathbb{1})$

n-particle bound state operator

Proposition

For each n, there is a small ϵ such that $P_n(\Delta^\epsilon \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1})P_n$ is self-adjoint.

proof)

- $\Delta^\epsilon \otimes \cdots \mathbb{1} + \mathbb{1} \otimes \Delta^\epsilon \cdots \otimes \mathbb{1} + \mathbb{1} \otimes \cdots \otimes \Delta^\epsilon$ is self-adjoint.
- $\sum D_n(\rho_k)(\Delta^\epsilon \otimes \cdots \mathbb{1})D_n(\rho_k)^{-1} + \mathbb{1} \otimes \Delta^\epsilon \cdots \otimes \mathbb{1} + \mathbb{1} \otimes \cdots \otimes \Delta^\epsilon$ is self-adjoint.
- Lemma: if A, B positive, $A + B$ self-adjoint, $\langle A\Psi, B\Psi\rangle \geq 0$, then A and B are essentially self-adjoint on the domain of $A + B$ (apply Wüst's theorem).
- $\sum D_n(\rho_k)(\Delta^\epsilon \otimes \cdots \mathbb{1})D_n(\rho_k)^{-1}$ is self-adjoint, $\rho_k = (k, 1, \cdots, k-1)$.
- $P_n(\Delta^\epsilon \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1})P_n$ is self-adjoint.

Lesson

The result is sensitive to ϵ. Poles and zeros play a crucial role.

Summary

Open problem:

- self-adjointness of $\chi(h) + \chi'(g)$

Consequence:

- Strong commutativity between $\widetilde{\phi}(f) = \phi(f) + \chi(f)$ and $\widetilde{\phi}'(g) = \phi'(g) + \chi'(g)$.
- Haag-Kaslter net with minimal radius.
- Factorizing S-matrix S (work in progress).

Outlook:

- more models (sine-Gordon, Toda field theories, $Z(N)$-Ising models): no positivity of $\chi(f)$...