Scaling limits of lattice quantum fields by wavelets

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Constructing continuum quantum field from lattice

- A classical field is a function on the spacetime $\mathbb{R}^{d+1}$. E.g. Φ^4-theory:
  $(\partial_t^2 - \nabla^2)\Phi(x) = m^2\Phi(x) + 4\lambda\Phi(x)^3$.
- A quantum field is an operator-valued distribution $\Phi(x)$: for a test function f, $\Phi(f)$ gives an (unbounded) operator.
- Quantization of fields requires renormalization: If $\Phi(x)$ a quantum field on $\mathbb{R}^d$, $\Phi(x)^3$ makes no sense.
- Constructive QFT: regularize $\Phi(x)$ by restricting x to lattices, then $\Phi(x)$ is an operator. Then take the continuum limit.
- Osterwalder-Schrader axioms $\Rightarrow$ Wightman axioms $\Rightarrow$ Haag-Kastler axioms... (usually this is a very indirect procedure: correlation functions $\Rightarrow$ Wick rotation $\Rightarrow$ reconstruction...)
- Can we obtain the continuum quantum field from the lattice quantum field more directly? **Yes**, at least for the massive free field at time zero. Cf. Jones' no-go result.

Guessing the scaling limit of lattice fields

Take a lattice Λ_N in the time zero space $\mathbb{R}^d$. On each point $x \in \Lambda_N$ there should be an operator $\Phi(x)$.
We take $\ell^2(\Lambda_N)$ as the one-particle space and consider the Weyl algebra, represented on the Fock space, so we have the field $\Phi(x)$ and the momentum $\pi(x)$, satisfying the canonical commutation relation (CCR):

$$[\Phi(x), \Pi(y)] = i\delta_{x,y}.$$

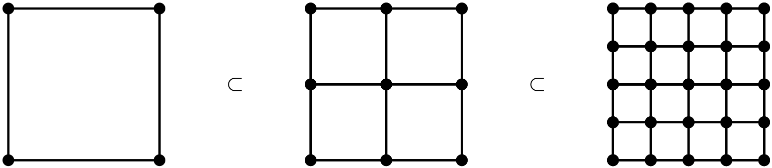
If a continuum limit could be constructed from the lattice algebras,

- the fields on different points should commute.
- the field on one point should decompose into the fields of more points in the finer scale.
- each element in the point should be some observable in the continuum.

The first two conditions can be satisfied in many ways, but the last one is not straightforward...

Guessing the scaling limit of lattice free fields

We consider the spacelike dimension $d \geq 1$ and take a sequence of gradually refined lattices.

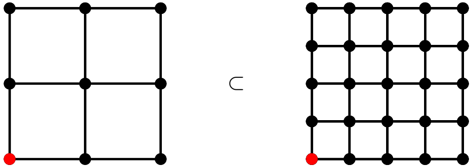


Refinement of lattices for $d = 2$.

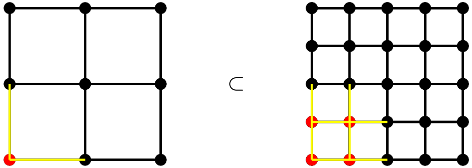
On each lattice point there should be a field.
When a lattice is included in a finer lattice, this algebra should be mapped to somewhere. But where?

Pointlike / block spin scaling (failed attempts)

The simplest possibility is the simple inclusion $\Lambda_N \subset \Lambda_{N+1}$:



then there would have to be a field operator $\Phi(x)$ in the continuum...



If we think of a point as a characteristic function χ_I, there would have to be the operator $\Pi(\chi_I)$ (does not...)

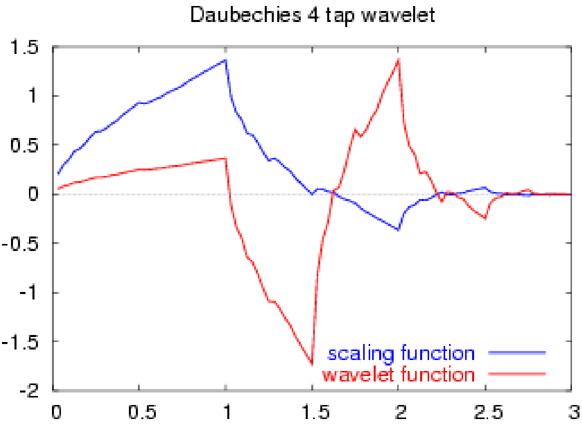
Wavelet scaling: a generalization

Let $\phi_{N,k}(x) = \phi(2^N x - k)$. A **scaling function** ϕ should satisfy:

- $\{\phi_{N,k}\}_{k\in\Lambda_N}$ for a fixed N is an orthonormal system.
- $\phi = \phi_{0,0}$ can be written as a linear combination of $\phi_{1,k}$.
- $\{\phi_{N,k}\}_{N\in\mathbb{N}_0,k\in\Lambda_N}$ span $L^2(\mathbb{T})$.

For any given $K \in \mathbb{N}$, there is the scaling function ${}_K\phi$ that generates **Daubechies wavelets**. ${}_K\phi \in H^{\frac{1}{2}}_{\mathbb{R}}(\mathbb{T})$ (actually in $H^{0.839}_{\mathbb{R}}(\mathbb{T})$).
The scaling function ${}_K\phi$ is supported in $[0, 2K-1]$. By scaling and normalizing appropriately, we obtain a scaling function compactly supported in $\mathbb{T}$.

Daubechies wavelets



Daubechies' scaling function with $K = 2$ ($D = 4$) (taken from commons.wikimedia.org/wiki/File:Daubechies4-functions.png created by LutzL)

Wavelet scaling

This amounts to identify the lattice fields $\Phi_N(k)$ and $\Pi_N(k)$ with $\Phi_{\mathrm{ct}}(\phi_{N,k}), \Pi_{\mathrm{ct}}(\phi_{N,k})$. As Daubechies scaling function is sufficiently regular, these operators make sense. With $\mathcal{W}_N(\Lambda_N)$ the Weyl algebra on lattice and $\mathcal{W}_{\mathrm{ct}}(\mathbb{T})$ the Weyl algebra of the massive free field,

Theorem

There is the embedding

$$\mathcal{W}_N(\Lambda_N) \subset \mathcal{W}_{N+1}(\Lambda_{N+1}) \subset \cdots \subset \mathcal{W}_{\mathrm{ct}}(\mathbb{T}),$$

where the field $\Phi_N(k)$ in one point is mapped to the continuum field $\Phi_{\mathrm{ct}}(\phi_{N,k})$ smeared with the scaling function $\phi_{N,k}$. The ground states of the massive lattice fields converge to the continuum massive free state in the weak* topology.

Extending this to $d > 1$ is straightforward by using the tensor product of scaling functions.

Outlook

- Interacting QFT?
  - Change the Hamiltonian, change the states.
  - Locality from Lieb-Robinson bound?
- De Sitter space (**wavelets on the sphere**)?
- σ-models? **Gauge theories**?
- Free (massless) fermion, CFT?
- Entropy on lattice algebras?