Lattice Green functions for pedestrians: Exponential decay

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Abstract

The exponential decay of lattice Green functions is one of the main technical ingredients of the Bałaban's approach to renormalization. We give here a self-contained proof, whose various ingredients were scattered in the literature. The main sources of exponential decay are the Combes-Thomas method and the analyticity of the Fourier transforms. They are combined using a renormalization group equation and the method of images.

Contents

1 Introduction

The approach of Bałaban, which builds on the Wilsonian renormalization of lattice quantum field theory (QFT), stands out as a charted route towards a construction of non-trivial just-renormalizable QFT. The strategy, outlined e.g. in [BJ85, Ba90], can be divided into four main steps: (1) the exponential decay of lattice Green functions [BJ85, II.9,II.10,IV], (2) variational problem identifying the background field [BJ85, pp. 242–243], (3) Wilsonian renormalization in the small field region by an expansion around the background field [BJ85, III.1–III.5], (4) the large field problem [BJ85, III.6][Ba90]. Unfortunately, most of the literature on this topic was written in a style suitable for a small circle of experts and, after several decades, is difficult to comprehend for interested readers. The main obstacle, as we see it, is the absence of systematic expositions of the methods used and developed in these works. Relevant pieces of information are scattered over many papers treating different models and a substantial part of the discussion is left to the reader. As a valuable exception we would like to mention relatively recent papers of Dimock on the UV stability of the ϕ_3^4 model [Di13, Di13.1, Di14], which are largely self-contained and readable line by line. But even these useful works on a super-renormalizable model do not provide the reader with sufficient background to delve into the literature on just-renormalizable theories. One reason is a rather brief discussion of the lattice Green functions in [Di13], which covers only the more elementary part of the subject, leaving the rest to rather cryptic references. As the lattice Green functions are at the foundation of the entire subject and their detailed control is needed already at the level of the variational problem in some just-renormalizable models, we decided to write this systematic account. We recall that the need to present Bałaban's method in a more comprehensible manner was stressed already in [Ri00].

Let us describe the content of our article in an informal way: Let Ω be a finite square lattice in d-dimensions with lattice spacing η. In the Wilsonian spirit, we divide this lattice into boxes $B(y)$ of linear size L. The points y, which label the boxes, form the coarse lattice $\Omega_1 := (L\Omega)\cap\Omega$. We can repeat this procedure k times, denoting the resulting boxes $B_k(y)$ and the coarse lattice $\Omega_k := (L^k\Omega)\cap\Omega$. We set in the following $\eta = L^{-k}$, in which case Ω_k has a unit spacing. The operator of k-times averaging $Q_{\Omega,k} : \mathcal{L}^2(\Omega) \to \mathcal{L}^2(\Omega_k)$ is given by

$$(Q_{\Omega,k}f)(y) = \frac{1}{L^{kd}} \sum_{x\in B_k(y)} f(x) \tag{1.1}$$

and the relevant propagator has the form

$$G_k(\Omega) := (-\Delta_\Omega^\eta + \bar{\mu}_k + a_k Q_{\Omega,k}^* Q_{\Omega,k})^{-1}. \tag{1.2}$$

Here Δ_Ω^η is the discrete Laplacian on Ω with Neumann boundary conditions, $\{a_k\}_{k\in\mathbb{N}}$ is a concrete bounded sequence which stays away from zero, see formula (2.75), and $\bar{\mu}_k = L^{2k}\bar{\mu}_0$ are mass parameters. It turns out that $Q_{\Omega,k}^* Q_{\Omega,k}$ is a projection operator on the subspace of functions in $\mathcal{L}^2(\Omega)$ which are constant on the blocks $B_k(y)$. The kernel $(x,x') \mapsto G_k(\Omega)(x,x')$ is called a Green function.

Leaving the question of motivation of definition (1.2) aside for a moment, let us justify the existence of the inverse. Here the key observation is that the Laplacian vanishes only on constant functions, whereas the averaging operator leaves such functions invariant (cf. Lemma 2.10). Moving towards the exponential decay of Green functions, one observes that the denominator in (1.2) resembles a simple Schrödinger operator consisting of a Laplacian perturbed by a finite dimensional projection. Using the Combes-Thomas method [CT74] from the Schrödinger operator theory, one easily obtains an $\mathcal{L}^2$ bound (cf. Lemma 2.12)

$$|\langle f', G_k(\Omega)f\rangle| \leq c\mathrm{e}^{-c_1|y-y'|}\|f'\|_2\|f\|_2, \tag{1.3}$$

where $\operatorname{supp} f \subset B_k(y)$, $\operatorname{supp} f' \subset B_k(y')$. However, the small field conditions of the Balaban's approach single out the supremum norm, in particular in the analysis of the variational problem. This leads us to the main theorem of this paper:

Theorem A. *There is $c_1 > 0$ s.t. for all $f \in \mathcal{L}^\infty(\Omega)$*

$$|(G_k(\Omega)f)(x)| \leq cL^{2d}\mathrm{e}^{-c_1 \mathrm{d}(x,\mathrm{supp}(f))}\|f\|_\infty, \tag{1.4}$$

where $\mathrm{d}(x, \mathrm{supp}(f)) := \inf_{y\in\mathrm{supp}(f)}|x-y|$ and c, c_1 are constants depending only on dimension d of the lattice. (In particular they are independent of the lattice spacing, the size of the lattice and the parameter L).

We note that estimate (1.4) is non-trivial only because the constants c, c_1 are independent of the lattice size. This result is stated in many references including [Ba83] and [Di13] but we could not find a complete and verifiable proof. Theorem A will be proven as Theorem 2.25 below. We summarize the proof in the later part of this introduction.

We mention as an aside that (1.4) is not a direct consequence of (1.3): Setting $f' = \frac{1}{\eta^d}\delta_x$ in (1.3) and then estimating the $\mathcal{L}^2$-norm of f by the supremum norm, we immediately obtain a dependence of the constants on the lattice spacing and the volume of the lattice. Furthermore, (1.4) does not imply that $(x, x') \mapsto G_k(\Omega)(x, x')$ is a bounded function uniformly in the lattice spacing: by setting $f = \frac{1}{\eta^d}\delta_x$ in (1.4) we obtain again η-dependence of the constants. The latter fact reflects the well-known singularity at coinciding points of the free Green function in the continuum, e.g. for $d \geq 3$

$$G(x, x') := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}p\cdot(x-x')}\frac{1}{p^2+m^2}dp = \frac{1}{4\pi}\frac{1}{|x-x'|^{d-2}}\mathrm{e}^{-m|x-x'|}. \tag{1.5}$$

To explain how the propagators defined in (1.2) appear in QFT on the lattice, let us define the following family of functions on $\mathcal{L}^2(\Omega)$

$$\rho_0^J(\phi) := \mathrm{e}^{-\frac{1}{2}\langle\phi,\Delta^{(0),\eta}(\Omega)\phi\rangle}\mathrm{e}^{\mathrm{i}\langle J,\phi\rangle}, \quad \Delta^{(0),\eta}(\Omega) = -\Delta_\Omega^\eta + \bar{\mu}_k, \tag{1.6}$$

which, after integration in ϕ, gives the generating functional of the theory in the variable $J \in \mathcal{L}^2(\Omega)$. Denote by $T^\eta_{a_j,L^j}$ the renormalization group transformation, acting by averaging j-times over boxes, defined precisely in (2.74). The averaged function $\rho_j^J := T^\eta_{a_j,L^j}[\rho_0^J]$ can be computed via Gaussian integration and has the form:

$$\rho_j^J(\psi) = Z_j^\eta(\Omega)\mathrm{e}^{-\frac{1}{2}\langle\psi,\Delta^{(j),L^j\eta}(\Omega)\psi\rangle}\mathrm{e}^{\mathrm{i}\langle J,\mathcal{H}_j^\eta(\Omega)\psi\rangle}\mathrm{e}^{-\frac{1}{2}\langle J,G_j^\eta(\Omega)J\rangle}. \tag{1.7}$$

By setting $J = 0$, we see that the covariance of the averaged theory is the inverse of

$$\Delta^{(j),L^j\eta}(\Omega) := c_{(j)} - c_{(j)}^2 Q_{\Omega,j}G_j^\eta(\Omega)Q^*_{\Omega,j}, \tag{1.8}$$

where $c_{(j)} := a_j(L^j\eta)^{-2}$. The propagator $G_j^\eta(\Omega)$ is defined as in (1.2), with a_j replaced by $c_{(j)}$, and it is related to $G_j(\Omega)$ by a simple scaling transformation, cf. (2.65). It appears not only in (1.8), but also in another ingredient of (1.7)

$$\mathcal{H}_j^\eta(\Omega) := c_{(j)}G_j^\eta(\Omega)Q^*_{\Omega,j} \tag{1.9}$$

which will be important below. Thus the Green functions we study in this paper do not have an immediate meaning as correlations of some physical system. Instead, they are convenient building blocks to express various natural quantities appearing in the process of renormalization.

The semigroup property of the renormalization group transformation (cf. Lemma 2.16) turns out to be useful for proving Theorem A. On the one hand we have $\rho_{j+1}^J := T^\eta_{a_{j+1},L^{j+1}}[\rho_0^J]$, on the other hand $\rho_{j+1}^J := T^{L^j\eta}_{a,L}[\rho_j^J]$. The latter formula, stated explicitly in (2.95), can be compared with (1.7), which results in the so called renormalization group formula

$$G_{j+1}^\eta(\Omega) = \mathcal{H}_j^\eta(\Omega)C^{(j),L^j\eta}(\Omega)\mathcal{H}_j^\eta(\Omega)^* + G_j^\eta(\Omega), \quad C^{(j),L^j\eta}(\Omega) := (\Delta^{(j),L^j\eta}(\Omega) + \frac{a}{(L^{j+1}\eta)^2}Q^*_{\Omega_j}Q_{\Omega_j})^{-1}. \tag{1.10}$$

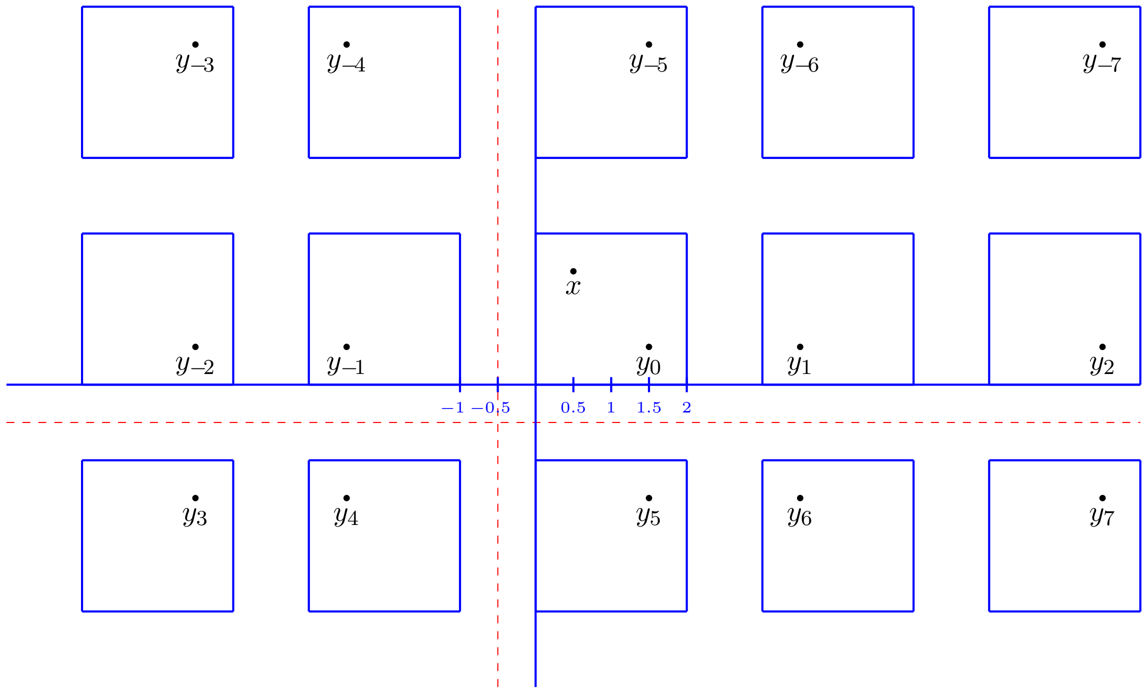

Figure 1: Set of image points y_j of the argument y in (1.12). The square containing the origin is the set Ω.

This formula appears (without proof) already in [Ba82], the proof sketched above is taken from [Di04]. It turns out that an exponential decay of the kernel of $C^{(j),L^j\eta}(\Omega)$ can be obtained by the Combes-Thomas method [CT74]. The exponential decay of the kernel of $\mathcal{H}^\eta_j(\Omega)$ also holds, but its proof is a separate story, which we will discuss below. The propagator $G^\eta_{j=1}(\Omega)$ can be rescaled to $G_{j=1}(L^{k-1}\Omega)$, for which the bound (1.4) is easy to prove, again by the Combes-Thomas method, since the lattice $L^{k-1}\Omega$ has spacing L^{-1} and L is fixed in the discussion. It should be clear from these remarks, that the renormalization group formula (1.10) allows for an iterative proof of Theorem A.

To conclude the discussion of the proof of Theorem A, let us comment on the exponential decay of the kernel of $\mathcal{H}^\eta_j(\Omega)$. As usual, it suffices to study its rescaled variant $\mathcal{H}_k(\Omega) := a_k G_k(\Omega) Q^*_{\Omega,k}$. The first step is to express the Green functions $G_k(\Omega)$ on the finite lattice Ω by their counterparts G_k on an infinite lattice. This is achieved using the method of images:

$$G_k(\Omega)(x,y) = \sum_{y_j \in \mathrm{Img}} G_k(x, y_j), \tag{1.11}$$

where Img is the set of images of y as indicated in Figure 1. (Think of the problem of determining the electric field of a charge near infinitely extended conductive planes from basic electrostatic). Once stated, formula (1.11) is not difficult to prove, it suffices to check that the r.h.s. is indeed the inverse of $(-\Delta^\eta_\Omega + \bar{\mu}_k + a_k Q^*_{\Omega,k} Q_{\Omega,k})$. The advantage of translating the problem on an infinite lattice is that we can use the Fourier transform to obtain the following representation

$$(G_k Q^*_k)(x,y) = (2\pi)^{-d} \int_{[-\pi/\eta, \pi/\eta[^{\times d}} dp\, \mathrm{e}^{\mathrm{i}p\cdot(x-y)} h(p). \tag{1.12}$$

If h can be analytically continued to a bounded function on a strip in a complex plane, then exponential decay of this expression follows by standard arguments. The difficult part is in showing that the width of this strip can be chosen uniformly in the lattice spacing and other relevant parameters. This step is completely skipped in [Ba83, p. 586], a related discussion appears in [BOS89] in a different context, but we did not find a complete proof of this fact in the literature. We provide such a proof in Appendix F, which is the main technical result of this article. The main idea of the proof is described below Lemma 3.19.

The proof of Theorem A can thus be summarized as follows: there are two main sources of exponential decay: the Combes-Thomas method (1.3) and analytic continuation of the Fourier transform (1.12). To combine them, two identities are used: the renormalization group equation (1.10) and the method of images formula (1.11).

To keep this article within reasonable limits, we deliberately omit the topic of random walk expansions. They are useful, e.g., to translate Theorem A to periodic boundary conditions, cf. [Di13, Lemma 6], or to control the kernels of operators $(Q_k G_k(\Omega) Q_k^*)^{-1}$, which appear at the level of the variational problem of some just-renormalizable models. We plan to come back to this topic in future publications.

Our paper is organized as follows: In Section 2 we focus on the Laplacian with Neumann boundary conditions and provide a proof of Theorem A assuming the exponential decay of the kernel of $\mathcal{H}_k(\Omega)$. This part is mostly based on [Di04, Di13] but our discussion is more detailed. In Section 3 we prove such a decay for the counterpart of $\mathcal{H}_k(\Omega)$ on an infinite lattice, using the Fourier space representation (1.12). As this part is omitted in [Di13], we follow mostly [Ba83] and complete the discussion regarding analytic continuation. Finally, in Section 4 we use the method of images to prove formula (1.11) which is only stated in [Ba83]. This implies the required decay of the kernel of $\mathcal{H}_k(\Omega)$ and concludes the proof of Theorem A. More technical steps of the discussion are postponed to the appendices.

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Notation

1. By $c, c', c_1, c_2 \dots$ we denote numerical constants, independent of any parameters except for the dimension d. These constants may change from line to line.

2. We denote by η the lattice spacing. We introduce an odd positive integer $L > 1$ and set $\eta = L^{-k}$, $k \in \mathbb{N}$. To change the lattice spacing, we keep L fixed and change k.

3. We set $I = \eta[0, 1, \dots, n-1]$, $n - 1 = L^m$, so that the parameter $m \geq k$ controls the size of the interval. (We remark that the interesting case is m much larger than k, since Theorem A concerns large-distance properties).

4. We denote by $\Omega \subset \eta\mathbb{Z}^d$ a cube $\Omega := I^{\times d} = \eta[0, 1, \dots, L^m]^{\times d}$.

5. For $j = 1, \dots, m$ we denote by $\Omega_j \subset L^j \eta \mathbb{Z}^d$ the coarse lattices of the form $\Omega_j = L^j \eta [0, 1, \dots, L^{m-j}]^{\times d}$.

6. We will use $\mathrm{k}, \mathrm{m}, \dots \in \mathbb{Z}$ to number points of the lattice, as opposed to parameters k, m, which control the lattice spacing and size of the box.

7. The elements of the Hilbert space $\mathcal{L}^2(\eta\mathbb{Z})$ are complex-valued functions denoted f, f', g, g'. The scalar product has the form

$$\langle f, g \rangle = \eta \sum_{\mathrm{k} \in \mathbb{Z}} \bar{f}_{\mathrm{k}} g_{\mathrm{k}} = \eta \sum_{x \in \eta\mathbb{Z}} \bar{f}(x) g(x), \tag{1.13}$$

where we write $f(x)$ or f_{k} for $x = \eta\mathrm{k}$. We set $\|f\|_2^2 = \langle f, f \rangle$. We use complex (rather than real) $\mathcal{L}^2$-spaces for simplicity of Fourier analysis in Section 3.

8. The Hilbert spaces $\mathcal{L}^2(I)$, $\mathcal{L}^2(\eta\mathbb{Z}^d)$, $\mathcal{L}^2(\Omega_j)$ are defined analogously as $\mathcal{L}^2(\eta\mathbb{Z})$ above and the corresponding scalar products and norms will be denoted as for $\mathcal{L}^2(\eta\mathbb{Z})$. If there is a risk of confusion, we will add the region of summation as subscript, e.g. $\langle\,\cdot\,,\cdot\,\rangle_{\Omega_j}, \|\,\cdot\,\|_{2,\Omega_j}$. The subspaces of real-valued functions are denoted by $\mathcal{L}^2_{\mathbb{R}}(I)$, $\mathcal{L}^2_{\mathbb{R}}(\eta\mathbb{Z}^d)$, $\mathcal{L}^2_{\mathbb{R}}(\Omega_j)$.

9. For any region $O \subset \eta\mathbb{Z}^d$ we will write $\|f\|_{\infty,O} = \sup_{x\in O}|f(x)|$. If there is no risk of confusion, we will drop O. We denote by $\mathcal{L}^\infty(O)$ the Banach space of functions on O equipped with the norm $\|\,\cdot\,\|_{\infty,O}$.

10. For quantities or conditions X_μ depending on the spacetime index $\mu = 0, 1, \ldots, d-1$ we will often write $X_\bullet$ as an abbreviation for $X_\mu, \mu = 0, 1, \ldots, d-1$.

11. We define the boxes in the lattice
$$B_j(y) := \{\, x \in \eta\mathbb{Z}^d \,|\, y_\mu \le x_\mu < y_\mu + L^j\eta,\ \mu = 0, 1, \ldots, d-1\} \tag{1.14}$$
and call y the *label* of the box. For $j = k$ we obtain unit boxes $\triangle_y := B_k(y)$, since $\eta = L^{-k}$. We also define $\tilde{\triangle}_z := L^{-1}B_{k+1}(Lz)$, $z \in L^{-1}\Omega$, which are unit boxes in the lattice $L^{-1}\Omega$.

12. We denote by Δ^η_Ω, resp. Δ^η, the Laplacian on Ω with Neumann boundary conditions, resp. the Laplacian on $\eta\mathbb{Z}^d$ with free boundary conditions.

13. The mass parameters have the form $\bar{\mu}_k = L^{2k}\bar{\mu}_0$ for $\bar{\mu}_0 \ge 0$. This corresponds to canonical scaling if $\bar{\mu}_0$ is the mass squared. As the mass parameter will play a minor role in our discussion, we abbreviate
$$-\Delta^{\eta,\bar{\mu}_k}_\Omega := -\Delta^\eta_\Omega + \bar{\mu}_k, \quad -\Delta^{\eta,\bar{\mu}_k} := -\Delta^\eta + \bar{\mu}_k. \tag{1.15}$$
Furthermore, in Appendix F we will use the notation $\overset{*}{\sum}{}_{\mu=0}^{d-1} a_\mu = (\sum_{\mu=0}^{d-1} a_\mu) + \frac{1}{4}\bar{\mu}_0$.

14. $\mathrm{d}(x, x') := |x - x'| = \big(\sum_{\mu=0}^{d-1}(x_\mu - x'_\mu)^2\big)^{1/2}$, $\mathrm{d}(x, O) := \inf_{y\in G}|x - y|$ for $O \subset \eta\mathbb{Z}^d$, $|x - x'|_\infty := \sup_{\mu=0,\ldots,d-1}|x_\mu - x'_\mu|$.

15. $\mathbb{1}_O$ denotes the characteristic function of a set O.

2 Green functions with Neumann boundary conditions

2.1 Laplacian on interval with Neumann boundary conditions

We consider the finite dimensional Hilbert space $\mathcal{L}^2(I)$, where $I := \eta[0, 1, 2, \ldots, n-1]$, with the scalar product
$$\langle f, g\rangle = \eta \sum_{\mathrm{k}=0}^{n-1} \bar{f}_{\mathrm{k}} g_{\mathrm{k}}. \tag{2.1}$$
We note that $\mathrm{k} \mapsto \delta_{\eta;\mathrm{m},\mathrm{k}} = \frac{1}{\eta}\delta_{\mathrm{m},\mathrm{k}}$ plays the role of the Dirac delta, since $\langle \delta_{\eta;\mathrm{m}}, f\rangle = f_{\mathrm{m}}$. Let $A : \mathcal{L}^2(I) \to \mathcal{L}^2(I)$ be a linear map. We define its kernel by
$$A_{\mathrm{m},\mathrm{m}'} = \langle \delta_{\eta;\mathrm{m}}, A\delta_{\eta;\mathrm{m}'}\rangle. \tag{2.2}$$
We have $(Af)_{\mathrm{k}} = \langle \delta_{\eta;\mathrm{k}}, Af\rangle$. By writing $f_{\mathrm{l}} = \eta\sum_{\mathrm{k}'} \delta_{\eta;\mathrm{l},\mathrm{k}'} f_{\mathrm{k}'}$, we have
$$(Af)_{\mathrm{k}} = \eta \sum_{\mathrm{k}'\in\mathbb{Z}} A_{\mathrm{k},\mathrm{k}'} f_{\mathrm{k}'}. \tag{2.3}$$

As specific examples of A we define the discrete derivatives
$$(\partial^\eta_I f)_{\mathrm{k}} := \frac{1}{\eta}(f_{\mathrm{k}+1} - f_{\mathrm{k}}), \quad (\partial^{\eta,\dagger}_I f)_{\mathrm{k}} := -\frac{1}{\eta}(f_{\mathrm{k}} - f_{\mathrm{k}-1}) = -(\partial^\eta_I f)_{\mathrm{k}-1} \tag{2.4}$$
with Neumann boundary conditions $f_n := f_{n-1}$, $f_{-1} := f_0$, that is, $(\partial^{\eta,\dagger}_I f)_0 = 0 = (\partial^\eta_I f)_{n-1}$. By Lemma 2.1 we see that $\partial^{\eta,\dagger}_I$ is not the adjoint of ∂^η_I, given Neumann boundary conditions.

Lemma 2.1. *The Leibniz rule holds in the form*

$$(\partial_I^\eta(fg))_{\mathrm{k}} = (\partial_I^\eta f)_{\mathrm{k}} g_{\mathrm{k}+1} + f_{\mathrm{k}} (\partial_I^\eta g)_{\mathrm{k}}. \tag{2.5}$$

Integration by parts holds in the form

$$\begin{aligned}\langle f, \partial_I^\eta g\rangle &= -\langle \partial_I^\eta f, g_{\cdot+1}\rangle - \bar{f}_0 g_0 + \bar{f}_{n-1} g_{n-1} \\ &= \langle \partial_I^{\eta,\dagger} f, g\rangle - \bar{f}_0 g_0 + \bar{f}_{n-1} g_{n-1}.\end{aligned} \tag{2.6}$$

Proof. We compute

$$\begin{aligned}(\partial_I^\eta(fg))_{\mathrm{k}} &= \frac{1}{\eta}\big((fg)_{\mathrm{k}+1} - (fg)_{\mathrm{k}}\big) \\ &= \frac{1}{\eta}\big((f_{\mathrm{k}+1} - f_{\mathrm{k}}) g_{\mathrm{k}+1} + f_{\mathrm{k}}(g_{\mathrm{k}+1} - g_{\mathrm{k}})\big),\end{aligned} \tag{2.7}$$

which gives (2.5). Next, we note that

$$\eta \sum_{\mathrm{k}=0}^{n-1} (\partial_I^\eta(fg))_{\mathrm{k}} = -f_0 g_0 + f_{n-1} g_{n-1}. \tag{2.8}$$

On the other hand

$$\eta \sum_{\mathrm{k}=0}^{n-1} (\partial_I^\eta(fg))_{\mathrm{k}} = \langle (\partial_I^\eta f), g_{\cdot+1}\rangle + \langle f, (\partial_I^\eta g)\rangle. \tag{2.9}$$

Finally, we write

$$\begin{aligned}\langle (\partial_I^\eta f), g_{\cdot+1}\rangle &= \sum_{\mathrm{k}=0}^{n-1} (\partial_I^\eta f)_{\mathrm{k}} g_{\mathrm{k}+1} = \sum_{\mathrm{k}'=1}^{n} (\partial_I^\eta f)_{\mathrm{k}'-1} g_{\mathrm{k}'} \\ &= \sum_{\mathrm{k}'=0}^{n-1} (\partial_I^\eta f)_{\mathrm{k}'-1} g_{\mathrm{k}'} = -\langle \partial_I^{\dagger,\eta} f, g\rangle,\end{aligned} \tag{2.10}$$

where in the third step we made use of Neumann boundary conditions. □

The Laplacian with Neumann boundary conditions in one dimension on the lattice $I = \eta[0, 1, 2, \ldots, n-1]$ has the form

$$(\Delta_I^\eta f)_{\mathrm{k}} := ((\partial_I^\eta)(\partial_I^\eta f)_{\cdot-1})_{\mathrm{k}} = -((\partial_I^\eta)(\partial_I^{\eta,\dagger} f))_k = \frac{f_{\mathrm{k}+1} - 2f_{\mathrm{k}} + f_{\mathrm{k}-1}}{\eta^2} \tag{2.11}$$

with the boundary conditions $f_n := f_{n-1}$, $f_{-1} := f_0$. Although it is not manifest from the definition, $(-\Delta_I^\eta)$ is a positive operator:

Lemma 2.2. *We have*

$$\langle f, (-\Delta_I^\eta) f\rangle = \langle (-\Delta_I^\eta) f, f\rangle = \frac{1}{\eta^2} \sum_{\mathrm{k}=0}^{n-2} |f_{\mathrm{k}+1} - f_{\mathrm{k}}|^2. \tag{2.12}$$

Proof. We compute using (2.5)

$$\langle f, (\partial_I^\eta(\partial_I^\eta f)_{\cdot-1})\rangle = -\langle \partial_I^\eta f, \partial_I^\eta f\rangle - \bar{f}_0 (\partial_I^\eta f)_{-1} + \bar{f}_{n-1} (\partial_I^\eta f)_{n-1}. \tag{2.13}$$

Noting that $(\partial_I^\eta f)_{-1} = (\partial_I^\eta f)_{n-1} = 0$ by the Neumann boundary conditions and dropping the last term in the sum defining the scalar product we obtain the claim. □

Next, let us recall the standard computation of the spectrum of the Laplacian on a finite lattice[1]. This will be needed in the proof of existence of lattice Green functions in (2.51) below.

[1] See Wikipedia: Eigenvalues and eigenvectors of the second derivative.

Lemma 2.3. *The eigenvalues of* Δ_I^η *are*

$$\lambda^{(j)} = -\frac{4}{\eta^2}\sin^2\left(\frac{\pi j}{2n}\right), \tag{2.14}$$

where $j = 0, \dots, n-1$.

Proof. The eigenvalue equation reads

$$\frac{f_{\mathrm{k}+1} - 2f_{\mathrm{k}} + f_{\mathrm{k}-1}}{\eta^2} = \lambda f_{\mathrm{k}}, \quad \mathrm{k} = 0, \dots, n-1, \tag{2.15}$$

$f_{-1} = f_0$, $f_n = f_{n-1}$. We immediately note that $f = \text{const}$ and $\lambda = 0$ is a solution, so we can restrict attention to non-constant f. We make a change of variables

$$h_{\mathrm{k}} := f_{\mathrm{k}+1} - f_{\mathrm{k}}, \quad \mathrm{k} = -1, \dots n-1. \tag{2.16}$$

This gives

$$\begin{aligned} h_{\mathrm{k}+1} - h_{\mathrm{k}} &= \eta^2\lambda f_{\mathrm{k}} && (2.17)\\ &= \eta^2\lambda h_{\mathrm{k}-1} + \eta^2\lambda f_{\mathrm{k}-1} && (2.18)\\ &= \eta^2\lambda h_{\mathrm{k}-1} + (h_{\mathrm{k}} - h_{\mathrm{k}-1}), && (2.19)\\ h_{\mathrm{k}+1} &= (2+\eta^2\lambda)h_{\mathrm{k}} - h_{\mathrm{k}-1}, && (2.20) \end{aligned}$$

where in (2.19) we applied (2.17) with $\mathrm{k} \to \mathrm{k}-1$. The Neumann boundary conditions now read $h_{-1} = 0, h_{n-1} = 0$. Setting $2\alpha = 2 + \eta^2\lambda$ we have the recurrence

$$h_{\mathrm{k}+1} = 2\alpha h_{\mathrm{k}} - h_{\mathrm{k}-1}, \quad h_{-1} = 0,\ h_{n-1} = 0. \tag{2.21}$$

Assuming $h_0 = 1$ (to exclude that $f = \text{const}$) we get

$$h_{\mathrm{k}} = U_{\mathrm{k}}(\alpha), \tag{2.22}$$

where U_{k} is the k-th Chebyshev polynomial of the 2nd kind by Lemma 2.4 below. Now $h_{n-1} = 0$ gives

$$U_{n-1}(\alpha) = 0, \tag{2.23}$$

which holds for $\alpha_j = \cos\left(\frac{j}{n}\pi\right)$, $j = 1, \dots, n-1$ by (2.25) below. This, together with $2\alpha = 2 + \eta^2\lambda$, gives the remaining $n-1$ eigenvalues. □

Lemma 2.4. [2] *The Chebyshev polynomials of the 2nd kind, defined by the recurrence relation*

$$U_0(\alpha) = 1, \quad U_1(\alpha) = 2\alpha, \quad U_{\mathrm{k}+1}(\alpha) = 2xU_{\mathrm{k}}(\alpha) - U_{\mathrm{k}-1}(\alpha), \tag{2.24}$$

have the following property: Each U_n *has* n *different simple roots in* $[-1, 1]$ *given by*

$$\alpha_j = \cos\left(\frac{j}{n+1}\pi\right), \quad j = 1, \dots, n. \tag{2.25}$$

[2] See Wikipedia: Chebyshev polynomials.

2.2 Laplacian on $\Omega \subset \eta\mathbb{Z}^d$ with Neumann boundary conditions

Let $\Omega := I^{\times d}$ be a hypercube in the lattice, where $I = \eta[0, 1, 2, \ldots, n-1]$ and write $\mathrm{k} = (\mathrm{k}_0, \ldots, \mathrm{k}_{d-1}) = (\mathrm{k}_\mu)_{\mu=0,\ldots,d-1}$ for integer parameters labelling elements $x := \eta\mathrm{k}$ of Ω. The boundary $\partial\Omega \subset \Omega$ consists of $2d$ faces

$$\begin{aligned}\partial\Omega \;&=\; \bigcup_{\mu=0}^{d-1}(\underbrace{I\times\cdots\times\{0\}}_{\mu+1}\times\cdots\times I)\cup\bigcup_{\mu=0}^{d-1}(\underbrace{I\times\cdots\times\{n-1\}}_{\mu+1}\times\cdots\times I)\\ &=: \bigcup_{\mu=0}^{d-1}(\partial\Omega)_\mu \cup \bigcup_{\mu=0}^{d-1}(\partial\Omega)^\mu. \end{aligned}\tag{2.26}$$

We consider the Hilbert space $\mathcal{L}^2(\Omega) = \mathcal{L}^2(I)^{\otimes d}$ whose scalar product, in accordance with (2.1), has the form

$$\langle f, g\rangle = \eta^d \sum_{\eta\mathrm{k}\in\Omega} \bar{f}_\mathrm{k} g_\mathrm{k} = \eta^d \sum_{x\in\Omega} \bar{f}(x) g(x). \tag{2.27}$$

A basis in this space is formed by the delta functions

$$\delta_{\eta;\mathrm{k}} := \delta_{\eta;\mathrm{k}_0} \otimes \cdots \otimes \delta_{\eta;\mathrm{k}_{d-1}}. \tag{2.28}$$

In terms of lattice points $x := \eta\mathrm{k}$ we denote $\delta^\eta_x := \delta_{\eta;\mathrm{k}}$. For a linear map $A : \mathcal{L}^2(\Omega) \to \mathcal{L}^2(\Omega)$ we define its kernel via

$$A(x, x') = A_{\mathrm{k},\mathrm{l}} = \langle \delta_{\eta;\mathrm{k}}, A\delta_{\eta;\mathrm{l}}\rangle = \langle \delta^\eta_x, A\delta^\eta_{x'}\rangle. \tag{2.29}$$

The Laplacian on $\mathcal{L}^2(\Omega)$ with Neumann boundary conditions is defined by

$$\Delta^\eta_\Omega := \sum_{\mu=0}^{d-1} 1 \otimes \cdots \otimes \Delta^\eta_I \otimes \cdots \otimes 1. \tag{2.30}$$

This is a trivial generalization of the one-dimensional case, because it is a sum of commuting operators. Next, we note the following consequence of Lemma 2.2:

Lemma 2.5. *Let $f \in \mathcal{L}^2(\Omega)$. Then*

$$\langle f, (-\Delta^\eta_\Omega) f\rangle = \frac{1}{\eta^2}\sum_b \langle \partial f(b), \partial f(b)\rangle, \tag{2.31}$$

where the sum is over all the bonds in the lattice Ω and $(\partial f)(b) = f(b_-) - f(b_+)$.

Next, we study the behaviour of the Laplacian under scaling. Let O be some subset of $\tilde{\eta}\mathbb{Z}^d$ for $\tilde{\eta} > 0$. (For future convenience we distinguish $\tilde{\eta}$ from η which will be later set equal to L^{-k}). For $\lambda > 0$ we define a scaling transformation $S^O_\lambda : \mathcal{L}^2(O) \to \mathcal{L}^2(\lambda O)$ by

$$S^O_\lambda f = \lambda^{-d/2} f_\lambda, \quad f_\lambda(x) := f(\lambda^{-1}x). \tag{2.32}$$

It is easy to see that S^O_λ is a unitary map and $(S^O_\lambda)^* = S^{\lambda O}_{\lambda^{-1}}$. In particular:

$$\|(S^O_\lambda f)\|^2_{2,\lambda O} = (\lambda\tilde{\eta})^d \sum_{x\in\lambda O} |(S^O_\lambda f)(x)|^2 \;=\; \tilde{\eta}^d \sum_{x'\in O} |f(x')|^2 = \|f\|^2_{2,O}. \tag{2.33}$$

We also note the semigroup property: $S^{\lambda_1 O}_{\lambda_2} S^O_{\lambda_1} = S^O_{\lambda_2\lambda_1}$. For $O = \Omega$ we will often skip the superscript of S^O_λ.

We check the behaviour of the Laplacian under scaling:

Lemma 2.6. *We have, for $\lambda > 0$,*

$$\Delta^{\eta}_{\Omega} = \lambda^2 (S^{\Omega}_{\lambda})^* \Delta^{\lambda\eta}_{\lambda\Omega} S^{\Omega}_{\lambda}. \tag{2.34}$$

Proof. We note the relation

$$(\Delta^{\lambda\eta}_{\lambda\Omega} f_\lambda)(\lambda x) = \sum_{\mu=0}^{d-1} \frac{f_\lambda(\lambda x + e_\mu \lambda\eta) - 2 f_\lambda(\lambda x) + f_\lambda(\lambda x - e_\mu \lambda\eta)}{(\lambda\eta)^2} = \lambda^{-2} (\Delta^{\eta}_{\Omega} f)(x), \tag{2.35}$$

where $\{e_\mu\}_{\mu=0,1,\ldots,d-1}$, are basis vectors. By rewriting this expression using S_λ we obtain the formula. Clearly, modifications at the boundaries of the region do not change the result. □

2.3 Averaging operator on Ω

Recall that the finite lattice in one dimension has the form $I := \eta[0, 1, 2, \ldots, n-1]$. We fix an odd[3] integer $L > 1$, set $\eta = L^{-k}$ and $n - 1 = L^m$ for some fixed $m \geq k$. Now we define the coarse lattice for $1 \leq j \leq k$:

$$\begin{aligned} I_j = (L^j I) \cap I \; &= \; (L^j \eta)[0, 1, 2, \ldots, L^m] \cap \eta[0, 1, 2, \ldots, L^m] \\ &= \; (L^j \eta)[0, 1, 2, \ldots, L^{m-j}]. \end{aligned} \tag{2.36}$$

Next, consider the d-dimensional case with $\Omega = I^{\times d}$. The coarse finite lattice has the form

$$\Omega_j = L^j \Omega \cap \Omega = (L^j \eta)[0, 1, 2, \ldots, L^{m-j}]^{\times d}. \tag{2.37}$$

For future reference, we note the following lemma, which checks that scaling is compatible with the procedure of making the lattice coarser.

Lemma 2.7. *With $\Omega = L^{-k}[0, 1, 2, \ldots, L^m]^{\times d}$, we have for any $\ell \in \mathbb{Z}$,*

$$(L^\ell \Omega)_j = L^\ell(\Omega_j). \tag{2.38}$$

Proof. We note that $L^\ell \Omega$ is obtained from Ω by changing the lattice spacing η to $L^\ell \eta$. Thus

$$(L^\ell \Omega)_j = L^j (L^\ell \Omega) \cap (L^\ell \Omega) = (L^j L^\ell \eta)[0, 1, 2, \ldots, L^{m-j}]^{\times d} = L^\ell(\Omega_j) \tag{2.39}$$

as claimed. □

We define the averaging operator $Q : \mathcal{L}^2(\Omega) \to \mathcal{L}^2(\Omega_1)$ by

$$(Qf)(y) \;=\; \frac{1}{L^d} \sum_{y_\bullet \leq x'_\bullet < y_\bullet + L\eta} f(x'), \tag{2.40}$$

where the condition under the sum is an abbreviation for $y_\mu \leq x'_\mu < y_\mu + L\eta, \; \mu = 0, 1, \ldots, d-1$. We can iterate this procedure, remembering that upon second application Q acts between different spaces, namely $Q : \mathcal{L}^2(\Omega_1) \to \mathcal{L}^2(\Omega_2)$. We obtain

$$\begin{aligned} (Q_2 f)(z) \;&=\; (Q^2 f)(z) = \frac{1}{L^d} \sum_{z_\bullet \leq y'_\bullet < z_\bullet + L^2\eta} (Qf)(y') \\ &= \frac{1}{L^{2d}} \sum_{z_\bullet \leq y'_\bullet < z_\bullet + L^2\eta} \; \sum_{y'_\bullet \leq x'_\bullet < y'_\bullet + L\eta} f(x') = \frac{1}{L^{2d}} \sum_{z_\bullet \leq x'_\bullet < z_\bullet + L^2\eta} f(x'). \end{aligned} \tag{2.41}$$

[3]The assumption that L is odd is used in Lemma 3.13. It is consistent with [Di13].

By iterating, we obtain $Q_j : \mathcal{L}^2(\Omega) \to \mathcal{L}^2(\Omega_j)$:

$$(Q_j f)(z) = \frac{1}{L^{jd}} \sum_{z_\bullet \leq x'_\bullet < z_\bullet + L^j \eta} f(x'). \tag{2.42}$$

We denote the region of summation in (2.42) $B_j(z)$ and call z the label of this box. For $j = k$ we have $L^k \eta = 1$, hence $\triangle_z := B_k(z)$ is a unit box. If there is a risk of confusion, we will denote Q_j by $Q_{\Omega,j}$.

It is easy to see that the adjoint $Q_j^* : \mathcal{L}^2(\Omega_j) \to \mathcal{L}^2(\Omega)$ has the form

$$(Q_j^* h)(x) = h(y_x), \tag{2.43}$$

where $y_{x,\mu} := \left[\frac{x_\mu}{L^j \eta}\right](L^j \eta)$ is an element of the coarser lattice and $[a]$ denotes the integer part of $a \in \mathbb{R}$. (Equivalently, one can say that y_x is the unique element of the coarser lattice Ω_j s.t. $x \in B_j(y_x)$. That is y_x is the label of the box to which x belongs). To verify that (2.43) defines the adjoint of Q_j we compute

$$\begin{aligned} \langle h, Q_j f \rangle_{\Omega_j} &= (L^j \eta)^d \sum_{y \in (L^j \eta)\mathbb{Z}^d} \overline{h}(y) \frac{1}{L^{jd}} \sum_{y_\bullet \leq x'_\bullet < y_\bullet + L^j \eta} f(x') = \eta^d \sum_{y \in (L^j \eta)\mathbb{Z}^d} \sum_{y_\bullet \leq x'_\bullet < y_\bullet + L^j \eta} \overline{h}(y) f(x') \\ &= \eta^d \sum_{x' \in \eta \mathbb{Z}^d} \overline{h}(y_{x'}) f(x') = \langle Q_j^* h, f \rangle_\Omega. \end{aligned} \tag{2.44}$$

Thus we get for $Q_j^* Q_j : \mathcal{L}^2(\Omega) \to \mathcal{L}^2(\Omega)$

$$(Q_j^* Q_j f)(x) = \frac{1}{L^{jd}} \sum_{y_{x,\bullet} \leq x'_\bullet < y_{x,\bullet} + L^j \eta} f(x') = \frac{1}{L^{jd}} \sum_{[\frac{x_\bullet}{L^j \eta}](L^j \eta) \leq x'_\bullet < [\frac{x_\bullet}{L^j \eta}](L^j \eta) + L^j \eta} f(x'). \tag{2.45}$$

This operator is the main building block of the lattice Green functions in Subsection 2.4.

Remark 2.8. *Since $Q_j Q_j^* = 1$, the operator $Q_j^* Q_j$ is an orthogonal projection, hence $\|Q_j\| = \|Q_j^*\| = 1$. Its range is the subspace of 'block-constant functions', i.e., step functions which are constant on $B_j(y)$, $y \in \Omega_j$. The statement remains true* mutatis mutandis *in the case of the infinite lattice studied in Subsection 3.4.*

Now we note the behaviour of the averaging operators under the scaling transformations (2.32).

Lemma 2.9. *We have, for $\lambda = L^\ell$, $\ell \in \mathbb{Z}$,*

$$Q_{\lambda\Omega,j} S_\lambda^\Omega = S_\lambda^{\Omega_j} Q_{\Omega,j}. \tag{2.46}$$

Proof. We compute the l.h.s. on $f \in \mathcal{L}^2(\Omega)$

$$(Q_{\lambda\Omega} S_\lambda^\Omega f)(y) = \frac{1}{L^{jd}} \sum_{y_\bullet \leq x_\bullet < y_\bullet + \lambda L^j \eta} (S_\lambda^\Omega f)(x) = \frac{1}{L^{jd}} \sum_{\lambda^{-1} y_\bullet \leq \lambda^{-1} x_\bullet < \lambda^{-1} y_\bullet + L^j \eta} \lambda^{-d/2} f(\lambda^{-1} x). \tag{2.47}$$

Now the r.h.s. gives

$$\begin{aligned} (S_\lambda^{\Omega_j} Q_{\Omega,j} f)(y) &= \lambda^{-d/2} (Q_{\Omega,j} f)(\lambda^{-1} y) = \lambda^{-d/2} \frac{1}{L^{jd}} \sum_{\lambda^{-1} y_\bullet \leq x_\bullet < \lambda^{-1} y_\bullet + L\eta} f(x) \\ &= \frac{1}{L^{jd}} \sum_{\lambda^{-1} y_\bullet \leq \lambda^{-1} x'_\bullet < \lambda^{-1} y_\bullet + L^j \eta} \lambda^{-d/2} f(\lambda^{-1} x'). \end{aligned} \tag{2.48}$$

This concludes the proof. □

2.4 Green functions with Neumann boundary conditions

Now we define the propagator with Neumann boundary conditions as a map on $\mathcal{L}^2(\Omega)$:

$$G_k(\Omega) := (-\Delta^\eta_\Omega + \bar{\mu}_k + a_k Q^*_{\Omega,k} Q_{\Omega,k})^{-1} =: \big[- \Delta^\eta + \bar{\mu}_k + a_k Q^*_k Q_k \big]^{-1}_\Omega, \tag{2.49}$$

where $0 < c \le a_k \le c' < \infty$ will be specified in (2.75) below. The mass parameters have the form $\bar{\mu}_k = L^{2k}\bar{\mu}_0$ for $\bar{\mu}_0 \ge 0$. As the mass parameter will play a minor role in our discussion, we abbreviate

$$-\Delta^{\eta,\bar{\mu}_k}_\Omega := -\Delta^\eta_\Omega + \bar{\mu}_k. \tag{2.50}$$

We first have to show that the inverse exists:

Lemma 2.10. [Di13] *The following holds:*

1. *For a unit cube $\triangle$, as operators on $\mathcal{L}^2(\triangle)$*

$$-\Delta^{\eta,\bar{\mu}_k}_\triangle + a_k Q^*_{\triangle,k} Q_{\triangle,k} \ge c(-\Delta^\eta_\triangle + 1), \tag{2.51}$$

   where $\Delta^\eta_\triangle$ has Neumann boundary conditions on $\triangle$.

2. *For Ω a union of disjoint unit cubes the following inequality holds as operators on $\mathcal{L}^2(\Omega)$*

$$-\Delta^{\eta,\bar{\mu}_k}_\Omega + a_k Q^*_{\Omega,k} Q_{\Omega,k} \ge c(-\Delta^\eta_\Omega + 1), \tag{2.52}$$

where $c > 0$ is independent of η and of the size of the box.

Remark 2.11. *The choice of Neumann boundary conditions is used in the proof of the independence of the constant in (2.52) of the size of the box Ω.*

Proof. If $f \in \mathcal{L}^2(\triangle)$ is constant then $-\Delta^\eta_\triangle f = 0$ and

$$\langle f, a_k Q^*_{\triangle,k} Q_{\triangle,k} f\rangle = a_k \|f\|_2^2 \ge c\|f\|_2^2. \tag{2.53}$$

If $f \in \mathcal{L}^2(\triangle)$ is orthogonal to constant functions, then $-\Delta^\eta_\triangle$ is strictly positive with lowest eigenvalue given by (2.14) and (2.30):

$$(-\lambda^{(1)}) = \frac{4}{\eta^2}\sin^2\left(\frac{\pi}{2n}\right) = \frac{4}{\eta^2}\sin^2\left(\frac{\pi\eta}{2(\eta+1)}\right), \tag{2.54}$$

where we used that we are in a unit box, so $(n-1)\eta = 1$. We have $(-\lambda^{(1)}) \ge c > 0$ uniformly in $\eta \le 1$. Therefore,

$$\langle f, (-\Delta^\eta_\triangle) f\rangle \ge c\|f\|_2^2 \tag{2.55}$$

and since $Q^*_{\triangle,k} Q_{\triangle,k}$ is also positive and $\bar{\mu}_k \ge 0$

$$\langle f, (-\Delta^{\eta,\bar{\mu}_k}_\triangle + a_k Q^*_{\triangle,k} Q_{\triangle,k}) f\rangle \ge c\|f\|_2^2. \tag{2.56}$$

This proves (2.51).

Now let $f \in \mathcal{L}^2(\Omega)$ and set $f_\triangle = f|_\triangle$. We choose as $\triangle$ the boxes $B_k(y), y \in \Omega_k$, to ensure that $Q^*_k Q_k f_\triangle$ is supported in $\triangle$. Since $\triangle \subset \Omega$ is a unit box, we have

$$\langle f, (-\Delta^{\eta,\bar{\mu}_k}_\Omega + a_k Q^*_{\Omega,k} Q_{\Omega,k}) f\rangle \ge \sum_{\triangle\subset\Omega} \big\langle f_\triangle, \big(- \Delta^{\eta,\bar{\mu}_k}_\triangle + a_k Q^*_{\triangle,k} Q_{\triangle,k}\big) f_\triangle\big\rangle \tag{2.57}$$

$$\ge c \sum_{\triangle\subset\Omega} \|f_\triangle\|_2^2 = c\|f\|_2^2. \tag{2.58}$$

Here in the first inequality we used Lemma 2.5 and the Neumann boundary conditions to justify that we can drop the bonds linking different unit boxes $\triangle$. □

Now it is easy to obtain exponential decay of $G_k(\Omega)$ in the $\mathcal{L}^2$ sense by a Combes-Thomas [CT74] argument. Our efforts in later sections will aim at improving this decay from $\mathcal{L}^2$ to L^∞. We recall that $\triangle_y$ denotes the unit box with label y.

Lemma 2.12. [Di13] *Let* $\mathrm{supp}(f) \subset \triangle_y$, $\mathrm{supp}(f') \subset \triangle_{y'}$ *with* $\triangle_y, \triangle_{y'} \subset \Omega$ *unit boxes with labels* y, y'. *Then*

$$|\langle f, G_k(\Omega) f' \rangle| \leq c\mathrm{e}^{-c_1|y-y'|} \|f\|_2 \|f'\|_2 \tag{2.59}$$

for some numerical constants c *and* $c_1 > 0$. *These constants are, in particular, independent of the size of* Ω.

Proof. See Appendix A. □

As this proof is rather technical, let us explain the basics of the Combes-Thomas method in a simple example. Consider a Schrödinger operator $H = -\Delta + V(x)$ on $\mathbb{R}^d$, where V is a measurable function s.t. $V \geq 1$. Let $\triangle_y$, $y \in \mathbb{Z}^d$, be unit boxes of a unit lattice, which we draw on $\mathbb{R}^d$. Then there exists $\delta > 0$ s.t.

$$|\langle f_1, H^{-1} f_2 \rangle| \leq c\mathrm{e}^{-\delta|y_1-y_2|} \|f_1\|_2 \|f_2\|_2, \tag{2.60}$$

for all square-integrable f_1, f_2 s.t. $\mathrm{supp}(f_1) \subset \triangle_{y_1}$, $\mathrm{supp}(f_2) \subset \triangle_{y_2}$.

First, set $H_q = \mathrm{e}^{qx} H \mathrm{e}^{-qx}$ and assume that $\|H_q^{-1} h\|_2 \leq c\|h\|_2$ uniformly in $|q| \leq \delta \leq 1$. Then, we immediately get (2.60) by computing

$$|\langle f_1, H^{-1} f_2 \rangle| = |\langle \mathrm{e}^{-qx} f_1, H_q^{-1} \mathrm{e}^{qx} f_2 \rangle| \leq c\|\mathrm{e}^{-qx} f_1\|_2 \|\mathrm{e}^{qx} f_2\|_2 \leq c' \mathrm{e}^{-q(y_1-y_2)} \|f_1\|_2 \|f_2\|_2. \tag{2.61}$$

and setting $q = \delta \frac{(y_1-y_2)}{|y_1-y_2|}$.

In order to justify our assumption, we note that $H_q = (p + \mathrm{i}q)^2 + V(x)$, where $p_\mu = -\mathrm{i}\partial_{x_\mu}$. Hence $H_q - H = \mathrm{i}2p \cdot q - q^2$ and

$$\|(-\Delta + 1)^{-1/2} (H_q - H)(-\Delta + 1)^{-1/2}\| \leq cq, \tag{2.62}$$

since $(-\Delta + 1)^{-1/2} p_\mu (-\Delta + 1)^{-1/2}$ is a bounded operator. Consequently,

$$|\langle f, H_q f \rangle| \geq |\langle f, Hf \rangle| - |\langle f, (H_q - H) f \rangle| \geq \langle f, (-\Delta + 1) f \rangle - cq\langle f, (-\Delta + 1) f \rangle \geq c' \|f\|_2^2. \tag{2.63}$$

Setting q small we get $c' > 0$. Now for $f = H_q^{-1} h$

$$\|h\|_2 \|H_q^{-1} h\|_2 \geq |\langle h, H_q^{-1} h \rangle| \geq c_1 \|H_q^{-1} h\|_2^2 \tag{2.64}$$

which gives $\|H_q^{-1} h\|_2 \leq c\|h\|_2$.

Coming back to the main course of our discussion, we note the following lemma for future reference in (2.82):

Lemma 2.13. *For* $\lambda_j = L^{k-j}$, $k \geq j \geq 1$,

$$G_j^\eta(\Omega) := \lambda_j^{-2} (S_{\lambda_j}^\Omega)^* G_j(\lambda_j \Omega) S_{\lambda_j}^\Omega = [-\Delta_\Omega^{\eta,\bar{\mu}_k} + a_j (L^j \eta)^{-2} Q_{\Omega,j}^* Q_{\Omega,j}]^{-1}, \tag{2.65}$$

$$\tilde{G}_{j+1}(\lambda_j \Omega) := \lambda_j^2 (S_{\lambda_j}^\Omega) G_{j+1}^\eta(\Omega) (S_{\lambda_j}^\Omega)^* = [-\Delta_{\lambda_j \Omega}^{\lambda_j \eta, \bar{\mu}_j} + \frac{a_{j+1}}{L^2} Q_{\lambda_j \Omega, j+1}^* Q_{\lambda_j \Omega, j+1}]^{-1}. \tag{2.66}$$

Remark 2.14. *The notation* $\tilde{G}_{j+1}$ *corresponds to* G_{j+1}^0 *in* [Di13].

Proof. Formula (2.65) is obtained from Lemmas 2.9, 2.6 and the following computation

$$\begin{aligned} G_j^{\eta}(\Omega)^{-1} &= (-\Delta_{\Omega}^{\eta,\bar{\mu}_k} + a_j(L^j\eta)^{-2}Q^*_{\Omega,j}Q_{\Omega,j}) \\ &= S^*_{\lambda_j}(-\lambda_j^2\Delta_{\lambda_j\Omega}^{\lambda_j\eta,\lambda_j^{-2}\bar{\mu}_k} + a_j(L^j\eta)^{-2}Q^*_{\lambda_j\Omega,j}Q_{\lambda_j\Omega,j})S_{\lambda_j} \\ &= \lambda_j^2 S^*_{\lambda_j}(-\Delta_{\lambda_j\Omega}^{\lambda_j\eta,\lambda_j^{-2}\bar{\mu}_k} + a_jQ^*_{\lambda_j\Omega,j}Q_{\lambda_j\Omega,j})S_{\lambda_j} = \lambda_j^2 S^*_{\lambda_j}G_j(\lambda_j\Omega)^{-1}S_{\lambda_j}, \end{aligned} \tag{2.67}$$

where we used $L^j\eta = \lambda_j^{-1}$ and $\lambda_j^{-2}\bar{\mu}_k = \bar{\mu}_j$.

Now we consider (2.66): Since $Q_{\Omega,j+1}(S^{\Omega}_{\lambda_j})^* = (S^{\Omega_{j+1}}_{\lambda_j})^*Q_{\lambda_j\Omega,j+1}$

$$\begin{aligned} S^{\Omega}_{\lambda_j}G^{\eta}_{j+1}(\Omega)(S^{\Omega}_{\lambda_j})^* &= S^{\Omega}_{\lambda_j}[-\Delta_{\Omega}^{\eta,\bar{\mu}_k} + a_{j+1}(L^{j+1}\eta)^{-2}Q^*_{\Omega,j+1}Q_{\Omega,j+1}]^{-1}(S^{\Omega}_{\lambda_j})^* \\ &= [-\lambda_j^2\Delta_{\lambda_j\Omega}^{\lambda_j\eta,\lambda_j^{-2}\bar{\mu}_k} + a_{j+1}(L^{j+1}\eta)^{-2}Q^*_{\lambda_j\Omega,j+1}Q_{\lambda_j\Omega,j+1}]^{-1} \\ &= \lambda_j^{-2}[-\Delta_{\lambda_j\Omega}^{\lambda_j\eta,\lambda_j^{-2}\bar{\mu}_k} + \frac{a_{j+1}}{L^2}Q^*_{\lambda_j\Omega,j+1}Q_{\lambda_j\Omega,j+1}]^{-1}. \end{aligned} \tag{2.68}$$

This concludes the proof. □

Now we note the following corollary of Lemma 2.12. A variant of estimate (2.69) is stated without detailed justification in [Di13, formula (364)].

Lemma 2.15. *Let $f, f' \in \mathcal{L}^2(\Omega)$ and $\tilde{\triangle}_y, \tilde{\triangle}_{y'}$ unit boxes in $L^{-1}\Omega$. For* $\mathrm{supp}(f_{L^{-1}}) \subset \tilde{\triangle}_y$ *and* $\mathrm{supp}(f'_{L^{-1}}) \subset \tilde{\triangle}_{y'}$ *we have*

$$|\langle f, \tilde{G}_{k+1}(\Omega)f'\rangle| \le cL^2\mathrm{e}^{-c_1|y-y'|}\|f\|_2\|f'\|_2 \tag{2.69}$$

for some numerical constants c and $c_1 > 0$. These constants are, in particular, independent of the size of Ω.

Proof. By Lemma 2.13, we can write $\tilde{G}_{j+1}(\lambda\Omega)$ as

$$\begin{aligned} \tilde{G}_{j+1}(\lambda_j\Omega) &= (\lambda_j/\lambda_{j+1})^2 S^{\Omega}_{\lambda_j}(S^{\Omega}_{\lambda_{j+1}})^*G_{j+1}(\lambda_{j+1}\Omega)S^{\Omega}_{\lambda_{j+1}}(S^{\Omega}_{\lambda_j})^* \\ &= L^2(S^{\lambda_j\Omega}_{L^{-1}})^*G_{j+1}(\lambda_{j+1}\Omega)S^{\lambda_j\Omega}_{L^{-1}}, \end{aligned} \tag{2.70}$$

where $\lambda_j := L^{k-j}$. By setting $j = k$, we get

$$\tilde{G}_{k+1}(\Omega) = L^2(S^{\Omega}_{L^{-1}})^*G_{k+1}(L^{-1}\Omega)S^{\Omega}_{L^{-1}}. \tag{2.71}$$

Thus we can write

$$\begin{aligned} |\langle f, \tilde{G}_{k+1}(\Omega)f'\rangle| &= L^2|\langle (S^{\Omega}_{L^{-1}})f, G_{k+1}(L^{-1}\Omega)S^{\Omega}_{L^{-1}}f'\rangle| \\ &= L^{d+2}|\langle f_{L^{-1}}, G_{k+1}(L^{-1}\Omega)f'_{L^{-1}}\rangle| \\ &\le cL^{d+2}\mathrm{e}^{-c_1|y-y'|}\|f_{L^{-1}}\|_2\|f'_{L^{-1}}\|_2 \\ &\le cL^2\mathrm{e}^{-c_1|y-y'|}\|f\|_2\|f'\|_2, \end{aligned} \tag{2.72}$$

where in the third step we applied Theorem 2.12 with $(\Omega, \eta, k, \triangle_y)$ replaced with $(L^{-1}\Omega, L^{-1}\eta, k+1, \tilde{\triangle}_y)$. This is legitimate, because $(L^{-1}\eta) = L^{-(k+1)}$, $\mathrm{supp}(f_{L^{-1}}) \subset \tilde{\triangle}_y$ and constants in Theorem 2.12 are independent of the lattice spacing and size of the lattice. In the last step we also exploited

$$\|f_{L^{-1}}\|^2_{2,L^{-1}\Omega} = L^{(-1-k)d}\sum_{x\in L^{-1}\Omega}|f(Lx)|^2 = L^{-d}\|f\|^2_{2,\Omega}, \tag{2.73}$$

which concludes the proof. □

2.5 Renormalization group formula

In this section we derive a key formula, stated in (2.86) below, which allows us to conclude exponential decay of the kernels of Green functions. We follow the discussion from [Ba82], where, however, the actual proof is left to the reader. The proof below is adapted from [Di04, Section 1.2.3]. As before, we work on a finite lattice Ω with spacing $\eta = L^{-k}$ and Neumann boundary conditions and denote by Ω_j the coarse lattices.

We define the renormalization transformation which maps a measurable function[4] $\rho_0 : \mathcal{L}^2_{\mathbb{R}}(\Omega) \to \mathbb{C}$ to a new function $\rho_j : \mathcal{L}^2_{\mathbb{R}}(\Omega_j) \to \mathbb{C}$. We set ρ_0 unspecified, only require that $\int d\phi\, |\rho_0(\phi)| := \int \prod_{x\in\Omega} d\phi(x) |\rho_0(\phi(\{x\}_{x\in\Omega})|$ is finite. We set for $j = 1, 2, 3, \dots$

$$T^{\eta}_{a_j,L^j}[\Omega, \rho_0](\psi) := \rho_j(\psi) = \left(\frac{b^{\eta}_j}{2\pi}\right)^{\frac{|\Omega_j|}{2}} \int d\phi\, \mathrm{e}^{-\frac{1}{2}b^{\eta}_j \sum_{y\in\Omega_j} |\psi(y)-(Q_{\Omega,j}\phi)(y)|^2} \rho_0(\phi), \tag{2.74}$$

where $b^{\eta}_j := a_j(L^j\eta)^{d-2}$. (We note that for $j = 0$ the r.h.s. of (2.74) is undefined, in particular it does not reproduce ρ_0). In the notation T^{η}_{a,L^j} the superscript η indicates the lattice spacing of Ω, the subscript L^j defines the size of the averaging box and we set for some $a > 0$

$$a_1 = a, \quad a_{j+1} = \frac{a a_j}{aL^{-2} + a_j} \quad \Rightarrow \quad a_j = a\frac{1 - L^{-2}}{1 - L^{-2j}}. \tag{2.75}$$

The normalization constant $\big(b^{\eta}_j/(2\pi)\big)^{\frac{|\Omega_j|}{2}}$ ensures that

$$\int d\psi \rho_j(\psi) = \int d\phi\, \rho_0(\phi). \tag{2.76}$$

We note the following:

Lemma 2.16. *We have* $T^{L^j\eta}_{a,L} T^{\eta}_{a_j,L^j} = T^{\eta}_{a_{j+1},L^{j+1}}$ *and similarly*

$$T^{L^{j-1}\eta}_{a,L} \dots T^{L\eta}_{a,L} T^{\eta}_{a,L} = T^{\eta}_{a_j,L^j}. \tag{2.77}$$

Proof. See Appendix B. □

Next, we choose ρ_0 as a function whose integral gives the generating functional of the free field theory:

$$\rho^J_0(\phi) := \mathrm{e}^{-\frac{1}{2}\langle\phi,\Delta^{(0),\eta}(\Omega)\phi\rangle} \mathrm{e}^{\mathrm{i}\langle J,\phi\rangle}, \quad \Delta^{(0),\eta}(\Omega) = -\Delta^{\eta}_{\Omega} + \bar{\mu}_k, \tag{2.78}$$

where $J \in \mathcal{L}^2_{\mathbb{R}}(\Omega)$. Here we set $\bar{\mu}_k > 0$ to ensure that the integral of ρ^J_0 is finite. This assumption will be relaxed to $\bar{\mu}_k \geq 0$ in Theorem 2.18 below.

We define ρ^J_j, $j = 1, 2, 3\dots$, iteratively, using the renormalization transformation

$$\rho^J_{j+1}(\psi) := T^{L^j\eta}_{a,L}[\Omega_j, \rho^J_j](\psi) = T^{\eta}_{a_{j+1},L^{j+1}}[\Omega, \rho^J_0](\psi), \tag{2.79}$$

where in the second step we made use of (2.77).

Lemma 2.17. *The following identity holds for* $j = 1, 2, 3\dots$

$$\rho^J_j(\psi) = Z^{\eta}_j(\Omega)\mathrm{e}^{-\frac{1}{2}\langle\psi,\Delta^{(j),L^j\eta}(\Omega)\psi\rangle} \mathrm{e}^{\mathrm{i}\langle J,\mathcal{H}^{\eta}_j(\Omega)\psi\rangle} \mathrm{e}^{-\frac{1}{2}\langle J,G^{\eta}_j(\Omega)J\rangle}, \tag{2.80}$$

where

$$\Delta^{(j),L^j\eta}(\Omega) := a_j(L^j\eta)^{-2} - a_j^2(L^j\eta)^{-4} Q_{\Omega,j} G^{\eta}_j(\Omega) Q^*_{\Omega,j}, \tag{2.81}$$

$$G^{\eta}_j(\Omega) := (-\Delta^{\eta}_{\Omega} + \bar{\mu}_k + a_j(L^j\eta)^{-2} Q^*_{\Omega,j} Q_{\Omega,j})^{-1}, \tag{2.82}$$

$$\mathcal{H}^{\eta}_j(\Omega) := \frac{a_j}{(L^j\eta)^2} G^{\eta}_j(\Omega) Q^*_{\Omega,j} \tag{2.83}$$

and $Z^{\eta}_j(\Omega) = \sqrt{(2\pi)^{|\Omega|}\det(G^{\eta}_j(\Omega))}$ *is determined by (2.76).*

[4] We identify here $\mathcal{L}^2_{\mathbb{R}}(\Omega)$ with $\mathbb{R}^{|\Omega|}$ and assume that ρ_0 is Borel measurable.

Proof. See Appendix C. □

Finally, we define the following map on $\mathcal{L}^2(\Omega_j)$

$$C^{(j),L^j\eta}(\Omega) := (\Delta^{(j),L^j\eta}(\Omega) + \frac{a}{(L^{j+1}\eta)^2} Q^*_{\Omega_j} Q_{\Omega_j})^{-1}. \tag{2.84}$$

It governs the step from ρ^J_j to ρ^J_{j+1} as we will see in the proof of the following theorem.

Theorem 2.18. *The following relations hold for $\bar{\mu}_k \geq 0$*

$$G^\eta_{j+1}(\Omega) = a_j^2 (L^j\eta)^{-4} G^\eta_j(\Omega) Q^*_{\Omega,j} C^{(j),L^j\eta}(\Omega) Q_{\Omega,j} G^\eta_j(\Omega) + G^\eta_j(\Omega), \tag{2.85}$$

$$G^\eta_k(\Omega) = \sum_{j=1}^{k-1} a_j^2 (L^j\eta)^{-4} G^\eta_j(\Omega) Q^*_{\Omega,j} C^{(j),L^j\eta}(\Omega) Q_{\Omega,j} G^\eta_j(\Omega) + G^\eta_1(\Omega), \tag{2.86}$$

and the sum in (2.86) should be skipped for $k = 1$.

Proof. We note that (2.86) follows by iteration from (2.85), so it suffices to prove the latter equation. By Lemma 2.17 we can write

$$\rho^J_{j+1}(\psi) = Z^\eta_{j+1}(\Omega) \mathrm{e}^{-\frac{1}{2}\langle \psi, \Delta^{(j+1),L^{j+1}\eta}(\Omega)\psi\rangle} \mathrm{e}^{\mathrm{i}\langle J, \mathcal{H}^\eta_{j+1}(\Omega)\psi\rangle} \mathrm{e}^{-\frac{1}{2}\langle J, G^\eta_{j+1}(\Omega) J\rangle}. \tag{2.87}$$

On the other hand, Lemma 2.16 gives

$$\rho^J_{j+1}(\psi) = T^{L^j\eta}_{a,L}[\Omega_j, \rho^J_j](\psi). \tag{2.88}$$

Thus using (2.88), (2.80) and definition (2.74), we obtain

$$\begin{aligned} \rho^J_{j+1}(\psi) &= \left(\frac{b_1^{L^j\eta}}{2\pi}\right)^{\frac{|\Omega_{j+1}|}{2}} \int d\tilde{\psi}\, \mathrm{e}^{-\frac{1}{2} b_1^{L^j\eta} \sum_{y\in\Omega_{j+1}} |\psi(y) - (Q_{\Omega_j}\tilde{\psi})(y)|^2} \rho^J_j(\tilde{\psi}) \\ &= Z_j \mathrm{e}^{-\frac{1}{2}\langle J, G^\eta_j(\Omega) J\rangle} \int d\tilde{\psi}\, \mathrm{e}^{-\frac{1}{2} b_1^{L^j\eta} \sum_{y\in\Omega_{j+1}} |\psi(y) - (Q_{\Omega_j}\tilde{\psi})(y)|^2} \mathrm{e}^{-\frac{1}{2}\langle\tilde{\psi}, \Delta^{(j),L^j\eta}(\Omega)\tilde{\psi}\rangle} \mathrm{e}^{\mathrm{i}\langle J, \mathcal{H}^\eta_j(\Omega)\tilde{\psi}\rangle}, \end{aligned} \tag{2.89}$$

where $Z_j := \big(\frac{b_1^{L^j\eta}}{2\pi}\big)^{\frac{|\Omega_{j+1}|}{2}} Z^\eta_j(\Omega)$ is an inessential constant. Let us now diagonalize the quadratic form in the exponent above. We define the function

$$F(\tilde{\psi}) := \frac{1}{2} b_1^{L^j\eta} \sum_{y\in\Omega_{j+1}} |\psi(y) - (Q_{\Omega_j}\tilde{\psi})(y)|^2 + \frac{1}{2}\langle\tilde{\psi}, \Delta^{(j),L^j\eta}(\Omega)\tilde{\psi}\rangle \tag{2.90}$$

and compute its derivatives as in (B.7)

$$\begin{aligned} \frac{\partial}{\partial\tilde{\psi}(\tilde{y})} F(\tilde{\psi}) &= -\frac{b_1^{L^j\eta}}{L^d}\big(\psi(y_{\tilde{y}}) - (Q_{\Omega_j}\tilde{\psi})(y_{\tilde{y}})\big) + (L^j\eta)^d (\Delta^{(j),L^j\eta}(\Omega)\tilde{\psi})(\tilde{y}) \\ &= -\frac{b_1^{L^j\eta}}{L^d}\big((Q^*_{\Omega_j}\psi)(\tilde{y}) - (Q^*_{\Omega_j} Q_{\Omega_j}\tilde{\psi})(\tilde{y})\big) + (L^j\eta)^d (\Delta^{(j),L^j\eta}(\Omega)\tilde{\psi})(\tilde{y}), \end{aligned} \tag{2.91}$$

$$\begin{aligned} \frac{\partial}{\partial\tilde{\psi}(\tilde{y}')}\frac{\partial}{\partial\tilde{\psi}(\tilde{y})} F(\tilde{\psi}) &= [\frac{b_1^{L^j\eta}}{L^d}(L^j\eta)^d Q^*_{\Omega_j} Q_{\Omega_j} + (L^j\eta)^{2d} \Delta^{(j),L^j\eta}(\Omega)](\tilde{y}', \tilde{y}) \\ &= (L^j\eta)^{2d} (C^{(j),L^j\eta}(\Omega))^{-1}(\tilde{y}', \tilde{y}). \end{aligned} \tag{2.92}$$

We compute $\tilde{\psi}_0$ at which the first derivative (2.91) vanishes:

$$-\frac{a}{(L^{j+1}\eta)^2}\big(Q^*_{\Omega_j}\psi - Q^*_{\Omega_j} Q_{\Omega_j}\tilde{\psi}_0\big) + \Delta^{(j),L^j\eta}(\Omega)\tilde{\psi}_0 = 0 \quad \Rightarrow \quad \tilde{\psi}_0 = \frac{a}{(L^{j+1}\eta)^2} C^{(j),L^j\eta}(\Omega) Q^*_{\Omega_j}\psi. \tag{2.93}$$

Altogether, we can write for $\tilde{\psi} = \tilde{\psi}_0 + \tilde{\psi}_1$

$$F(\tilde{\psi}) = F(\tilde{\psi}_0) + \frac{1}{2}\langle \tilde{\psi}_1, C^{(j),L^j\eta}(\Omega)^{-1}\tilde{\psi}_1\rangle, \tag{2.94}$$

where the part linear in $\tilde{\psi}_1$ vanishes as $F(\tilde{\psi}_0)$ is a minimum. By substituting this to (2.89), we obtain, referring to (D.1),

$$\begin{aligned}\rho^J_{j+1}(\psi) &= Z_j \mathrm{e}^{-\frac{1}{2}\langle J, G^\eta_j(\Omega)J\rangle} \mathrm{e}^{-F(\tilde{\psi}_0)} \mathrm{e}^{\mathrm{i}\langle (\mathcal{H}^\eta_j(\Omega))^* J, \tilde{\psi}_0\rangle} \int d\tilde{\psi}_1\, \mathrm{e}^{-\frac{1}{2}\langle \tilde{\psi}_1, C^{(j),L^j\eta}(\Omega)^{-1}\tilde{\psi}_1\rangle} \mathrm{e}^{\mathrm{i}\langle (\mathcal{H}^\eta_j(\Omega))^* J, \tilde{\psi}_1\rangle} \\ &= Z'_j(\psi) \mathrm{e}^{\mathrm{i}\langle (\mathcal{H}^\eta_j(\Omega))^* J, \tilde{\psi}_0\rangle} \mathrm{e}^{-\frac{1}{2}\langle J, G^\eta_j(\Omega)J\rangle} \mathrm{e}^{-\frac{1}{2}\langle J, \mathcal{H}^\eta_j(\Omega) C^{(j),L^j\eta}(\Omega)(\mathcal{H}^\eta_j(\Omega))^* J\rangle},\end{aligned} \tag{2.95}$$

where $Z'_j(\psi)$ is independent of J. By comparing the expressions quadratic in J in (2.95) and (2.87), we arrive at (2.85) using the following fact: Suppose that j is the complex conjugation on $\mathcal{L}^2(\Omega)$ and A a bounded operator on $\mathcal{L}^2(\Omega)$ s.t. $\mathrm{j}A\mathrm{j} = A$ and $\langle J, AJ\rangle = 0$ for all $J \in \mathcal{L}^2_{\mathbb{R}}(\Omega)$. Then $A = 0$.

To check that this relation is valid also for $\bar{\mu}_k = 0$, we note that

$$G^{\eta,\bar{\mu}_k}_j - G^{\eta,\bar{\mu}'_k}_j = G^{\eta,\bar{\mu}_k}_j G^{\eta,\bar{\mu}'_k}_j (\bar{\mu}'_k - \bar{\mu}_k) \tag{2.96}$$

is a Cauchy sequence in the operator norm by Lemma 2.10, thus $G^{\eta,\bar{\mu}_k=0}_j = \lim_{\bar{\mu}'_k\to 0} G^{\eta,\bar{\mu}'_k}_j$ exists in norm. □

Let us recall definitions (2.81), (2.84) of the following mappings on $\mathcal{L}^2(\Omega_j)$:

$$\begin{aligned}\Delta^{(j),L^j\eta}(\Omega) &:= a_j(L^j\eta)^{-2} - a_j^2(L^j\eta)^{-4} Q_{\Omega,j} G^\eta_j(\Omega) Q^*_{\Omega,j}, \\ C^{(j),L^j\eta}(\Omega) &:= (\Delta^{(j),L^j\eta}(\Omega) + \frac{a(L^j\eta)^{-2}}{L^2} Q^*_{\Omega_j} Q_{\Omega_j})^{-1}.\end{aligned}$$

We have the following scaling properties:

Lemma 2.19. *For $\eta = L^{-k}$, $\lambda_j = L^{k-j}$ the following equalities hold,*

$$\Delta_j(\lambda_j\Omega) := \lambda_j^{-2} S^{\Omega_j}_{\lambda_j} \Delta^{(j),L^j\eta}(\Omega)(S^{\Omega_j}_{\lambda_j})^* = (a_j - a_j^2 Q_{\lambda_j\Omega,j} G_j(\lambda_j\Omega) Q^*_{\lambda_j\Omega,j}), \tag{2.97}$$

$$C_j(\lambda_j\Omega) := \lambda_j^2 S^{\Omega_j}_{\lambda_j} C^{(j),L^j\eta}(\Omega)(S^{\Omega_j}_{\lambda_j})^* = [\Delta_j(\lambda_j\Omega) + \frac{a}{L^2} Q^*_{\lambda_j\Omega_j} Q_{\lambda_j\Omega_j}]^{-1}, \tag{2.98}$$

where the maps $\Delta_j(\lambda_j\Omega), C_j(\lambda_j\Omega)$ act on $\mathcal{L}^2(\lambda_j\Omega_j)$.

Proof. We recall from (2.65) and (2.46) that

$$Q_{\lambda_j\Omega,j'} S^{\Omega}_{\lambda_j} = S^{\Omega_{j'}}_{\lambda_j} Q_{\Omega,j'}, \tag{2.99}$$

$$S^{\Omega}_{\lambda_j} G^\eta_j(\Omega)(S^{\Omega}_{\lambda_j})^* = \lambda_j^{-2} G_j(\lambda_j\Omega). \tag{2.100}$$

Hence

$$\begin{aligned}S^{\Omega_j}_{\lambda_j} \Delta^{(j),L^j\eta}(\Omega)(S^{\Omega_j}_{\lambda_j})^* &= (L^j\eta)^{-2}\big(a_j - a_j^2(L^j\eta)^{-2} S^{\Omega_j}_{\lambda_j} Q_{\Omega,j} G^\eta_j(\Omega) Q^*_{\Omega,j}(S^{\Omega_j}_{\lambda_j})^*\big) \\ &= (L^j\eta)^{-2}\big(a_j - a_j^2(L^j\eta)^{-2} Q_{\lambda_j\Omega,j} S^{\Omega}_{\lambda_j} G^\eta_j(\Omega)(S^{\Omega}_{\lambda_j})^* Q^*_{\lambda_j\Omega,j}\big) \\ &= (L^j\eta)^{-2}\big(a_j - a_j^2(L^j\eta)^{-2}\lambda_j^{-2} Q_{\lambda_j\Omega,j} G_j(\lambda_j\Omega) Q^*_{\lambda_j\Omega,j}\big) \\ &= \lambda_j^2\big(a_j - a_j^2 Q_{\lambda_j\Omega,j} G_j(\lambda_j\Omega) Q^*_{\lambda_j\Omega,j}\big).\end{aligned} \tag{2.101}$$

Now we compute

$$\begin{aligned}\lambda_j^2 S^{\Omega_j}_{\lambda_j} C^{(j),L^j\eta}(\Omega)(S^{\Omega_j}_{\lambda_j})^* &= \lambda_j^2 S^{\Omega_j}_{\lambda_j}(\Delta^{(j),L^j\eta}(\Omega) + \frac{a(L^j\eta)^{-2}}{L^2} Q^*_{\Omega_j} Q_{\Omega_j})^{-1}(S^{\Omega_j}_{\lambda_j})^* \\ &= (\Delta_j(\lambda_j\Omega) + \frac{a}{L^2} Q^*_{\lambda_j\Omega_j} Q_{\lambda_j\Omega_j})^{-1},\end{aligned} \tag{2.102}$$

where we used that by (2.99) and $(S_\lambda^O)^* = S_\lambda^{\lambda^{-1}O}$

$$Q_{\Omega_j,1}(S_{\lambda_j}^{\Omega_j})^* = Q_{\lambda_j^{-1}\lambda_j\Omega_j,1}S_{\lambda_j^{-1}}^{\lambda_j\Omega_j} = S_{\lambda_j^{-1}}^{\lambda_j\Omega_{j+1}}Q_{\lambda_j\Omega_j,1}. \tag{2.103}$$

This concludes the proof. □

We define the following objects:

$$\mathcal{H}_j(\Omega) := a_j G_j(\Omega)Q_{\Omega,j}^*, \quad C_j'(\Omega) := \mathcal{H}_j(\Omega)C_j(\Omega)\mathcal{H}_j(\Omega)^* \tag{2.104}$$

and prove a rescaled variant of the renormalization group formula (2.86). Relation (2.105) is stated without proof in [Di13, formula (373)], with unspecified $j = 0$ term. A proof is sketched in [Di04].

Theorem 2.20. *For $\eta = L^{-k}$ and $\lambda_j = L^{k-j}$, we have the following relation*

$$(G_k(\Omega)f)(x) = \sum_{j=1}^{k-1} \lambda_j^{-2}\big(C_j'(\lambda_j\Omega)f_{\lambda_j}\big)(\lambda_j x) + \lambda_1^{-2}(G_1(\lambda_1\Omega)f_{\lambda_1})(\lambda_1 x). \tag{2.105}$$

Remark 2.21. *If we tried to estimate the kernel of $G_k(\Omega)$ by substituting $f = \delta_{x'}^\eta$ the first term on the r.h.s. of (2.105) would stay regular, while the last term would acquire η^{-d}.*

Proof. As a preparation, we note that, by taking adjoints in (2.46), using $(S_\lambda^O)^* = S_{\lambda^{-1}}^{\lambda O}$ and rescaling using Lemma 2.7, we get

$$Q_{\lambda\Omega,j}S_\lambda^\Omega = S_\lambda^{\Omega_j}Q_{\Omega,j} \quad \Rightarrow \quad S_\lambda^\Omega Q_{\Omega,j}^* = Q_{\lambda\Omega,j}^* S_\lambda^{\Omega_j}. \tag{2.106}$$

We also recall the defining relations (2.65), (2.98)

$$G_j^\eta(\Omega) := \lambda_j^{-2}(S_{\lambda_j}^\Omega)^* G_j(\lambda_j\Omega)S_{\lambda_j}^\Omega, \tag{2.107}$$

$$C_j(\lambda_j\Omega) := \lambda_j^2 S_\lambda^{\Omega_j} C^{(j),L^j\eta}(\Omega)(S_\lambda^{\Omega_j})^*. \tag{2.108}$$

Now we restate formula (2.86)

$$G_k^\eta(\Omega) = \sum_{j=1}^{k-1} a_j^2\lambda_j^4 G_j^\eta(\Omega)Q_{\Omega,j}^* C^{(j),L^j\eta}(\Omega)Q_{\Omega,j}G_j^\eta(\Omega) + G_1^\eta(\Omega) \tag{2.109}$$

and note that $G_k^\eta(\Omega) = G_k(\Omega)$. Thus, by (2.107), (2.108), (2.106),

$$\begin{aligned} G_k(\Omega) &= \sum_{j=1}^{k-1} a_j^2(S_{\lambda_j}^\Omega)^* G_j(\lambda_j\Omega)S_{\lambda_j}^\Omega Q_{\Omega,j}^* C^{(j),L^j\eta}(\Omega)Q_{\Omega,j}(S_{\lambda_j}^\Omega)^* G_j(\lambda_j\Omega)S_{\lambda_j}^\Omega + G_1^\eta(\Omega) &(2.110)\\ &= \sum_{j=1}^{k-1} a_j^2(S_{\lambda_j}^\Omega)^* G_j(\lambda_j\Omega)Q_{\lambda_j\Omega,j}^* S_{\lambda_j}^{\Omega_j} C^{(j),L^j\eta}(\Omega)(S_{\lambda_j}^{\Omega_j})^* Q_{\lambda_j\Omega,j}G_j(\lambda_j\Omega)S_{\lambda_j}^\Omega + G_1^\eta(\Omega) &(2.111)\\ &= \sum_{j=1}^{k-1} a_j^2\lambda_j^{-2}(S_{\lambda_j}^\Omega)^* G_j(\lambda_j\Omega)Q_{\lambda_j\Omega,j}^* C_j(\lambda_j\Omega)Q_{\lambda_j\Omega,j}G_j(\lambda_j\Omega)S_{\lambda_j}^\Omega + \lambda_1^{-2}(S_{\lambda_1}^\Omega)^* G_1(\lambda_1\Omega)S_{\lambda_1}^\Omega. &(2.112)\end{aligned}$$

By evaluation on a function we obtain (2.105). □

In the following lemma we use notation $[\ldots]_O$ which was introduced in (2.49).

Lemma 2.22. *Let $\eta = L^{-k}$, $\lambda_j = L^{k-j}$. The following relation holds*

$$C_j(\lambda_j\Omega) = [\lambda_j^2 A_j + \tilde{a}_{j,j}^2 A_j Q_j \tilde{G}_{j+1}(\lambda_j\Omega) Q_j^* A_j]_{\lambda_j\Omega_j}, \tag{2.113}$$

where, $\tilde{a}_{j,\ell} := a_j(L^\ell\eta)^{-2}$ and, as in (D.6),

$$A_{j,\Omega_j} = \frac{1}{\tilde{a}_{j,j}} + \left(\frac{1}{\tilde{a}_{j,j} + \tilde{a}_{1,j}L^{-2}} - \frac{1}{\tilde{a}_{j,j}}\right) Q_{\Omega_j}^* Q_{\Omega_j}, \quad \tilde{a}_{j,\ell} = a_j(L^\ell\eta)^{-2}. \tag{2.114}$$

Proof. We have, according to Lemma 2.19,

$$C_j(\lambda_j\Omega) := \lambda_j^2 S_{\lambda_j}^{\Omega_j} C^{(j),L^j\eta}(\Omega)(S_{\lambda_j}^{\Omega_j})^*. \tag{2.115}$$

Now Lemma D.1 gives

$$C^{(j),L^j\eta}(\Omega) = [A_j + \tilde{a}_{j,j}^2 A_j Q_j G_{j+1}^\eta Q_j^* A_j]_{\Omega_j}. \tag{2.116}$$

Furthermore, in Lemma 2.13 we defined

$$\tilde{G}_{j+1}(\lambda_j\Omega) := \lambda_j^2 S_{\lambda_j}^{\Omega} G_{j+1}^\eta(\Omega)(S_{\lambda_j}^{\Omega})^*. \tag{2.117}$$

Moreover, we have by $(S_\lambda^O)^* = S_{\lambda^{-1}}^{\lambda O}$ and $Q_{\lambda\Omega,j'} S_\lambda^\Omega = S_\lambda^{\Omega_{j'}} Q_{\Omega,j'}$ (cf. (2.46))

$$Q_{\Omega_j,1}(S_{\lambda_j}^{\Omega_j})^* = (S_{\lambda_j}^{\Omega_{j+1}})^* Q_{\lambda_j\Omega_j,1}, \tag{2.118}$$

$$S_{\lambda_j}^{\Omega_j} Q_{\Omega_j,1}^* Q_{\Omega_j,1}(S_{\lambda_j}^{\Omega_j})^* = Q_{\lambda_j\Omega_j,1}^* Q_{\lambda_j\Omega_j,1}, \tag{2.119}$$

$$Q_{\Omega,j}^*(S_{\lambda_j}^{\Omega_j})^* = (S_{\lambda_j}^{\Omega})^* Q_{\lambda_j\Omega,j}^*. \tag{2.120}$$

Now we have all the information to compute (2.115):

$$\begin{aligned} C_j(\lambda_j\Omega) &= \lambda_j^2 S_{\lambda_j}^{\Omega_j}[A_{j,\Omega_j} + \tilde{a}_{j,j}^2 A_{j,\Omega_j} Q_{\Omega,j} G_{j+1}^\eta(\Omega) Q_{\Omega,j}^* A_{j,\Omega_j}](S_{\lambda_j}^{\Omega_j})^* \\ &= \lambda_j^2[A_{j,\lambda_j\Omega_j} + \tilde{a}_{j,j}^2 A_{j,\lambda_j\Omega} Q_{\lambda_j\Omega,j} S_{\lambda_j}^{\Omega} G_{j+1}^\eta(\Omega)(S_{\lambda_j}^{\Omega})^* Q_{\lambda_j\Omega,j}^* A_{j,\lambda_j\Omega_j}] \\ &= [\lambda_j^2 A_{j,\lambda_j\Omega_j} + \tilde{a}_{j,j}^2 A_{j,\lambda_j\Omega} Q_{\lambda_j\Omega,j} \tilde{G}_{j+1}(\lambda_j\Omega) Q_{\lambda_j\Omega,j}^* A_{j,\lambda_j\Omega_j}]. \end{aligned} \tag{2.121}$$

This concludes the proof. □

2.6 Exponential decay of lattice Green functions

In Theorem 2.25 below we give a proof of exponential decay of Green functions. The main ingredients of the proof are Lemmas 2.12, 2.15, formula (2.105) and the estimate

$$|\mathcal{H}_k(\Omega)(x,y)| \le c\mathrm{e}^{-c_1|x-y|}, \quad x \in \Omega, \quad y \in \Omega_k, \quad c_1 > 0, \tag{2.122}$$

on the expression $\mathcal{H}_k(\Omega) := a_k G_k(\Omega) Q_k^*$ which appeared in (2.104). It is the goal of Sections 3 and 4 to prove (2.122).

First, using Lemma 2.15, we obtain:

Lemma 2.23. *The following bound holds for $y, y' \in \Omega_k$*

$$|C_k(\Omega)(y,y')| \le cL^2\mathrm{e}^{-c_1 L^{-1}|y-y'|}, \quad y, y' \in \Omega_k. \tag{2.123}$$

Proof. By evaluating (2.113) for $j = k$, we get

$$C_k(\Omega) = [A_k + \tilde{a}_{k,k}^2 A_k Q_k \tilde{G}_{k+1}(\Omega) Q_k^* A_k]_{\Omega_k} \tag{2.124}$$

and we observe that Ω_k is a unit lattice. Writing $\delta_y := \delta_y^1$, (cf. (2.28) and the line below) we want to compute $\langle \delta_y, C_k(\Omega)\delta_{y'}\rangle$. First, we recall from (D.11)

$$A_k = [\tilde{a}_{k,k} + \frac{\tilde{a}_{1,k}}{L^2} Q^*Q]_{\Omega_k}^{-1} = [\frac{1}{\tilde{a}_{k,k}} - \frac{\tilde{a}_{k+1,k}}{\tilde{a}_{k,k}^2 L^2} Q^*Q]_{\Omega_k} = [\frac{1}{a_k} - \frac{a_{k+1}}{a_k^2 L^2} Q^*Q]_{\Omega_k}, \tag{2.125}$$

since Q^*Q is a projection and $\tilde{a}_{j,\ell} := a_j (L^\ell \eta)^{-2}$, $L^k\eta = 1$. Clearly, the kernel $A_k(y,y')$ vanishes for $|y-y'|_\infty > L$ so we only have to consider the second term in (2.124).

We note that $y \mapsto (A_k\delta_{y'})(y)$ is supported in an L-box $B_{k+1}(z)\cap\Omega_k$ with label z s.t. $|z-y'|_\infty \le L$. Now $g(x) := (Q_k^* A_k \delta_{y'})(x)$ is a function in $\mathcal{L}^2(\Omega)$ supported in $B_{k+1}(z) \subset \Omega$. It has the property that $g_{L^{-1}} \in \mathcal{L}^2(L^{-1}\Omega)$ is supported in $L^{-1}B_{k+1}(z) =: \tilde{\triangle}_{L^{-1}z} \subset L^{-1}\Omega$ and $|L^{-1}z - L^{-1}y'|_\infty \le 1$. Thus Lemma 2.15 and the triangle inequality give

$$\begin{aligned} |\langle \delta_y, A_k Q_k \tilde{G}_{k+1}(\Omega) Q_k^* A_k \delta_{y'}\rangle| &\le cL^2 \mathrm{e}^{-c_1|L^{-1}z - L^{-1}z'|} \|Q_k^* A_k \delta_{y'}\|_{2,\Omega} \|Q_k^* A_k \delta_y\|_{2,\Omega} \\ &\le cL^2 \mathrm{e}^{2c_1} \mathrm{e}^{-c_1 L^{-1}|y-y'|} \|Q_k^* A_k \delta_{y'}\|_{2,\Omega} \|Q_k^* A_k \delta_y\|_{2,\Omega} \\ &\le cL^2 \mathrm{e}^{2c_1} \mathrm{e}^{-c_1 L^{-1}|y-y'|} \|Q_k^* A_k \delta_{y'}\|_{2,\Omega} \|Q_k^* A_k \delta_y\|_{2,\Omega}. \end{aligned} \tag{2.126}$$

Now we estimate the norms on the r.h.s. of (2.126). Let us first consider the $1/a_k$-contribution from A_k in (2.125). Then $Q_k^*\delta_y$ is a characteristic function of a unit block in Ω, hence,

$$\|Q_k^*\delta_y\|_{2,\Omega} = 1. \tag{2.127}$$

Now we consider the Q^*Q contribution from A_k. Then $Q^*Q\delta_y$ is a function of value $1/L^d$ on one L-box $B_{k+1}(z)\cap\Omega_k$ of the unit lattice and zero everywhere else. The action of Q_k^* creates a block-constant function on Ω. We have

$$\|Q_k^* Q^* Q \delta_y\|_{2,\Omega} = \|Q\delta_y\|_{2,\Omega_{k+1}} = L^{-d/2} \le 1. \tag{2.128}$$

The last inequality also follows from the fact that $\|Q\| = \|Q^*\| = 1$, cf. Remark 2.8. This concludes the proof. □

Now we recall from (2.104) that $C_k'(\Omega) := \mathcal{H}_k(\Omega) C_k(\Omega) \mathcal{H}_k^*(\Omega)$ and prove the following lemma.

Lemma 2.24. *For* $\gamma_0 := \frac{1}{4}c_1 L^{-1}$, $c_1 > 0$,

$$|(C_k'(\Omega)f)(x)| \le cL^{2d+2} \mathrm{e}^{-\gamma_0 \mathrm{d}(x,\mathrm{supp}(f))} \|f\|_\infty, \tag{2.129}$$

where $\mathrm{d}(x,\mathrm{supp}(f)) := \inf_{y\in\mathrm{supp}(f)} |x-y|$.

Proof. We have

$$(C_k'(\Omega)f)(x) = \sum_{y,y'\in\Omega_k} \mathcal{H}_k(\Omega)(x,y) C_k(\Omega)(y,y') (\mathcal{H}_k^*(\Omega)f)(y'). \tag{2.130}$$

By (2.122) we have $\left|\mathcal{H}_k(\Omega)(x,y)\right| \le c\mathrm{e}^{-c_1|x-y|}$. Hence

$$\begin{aligned} |(\mathcal{H}_k^*(\Omega)f)(y')| \le \eta^d \sum_{x\in\Omega} |\langle \mathcal{H}_k(\Omega)\delta_{y'}, \delta_x^\eta\rangle f(x)| &\le c\eta^d \|f\|_\infty \mathrm{e}^{-\frac{c_1}{2}\mathrm{d}(y',\mathrm{supp}(f))} \sum_{z\in\mathbb{Z}^d} \mathrm{e}^{-\eta\frac{c_1}{2}|z|} \\ &\le c'\|f\|_\infty \mathrm{e}^{-\frac{c_1}{2}\mathrm{d}(y',\mathrm{supp}(f))}. \end{aligned} \tag{2.131}$$

Recalling from (2.123) that $|C_k(\Omega)(y,y')| \le cL^2 \mathrm{e}^{-c_1 L^{-1}\mathrm{d}(y,y')}$, we obtain

$$|(C_k'(\Omega)f)(x)| \le cL^2 \sum_{y,y'\in\Omega_k} \mathrm{e}^{-c_1\mathrm{d}(x,y)} \mathrm{e}^{-c_1 L^{-1}\mathrm{d}(y,y')} \mathrm{e}^{-\frac{c_1}{2}\mathrm{d}(y',\mathrm{supp}(f))} \|f\|_\infty. \tag{2.132}$$

By the triangle inequality

$$\begin{aligned} \frac{1}{4}c_1 L^{-1}\mathrm{d}(x,\mathrm{supp}(f)) &\le \frac{1}{4}c_1 L^{-1}[\mathrm{d}(x,y)+\mathrm{d}(y,y')+\mathrm{d}(y',\mathrm{supp}(f))] \\ &\le \frac{1}{2}[c_1\mathrm{d}(x,y)+c_1L^{-1}\mathrm{d}(y,y')+\frac{1}{2}c_1\mathrm{d}(y',\mathrm{supp}(f))]. \end{aligned} \tag{2.133}$$

Thus we obtain

$$\begin{aligned} |(C_k'(\Omega)f)(x)| &\le cL^2\mathrm{e}^{-\frac{1}{4}c_1L^{-2}\mathrm{d}(x,\mathrm{supp}(f))} \sum_{y,y'\in\mathbb{Z}^d} \mathrm{e}^{-\frac{1}{2}c_1\mathrm{d}(x,y)}\mathrm{e}^{-\frac{1}{2}c_1L^{-2}\mathrm{d}(y,y')}\|f\|_\infty \\ &\le cL^{2d+2}\mathrm{e}^{-\gamma_0\mathrm{d}(x,\mathrm{supp}(f))}\|f\|_\infty, \end{aligned} \tag{2.134}$$

where the L-dependence is determined by summing up the geometric series:

$$\sum_{y'\in\mathbb{Z}^d} \mathrm{e}^{-\frac{1}{2}c_1L^{-2}|y'|} \le c\left(\frac{1}{1-\mathrm{e}^{-\frac{1}{2}c_1L^{-2}}}\right)^d \le cL^{2d}. \tag{2.135}$$

This completes the proof. □

Theorem 2.25. *There is $c_1' > 0$ s.t. for all $f \in \mathcal{L}^\infty(\Omega)$,*

$$|(G_k(\Omega)f)(x)| \le cL^{2d}\mathrm{e}^{-c_1'\mathrm{d}(x,\mathrm{supp}(f))}\|f\|_\infty. \tag{2.136}$$

Proof. The proof is based on formula (2.105), which has the form

$$(G_k(\Omega)f)(x) = \sum_{j=1}^{k-1} \lambda_j^{-2}\big(C_j'(\lambda_j\Omega)f_{\lambda_j}\big)(\lambda_j x) + \lambda_1^{-2}(G_1(\lambda_1\Omega)f_{\lambda_1})(\lambda_1 x), \tag{2.137}$$

where $\lambda_j := L^{k-j}$. . Regarding the last term on the r.h.s. of (2.137), we apply Lemma 2.12:

$$\begin{aligned} |\lambda_1^{-2}(G_1(\lambda_1\Omega)f_{\lambda_1})(\lambda_1 x)| &= \lambda_1^{-2}|\sum_{y'\in(\lambda_1\Omega)_1} \langle \delta_{\lambda_1 x}^{L^{-1}}, G_1(\lambda_1\Omega)\mathbb{1}_{\triangle_{y'}}f_{\lambda_1}\rangle| \\ &\le \lambda_1^{-2}\sum_{y'\in(\lambda_1\Omega)_1} \mathrm{e}^{-c_1|y-y'|}\|\delta_{\lambda_1 x}^{L^{-1}}\|_{2,\lambda_1\Omega}\|\mathbb{1}_{\triangle_{y'}}f_{\lambda_1}\|_{2,\lambda_1\Omega}. \end{aligned} \tag{2.138}$$

We note that $\delta_{\lambda_1 x}^{L^{-1}}$ is supported in $\triangle_y$ s.t. $|y-\lambda_1 x|_\infty \le 1$. Furthermore, if $\mathbb{1}_{\triangle_{y'}}f_{\lambda_1} \ne 0$ then $|y'-\mathrm{supp}(f_{\lambda_1})|_\infty \le 1$. Consequently, by the inverse triangle inequality,

$$|y-y'| \ge |\lambda_1 x - \mathrm{supp}(f_{\lambda_1})| - c = \lambda_1|x-\mathrm{supp}(f)| - c. \tag{2.139}$$

Now we estimate the norms on the r.h.s. of (2.138):

$$\|\delta_{\lambda_1 x}^{L^{-1}}\|_{2,\lambda_1\Omega}^2 = L^{-d}\sum_{x'\in\lambda_1\Omega} \frac{1}{L^{-2d}}\delta_{\lambda_1 x,x'} = L^d, \tag{2.140}$$

$$\begin{aligned} \|\mathbb{1}_{\triangle_{y'}}f_{\lambda_1}\|_{2,\lambda_1\Omega}^2 &= L^{-d}\sum_{x'\in\lambda_1\Omega}\mathbb{1}_{\triangle_{y'}}(x')|f_{\lambda_1}(x')|^2 \\ &= L^{-d}\sum_{x''\in\Omega}\mathbb{1}_{\lambda_1^{-1}\triangle_{y'}}(x'')|f(x'')|^2 \\ &\le L^{-d}\|f\|_\infty^2\,\#(\lambda_1^{-1}\triangle_{y'}) \le c\|f\|_\infty^2, \end{aligned} \tag{2.141}$$

where we noticed that if $x'' \in \lambda_1^{-1}\triangle_{y'}$ then $|\lambda_1 x'' - y'|_\infty \leq 1$, i.e., $|x'' - \lambda_1^{-1}y'|_\infty \leq \lambda_1^{-1}$. The latter set of $x'' \in \Omega$ contains $(2L+1)^d \leq cL^d$ elements. (Here we used that $\lambda_1^{-1} = \eta L$ and the ball in supremum metric is a cube). Making use of (2.138), (2.139), (2.140), (2.141) we get

$$\begin{aligned} |\lambda_1^{-2}(G_1(\lambda_1\Omega)f_{\lambda_1})(\lambda_1 x)| &\leq c\lambda_1^{-2}\mathrm{e}^{-\frac{1}{2}c_1\lambda_1|x-\mathrm{supp}(f)|}\sum_{y'\in L^{-1}\mathbb{Z}^d}\mathrm{e}^{-\frac{1}{2}c_1|y-y'|}L^{d/2}\|f\|_\infty \\ &\leq c\lambda_1^{-2}L^{d/2}\|f\|_\infty\mathrm{e}^{-\frac{1}{2}c_1\lambda_1|x-\mathrm{supp}(f)|}\sum_{y''\in\mathbb{Z}^d}\mathrm{e}^{-\frac{1}{2}c_1L^{-1}|y''|} \\ &\leq c\lambda_1^{-2}L^{3d/2}\mathrm{e}^{-\frac{1}{2}c_1\lambda_1|x-\mathrm{supp}(f)|}\|f\|_\infty \qquad (2.142) \\ &\leq cL^{3d/2-2k+2}\mathrm{e}^{-\frac{1}{2}c_1|x-\mathrm{supp}(f)|}\|f\|_\infty, \qquad (2.143) \end{aligned}$$

where in (2.142) we argued as in (2.135). We also note that $L^{-2k+2} \leq 1$ since $k \in \mathbb{N}$.

Now we consider the sum in (2.137). Recall from Lemma 2.24 that

$$|(C_k'(\Omega)f)(x)| \leq cL^{2d+2}\mathrm{e}^{-\gamma_0 \mathrm{d}(x,\mathrm{supp}(f))}\|f\|_\infty. \qquad (2.144)$$

Hence,

$$\begin{aligned} |(C_j'(\lambda_j\Omega)f_{\lambda_j})(\lambda_j x)| \leq cL^{2d+2}\mathrm{e}^{-\gamma_0\mathrm{d}(\lambda_j x,\,\mathrm{supp}(f_{\lambda_j}))}\|f\|_\infty &\leq cL^{2d+2}\mathrm{e}^{-\gamma_0\lambda_j\mathrm{d}(x,\,\mathrm{supp}(f))}\|f\|_\infty \\ &\leq cL^{2d+2}\mathrm{e}^{-\gamma_0 L\mathrm{d}(x,\,\mathrm{supp}(f))}\|f\|_\infty, \qquad (2.145) \end{aligned}$$

where we used that

$$\mathrm{d}(\lambda_j x,\,\mathrm{supp}(f_{\lambda_j})) = \inf_{y\in\mathrm{supp}(f_{\lambda_j})}|\lambda_j x - y| = \inf_{y'\in\mathrm{supp}(f)}\lambda_j|x-y'| = \lambda_j\mathrm{d}(x,\mathrm{supp}(f)). \qquad (2.146)$$

Thus we obtain

$$\Big|\sum_{j=1}^{k-1}\lambda_j^{-2}\big(C_j'(\lambda_j\Omega)f_{\lambda_j}\big)(\lambda_j x)\Big| \leq cL^{2d}\mathrm{e}^{-\gamma_0 L\mathrm{d}(x,\,\mathrm{supp}(f))}\|f\|_\infty, \qquad (2.147)$$

where we used

$$\sum_{j=1}^{k-1}\lambda_j^{-2} = L^{-2} + (L^{-2})^2 + \cdots + (L^{-2})^{k-1} \leq L^{-2}\frac{1}{1-L^{-2}}. \qquad (2.148)$$

Now the claim follows, recalling that $\gamma_0 = c_1/4L$. □

3 Green functions with free boundary conditions

The goal of this and the next section is to prove estimate (2.122) which was used in the proof of Theorem 2.25.

3.1 Laplacian on the infinite lattice $\eta\mathbb{Z}$

Consider the Hilbert space $\mathcal{L}^2(\eta\mathbb{Z})$ equipped with the scalar product

$$\langle f, g\rangle = \eta\sum_{\mathrm{k}\in\mathbb{Z}}\overline{f}_{\mathrm{k}}g_{\mathrm{k}} = \eta\sum_{x\in\eta\mathbb{Z}}\overline{f(x)}g(x). \qquad (3.1)$$

As before, we use the notation $f_{\mathrm{k}} = f(\eta\mathrm{k})$, where k is an integer parameter corresponding to the point ηk of the lattice.

Now we define the discrete derivatives

$$(\partial^\eta f)_{\mathrm{k}} = \frac{1}{\eta}(f_{\mathrm{k}+1} - f_{\mathrm{k}}), \quad (\partial^{\eta,\dagger} f)_{\mathrm{k}} = -\frac{1}{\eta}(f_{\mathrm{k}} - f_{\mathrm{k}-1}) = -(\partial^\eta f)_{\mathrm{k}-1}. \tag{3.2}$$

For a fixed η these are clearly bounded operators. By Lemma 3.1 we obtain that $\partial^{\eta,\dagger} = (\partial^\eta)^*$. It is also easy to see that ∂^η is normal, see (3.5) below. By analogy with Lemma 2.1, we obtain the following:

Lemma 3.1. *The Leibniz rule holds in the form*

$$(\partial^\eta(fg))_{\mathrm{k}} = (\partial^\eta f)_{\mathrm{k}} g_{\mathrm{k}+1} + f_{\mathrm{k}}(\partial^\eta g)_{\mathrm{k}}. \tag{3.3}$$

Integration by parts holds in the form

$$\langle f, \partial^\eta g\rangle = -\langle \partial^\eta f, g_{\cdot+1}\rangle = \langle \partial^{\eta,\dagger} f, g\rangle. \tag{3.4}$$

The one-dimensional Laplacian has the familiar form

$$(\Delta^\eta_{d=1} f)_{\mathrm{k}} = (-\partial^\eta \partial^{\eta,\dagger} f)_{\mathrm{k}} = (-\partial^{\eta,\dagger}\partial^\eta f)_{\mathrm{k}} = \frac{f_{\mathrm{k}+1} - 2f_{\mathrm{k}} + f_{\mathrm{k}-1}}{\eta^2} \tag{3.5}$$

and is a bounded self-adjoint operator. We note the following:

Lemma 3.2. *Let* $\mathrm{k}_* \in \frac{1}{2}\mathbb{Z}$ *and* P *be the reflection operator*

$$(Pf)_{\mathrm{k}} = f_{2\mathrm{k}_* - \mathrm{k}}. \tag{3.6}$$

Then $P^2 = 1$ *and* $P = P^*$*. Furthermore,*

$$P\Delta^\eta_{d=1} P = \Delta^\eta_{d=1}. \tag{3.7}$$

(We note that P *depends on* k_**, although this is hidden in the notation).*

Proof. $P^2 = 1$ is clear. To check $P = P^*$ we write

$$\langle f, Pg\rangle = \eta\sum_{\mathrm{k}} f_{\mathrm{k}} g_{2\mathrm{k}_*-\mathrm{k}} = \eta\sum_{\mathrm{k}'} f_{2\mathrm{k}_*-\mathrm{k}'} g_{\mathrm{k}'} = \langle Pf, g\rangle. \tag{3.8}$$

Next, we note that

$$(\partial^\eta P f)_{\mathrm{k}} = \frac{1}{\eta}((Pf)_{\mathrm{k}+1} - (Pf)_{\mathrm{k}}) = \frac{1}{\eta}(f_{2\mathrm{k}_*-\mathrm{k}-1} - f_{2\mathrm{k}_*-\mathrm{k}}). \tag{3.9}$$

Hence

$$(P\partial^\eta P f)_{\mathrm{k}} = \frac{1}{\eta}(f_{2\mathrm{k}_*-(2\mathrm{k}_*-\mathrm{k})-1} - f_{2\mathrm{k}_*-(2\mathrm{k}_*-\mathrm{k})}) = (\partial^{\eta,\dagger} f)_{\mathrm{k}}. \tag{3.10}$$

Therefore $P\Delta^\eta_{d=1}P = -(P\partial^\eta P)(P\partial^{\eta,\dagger}P) = -\partial^{\eta,\dagger}\partial^\eta = \Delta^\eta_{d=1}$. $\Box$

Let us comment on the relation between the lattice Laplacians with free and Neumann boundary conditions in one dimension.

Definition 3.3. *We say that a function* $f \in \mathcal{L}^2(\eta\mathbb{Z})$ *satisfies Neumann boundary conditions on* I*, if the following relations hold on the boundary*

$$(\partial^{\eta,\dagger} f)(0) = 0, \tag{3.11}$$
$$(\partial^\eta f)(n-1) = 0. \tag{3.12}$$

We denote the subspace of such functions D_I*.*

Clearly, we have the following:

Lemma 3.4. *For any* $f \in D_I$

$$(\Delta_I^\eta f_I)(x) = (\Delta^\eta f)(x), \quad x \in I, \tag{3.13}$$

where $f_I \in \mathcal{L}^2(I)$ *is the restriction of* f *to* I.

Now we formulate a sufficient condition for $f \in \mathcal{L}^2(\eta\mathbb{Z})$ to be an element of D_I. Let P be a reflection with $\mathrm{k}_* = -1/2$ and $\overline{P}$ with $\mathrm{k}_* = n - 1/2$ (cf. (3.6)). Thus we obtain reflections w.r.t. lines passing in the $(1/2)\eta$ - distance to the boundary of I and denote the actions on the lattice points by the same symbols. The d-dimensional counterpart of Lemma 3.5 below (Lemma 3.10) will be needed in the context of the method of images (see Section 4 and Lemma 4.2).

Lemma 3.5. *Let* $f \in \mathcal{L}^2(\eta\mathbb{Z})$ *satisfy*

$$(Pf)(x) = f(x), \quad (\overline{P}f)(x) = f(x), \tag{3.14}$$

for x *at the ends of the interval* I. *Then* $f \in D_I$.

Remark 3.6. *This is analogous to the fact that a symmetric function has a vanishing derivative at zero.*

Proof. Let us check (3.11) for demonstration. We write for $x = 0$

$$f_0 = (Pf)_0 = f_{-1}, \tag{3.15}$$

which amounts to $(\partial^{\eta,\dagger} f)(0) = 0$. □

3.2 Laplacian on the infinite lattice $\eta\mathbb{Z}^d$

Now we work on $\mathcal{L}^2(\eta\mathbb{Z}^d) = \mathcal{L}^2(\eta\mathbb{Z})^{\otimes d}$ and write $\mathrm{k} = (\mathrm{k}_0, \ldots, \mathrm{k}_{d-1}) = (\mathrm{k}_\mu)_{\mu=0,\ldots,d-1}$ for elements of $\mathbb{Z}^d$ parametrizing points $x := \eta\mathrm{k}$ of $\eta\mathbb{Z}^d$. The scalar product in this space is given, accordingly, by

$$\langle f, g\rangle = \eta^d \sum_{\mathrm{k}\in\mathbb{Z}^d} \bar{f}_\mathrm{k} g_\mathrm{k}. \tag{3.16}$$

In the d-dimensional context the derivatives are denoted

$$\partial_\mu^\eta := 1 \otimes \cdots \otimes \partial^\eta \otimes \cdots \otimes 1, \quad \partial_\mu^{\eta,\dagger} := 1 \otimes \cdots \otimes \partial^{\eta,\dagger} \otimes \cdots \otimes 1. \tag{3.17}$$

Now the Laplacian is given by

$$\Delta^\eta := -\sum_{\mu=0}^{d-1} (\partial_\mu^{\eta,\dagger})(\partial_\mu^\eta). \tag{3.18}$$

We have the following corollary of Lemma 3.2.

Lemma 3.7. *Let* P *be a reflection w.r.t.* $\mathrm{k}_* \in \mathbb{Z}$ *as in Lemma 3.2 and denote by*

$$P_\mu = \underbrace{1 \otimes \cdots \otimes P}_{\mu+1} \otimes \cdots \otimes 1, \tag{3.19}$$

the corresponding reflections w.r.t. the hyperplanes

$$\{(\underbrace{\mathrm{k}'_0, \ldots, \mathrm{k}_*}_{\mu+1}, \ldots, \mathrm{k}'_{d-1}) \,|\, \mathrm{k}' \in \mathbb{Z}^{d-1}\}. \tag{3.20}$$

The following relation holds:

$$P_\mu \Delta^\eta P_\mu = \Delta^\eta, \quad \mu = 0, 1, \ldots, d-1. \tag{3.21}$$

(We note that P_μ *depends on* k_**, although this is hidden in the notation).*

We note as an aside that for any bounded Borel function $F:\mathbb{R}\to\mathbb{C}$ we can define the operator $F(\Delta^\eta)$ whose kernel satisfies

$$F(\Delta^\eta)(P_\mu x, P_\mu y) = F(\Delta^\eta)(x,y), \quad \mu = 0,1,\dots d-1. \tag{3.22}$$

Here we denoted the action of reflection on a lattice point x also by P_μ so that $P_\mu \delta^\eta_x = \delta^\eta_{P_\mu x}$ and used (2.29), (3.21).

Like in Subsection 3.1, we comment on the relation between the lattice Laplacians with free and Neumann boundary conditions in d dimensions.

Definition 3.8. *We say that a function $f\in\mathcal{L}^2(\eta\mathbb{Z}^d)$ satisfies Neumann boundary conditions on Ω, if the following relations hold on the respective subsets of the boundary (2.26):*

$$(\partial^{\eta,\dagger}_\mu f)(x) = 0 \quad \text{for} \quad x\in\partial\Omega_\mu, \tag{3.23}$$

$$(\partial^{\eta}_\mu f)(x) = 0 \quad \text{for} \quad x\in\partial\Omega^\mu, \tag{3.24}$$

$\mu = 0,1,\dots,d-1$. *We denote the subspace of such functions D_Ω.*

Clearly, we have the following:

Lemma 3.9. *For any $f\in D_\Omega$ we have*

$$(\Delta^\eta_\Omega f_\Omega)(x) = (\Delta^\eta f)(x), \quad x\in\Omega, \tag{3.25}$$

where $f_\Omega\in\mathcal{L}^2(\Omega)$ is the restriction of $f\in\mathcal{L}^2(\eta\mathbb{Z}^d)$ to Ω.

Now we formulate a sufficient condition for $f\in\mathcal{L}^2(\eta\mathbb{Z}^d)$ to be an element of D_Ω. We fix μ and let P be a reflection with $\mathrm{k}_* = -1/2$ and $\overline{P}$ with $\mathrm{k}_* = n-1/2$ (cf. Lemma 3.2). Thus we obtain reflections w.r.t. hyperplanes passing in the $(1/2)\eta$-distance to the boundary of Ω:

$$P_\mu = \underbrace{1\otimes\cdots\otimes P}_{\mu+1}\otimes\cdots\otimes 1, \quad \overline{P}_\mu = \underbrace{1\otimes\cdots\otimes \overline{P}}_{\mu+1}\otimes\cdots\otimes 1 \tag{3.26}$$

and denote their actions on the lattice points by the same symbols. We have:

Lemma 3.10. *Let $f\in\mathcal{L}^2(\eta\mathbb{Z}^d)$ satisfy*

$$(P_\mu f)(x) = f(x), \quad x\in\partial\Omega_\mu, \tag{3.27}$$

$$(\overline{P}_\mu f)(x) = f(x), \quad x\in\partial\Omega^\mu, \tag{3.28}$$

$\mu = 0,1,\dots,d-1$. *Then $f\in D_\Omega$.*

Proof. Let us check (3.23) for demonstration. We fix μ and write for $x=\eta\mathrm{k}\in(\underbrace{I\times\cdots\times\{0\}}_{\mu+1}\times\cdots\times I)$

$$f_{(\mathrm{k}_0,\dots,0,\dots,\mathrm{k}_{d-1})} = (P_\mu f)_{(\mathrm{k}_0,\dots,0,\dots\mathrm{k}_{d-1})} = f_{\mathrm{k}_0,\dots,-1,\dots\mathrm{k}_{d-1}}, \tag{3.29}$$

which amounts to $(\partial^{\eta,\dagger}_\mu f)(x) = 0$. □

Finally, we define the lattice translation operators

$$(T_{\mathrm{k}'} f)_{\mathrm{k}} = f_{\mathrm{k}-\mathrm{k}'}. \tag{3.30}$$

which form a unitary representation of $\mathbb{Z}^d$ acting on $\mathcal{L}^2(\eta\mathbb{Z}^d)$. We will also use the notation $T(x')$ for $x'=\eta\mathrm{k}'$. Since

$$\langle T_{\mathrm{k}'} g, f\rangle = \eta^d \sum_{\mathrm{k}\in\mathbb{Z}^d} \overline{T_{\mathrm{k}'} g_{\mathrm{k}}} f_{\mathrm{k}} = \eta^d \sum_{\mathrm{k}\in\mathbb{Z}^d} \overline{g}_{\mathrm{k}-\mathrm{k}'} f_{\mathrm{k}} = \eta^d \sum_{\mathrm{k}\in\mathbb{Z}^d} \overline{g}_{\mathrm{k}} f_{\mathrm{k}+\mathrm{k}'} = \langle g, T_{-\mathrm{k}'} f\rangle \tag{3.31}$$

we also have $T^*_{\mathrm{k}} = T_{-\mathrm{k}}$. By obvious computations using the definitions of ∂^η_μ and $\partial^{\eta,\dagger}_\mu$ we have

$$T_{\mathrm{k}}\Delta^\eta T^*_{\mathrm{k}} = \Delta^\eta. \tag{3.32}$$

3.3 Fourier transform

We define the Fourier transform and inverse Fourier transform on $\mathcal{L}^2(\eta\mathbb{Z}^d)$ by

$$(\mathcal{F}f)(p) := \hat{f}(p) := (2\pi)^{-d/2} \sum_{x\in\eta\mathbb{Z}^d} \eta^d \mathrm{e}^{-\mathrm{i}p\cdot x} f(x), \quad f(x) = (2\pi)^{-d/2} \int_{p\in\widehat{\eta\mathbb{Z}^d}} \mathrm{e}^{\mathrm{i}p\cdot x} \hat{f}(p)\, dp, \tag{3.33}$$

where $\widehat{\eta\mathbb{Z}^d}$ is the torus $[-\pi/\eta, \pi/\eta[^{\times d}$. We recall that the sum in (3.33) should not be taken literally: $\mathcal{F}$ is defined first on $\mathcal{L}^1(\eta\mathbb{Z}^d)$ and then extended to $\mathcal{L}^2(\eta\mathbb{Z}^d)$ using the isometry property checked below.

Lemma 3.11. *The Fourier transform (3.33) is a unitary $\mathcal{L}^2(\eta\mathbb{Z}^d) \to \mathcal{L}^2(\widehat{\eta\mathbb{Z}^d})$.*

Proof. We check here only the isometry property to verify normalization. We recall the formula

$$\int_{[-\pi,\pi[^{\times d}} d\tilde{p}\, \mathrm{e}^{\mathrm{i}\tilde{p}\mathrm{m}} = (2\pi)^d \delta_{\mathrm{m},0}, \quad \mathrm{m} \in \mathbb{Z}^d. \tag{3.34}$$

Now we compute

$$\begin{aligned} \langle \hat{f}_1, \hat{f}_2 \rangle &= (2\pi)^{-d} \int_{[-\pi/\eta,\pi/\eta[^{\times d}} dp \sum_{x_1,x_2\in\eta\mathbb{Z}^d} \eta^{2d} \mathrm{e}^{\mathrm{i}p\cdot(x_1-x_2)} \overline{f}_1(x_1) f_2(x_2) \\ &= (2\pi)^{-d} \int_{[-\pi,\pi[^{\times d}} d\tilde{p}\, \eta^{-d} \sum_{\mathrm{m}_1,\mathrm{m}_2\in\mathbb{Z}^d} \eta^{2d} \mathrm{e}^{\mathrm{i}\tilde{p}\cdot(\mathrm{m}_1-\mathrm{m}_2)} \overline{f}_1(\eta\mathrm{m}_1) f_2(\eta\mathrm{m}_2) \\ &= \sum_{\mathrm{m}_1,\mathrm{m}_2\in\mathbb{Z}^d} \eta^d \delta_{\mathrm{m}_1,\mathrm{m}_2} \overline{f}_1(\eta\mathrm{m}_1) f_2(\eta\mathrm{m}_2) = \langle f_1, f_2 \rangle. \qquad \Box \end{aligned} \tag{3.35}$$

Lemma 3.12. *The following relations hold*

$$\hat{\delta}^\eta_y(p) = \frac{1}{(2\pi)^{d/2}} \mathrm{e}^{-\mathrm{i}p\cdot y}, \tag{3.36}$$

$$\mathcal{F}\partial^\eta_\mu \mathcal{F}^{-1} = \{\frac{1}{\eta}(\mathrm{e}^{\mathrm{i}\eta p_\mu} - 1)\}_{p\in\widehat{\eta\mathbb{Z}^d}}, \tag{3.37}$$

$$\mathcal{F}\partial^{\eta,\dagger}_\mu \mathcal{F}^{-1} = \{-\frac{1}{\eta}(\mathrm{e}^{-\mathrm{i}\eta p_\mu} - 1)\}_{p\in\widehat{\eta\mathbb{Z}^d}}, \tag{3.38}$$

$$\mathcal{F}(-\Delta^\eta)\mathcal{F}^{-1} = \{\frac{2}{\eta^2} \sum_{\mu=0}^{d-1} \big(1 - \cos(p_\mu\eta)\big)\}_{p\in\widehat{\eta\mathbb{Z}^d}}, \tag{3.39}$$

where the r.h.s. in (3.37)–(3.39) denote multiplication operators on $\mathcal{L}^2(\widehat{\eta\mathbb{Z}^d})$.

Proof. Relation (3.36) is clear. To prove (3.37), we get for $x = \eta\mathrm{k}$, $e_\mu = (\underbrace{0, \dots, 1}_{\mu+1}, \dots 0)$

$$\begin{aligned} (\mathcal{F}\partial^\eta_\mu f)(p) &= (2\pi)^{-d/2} \sum_{\mathrm{k}\in\mathbb{Z}^d} \eta^d \mathrm{e}^{-\mathrm{i}\eta p\cdot\mathrm{k}} \frac{1}{\eta}(f_{\mathrm{k}+e_\mu} - f_\mathrm{k}) \\ &= (2\pi)^{-d/2} \sum_{\mathrm{k}\in\mathbb{Z}^d} \eta^d \frac{1}{\eta}(\mathrm{e}^{-\mathrm{i}\eta p\cdot(\mathrm{k}-e_\mu)} f_\mathrm{k} - \mathrm{e}^{-\mathrm{i}\eta p\cdot\mathrm{k}} f_\mathrm{k}) \\ &= \frac{1}{\eta}(\mathrm{e}^{\mathrm{i}\eta p_\mu} - 1)(\mathcal{F}f)(p). \end{aligned} \tag{3.40}$$

Relation (3.38) follows from (3.37) by taking adjoints and (3.39) follows from $\Delta^\eta = -\sum_{\mu=0}^{d-1} \partial^{\eta,\dagger}_\mu \partial^\eta_\mu$ via (3.37), (3.38). $\Box$

3.4 Averaging operators on $\eta\mathbb{Z}^d$

The averaging operators on $\eta\mathbb{Z}^d$ are defined analogously to the averaging operators on a finite lattice. Thus we will denote them by the same symbol Q_j and the discussion (2.40)–(2.45) can be repeated *mutatis mutandis*. The operator $Q_j : \mathcal{L}^2(\eta\mathbb{Z}^d) \to \mathcal{L}^2(L^j\eta\mathbb{Z}^d)$ is given by the familiar formula

$$(Q_j f)(z) = \frac{1}{L^{jd}} \sum_{z_\bullet \leq x'_\bullet < z_\bullet + L^j\eta} f(x') \tag{3.41}$$

and has norm equal to one, cf. Remark 2.8. Its adjoint has the form

$$(Q_j^* h)(x) = h(y_x), \tag{3.42}$$

where y_x is determined by $x \in B_j(y_x)$.

As a preparation, let us now write for $g \in \mathcal{L}^2(\mathbb{Z}^d)$

$$f(x) = (Q_k^* g)(x) = g(y_x), \tag{3.43}$$

where y_x was defined below (2.43). The Fourier transform has the form

$$\begin{aligned} \hat{f}(p) &= (2\pi)^{-d/2} \sum_{x \in \eta\mathbb{Z}^d} \eta^d \mathrm{e}^{-\mathrm{i}p\cdot x} g([x_\bullet]) \\ &= (2\pi)^{-d/2} \sum_{[x_\bullet] \in \mathbb{Z}} \sum_{\ell_\bullet = 0}^{L^k - 1} \eta^d \mathrm{e}^{-\mathrm{i}p_\nu([x_\nu] + \eta\ell_\nu)} g([x_\bullet]) \\ &= (2\pi)^{-d/2} u(p) \sum_{[x_\bullet] \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}p_\nu \cdot [x_\nu]} g([x_\bullet]) = u(p)\hat{g}(p), \end{aligned} \tag{3.44}$$

where u, the Fourier kernel of Q^*, has the form

$$u(p) := \eta^d \prod_{\mu=0}^{d-1} \frac{1 - \mathrm{e}^{-\mathrm{i}p_\mu}}{1 - \mathrm{e}^{-\mathrm{i}p_\mu \eta}}. \tag{3.45}$$

We note that $\hat{f}$ is a function on $[-\pi/\eta, \pi/\eta[^{\times d}$ after extending $\hat{g}$ from $[-\pi, \pi[^{\times d}$ to $[-\pi/\eta, \pi/\eta[^{\times d}$ by periodicity.

Furthermore, $Q_j^* Q_j : \mathcal{L}^2(\eta\mathbb{Z}^d) \to \mathcal{L}^2(\eta\mathbb{Z}^d)$ is given by

$$(Q_j^* Q_j f)(x) = \frac{1}{L^{jd}} \sum_{y_{x,\bullet} \leq x'_\bullet < y_{x,\bullet} + L^j\eta} f(x') = \frac{1}{L^{jd}} \sum_{[\frac{x_\bullet}{L^j\eta}](L^j\eta) \leq x'_\bullet < [\frac{x_\bullet}{L^j\eta}](L^j\eta) + L^j\eta} f(x'). \tag{3.46}$$

We want to compute the Fourier transform of this expression, cf. (3.33). In the following lemma it is used that L is odd and $L^k\eta = 1$.

Lemma 3.13. *For $j = k$ the Fourier transform of (3.46) has the form*

$$(\widehat{Q_k^* Q_k f})(p) = u(p) \sum_{\ell''_\bullet = -\frac{L^k-1}{2}}^{\frac{L^k-1}{2}} \overline{u(p + 2\pi\ell'')} \hat{f}(p + 2\pi\ell''), \tag{3.47}$$

where u was defined in (3.45).

Remark 3.14. *This agrees with formula (2.45) of* [Ba83].

Proof. See Appendix E. □

3.5 Green functions with free boundary conditions

This subsection serves to decipher the discussion [Ba83, (2.44)-(2.51)]: The propagators with free boundary conditions are operators on $\mathcal{L}^2(\eta\mathbb{Z}^d)$ given by

$$G_k := (-\Delta^{\eta,\bar{\mu}_k} + a_k Q_k^* Q_k)^{-1}. \tag{3.48}$$

The existence of the inverse is visible in the Fourier space and will follow from the discussion below (see Lemma 3.15). We consider the following equation for some $v, f \in \mathcal{L}^2(\eta\mathbb{Z}^d)$

$$(-\Delta^{\eta,\bar{\mu}_k} + a_k Q_k^* Q_k)v = f. \tag{3.49}$$

We take the Fourier transform, using (3.47) and (3.39)

$$\Delta^\eta(p)\hat{v}(p) + a_k u(p) \sum_{\ell''_\bullet = -\frac{L^k-1}{2}}^{\frac{L^k-1}{2}} \overline{u(p+2\pi\ell'')}\hat{v}(p+2\pi\ell'') = \hat{f}(p), \tag{3.50}$$

where $\Delta^\eta(p) := \frac{2}{\eta^2}\sum_{\mu=0}^{d-1}(1-\cos(p_\mu\eta)) + \bar{\mu}_k$, $u(p) := \eta^d \prod_{\mu=0}^{d-1} \frac{1-\mathrm{e}^{-\mathrm{i}p_\mu}}{1-\mathrm{e}^{-\mathrm{i}p_\mu\eta}}$ (cf. (3.45)) and we deliberately hide $\bar{\mu}_k$ in our notation as it will play a minor role in the following.

To solve equation (3.50), we set

$$\langle\!\langle u, \hat{v}\rangle\!\rangle(p) := \sum_{\ell''_\bullet = -\frac{L^k-1}{2}}^{\frac{L^k-1}{2}} \overline{u(p+2\pi\ell'')}\hat{v}(p+2\pi\ell''). \tag{3.51}$$

We note that $\langle\!\langle u, \hat{v}\rangle\!\rangle$ depends on k although this is hidden in the notation. By Lemma 3.18 below we have for any $\ell \in \mathbb{Z}^d$

$$\langle\!\langle u, \hat{v}\rangle\!\rangle(p) = \langle\!\langle u, \hat{v}\rangle\!\rangle(p+2\pi\ell). \tag{3.52}$$

Now we obtain from (3.50) using notation (3.51) and $u_\Delta(p) := \frac{u(p)}{\Delta^\eta(p)}$, $\hat{f}_\Delta(p) := \frac{\hat{f}(p)}{\Delta^\eta(p)}$

$$\hat{v}(p) + a_k u_\Delta(p)\langle\!\langle u, \hat{v}\rangle\!\rangle(p) = \hat{f}_\Delta(p). \tag{3.53}$$

Using (3.52), we obtain

$$\langle\!\langle u, \hat{v}\rangle\!\rangle(p) + a_k\langle\!\langle u, u_\Delta\rangle\!\rangle(p)\,\langle\!\langle u, \hat{v}\rangle\!\rangle(p) = \langle\!\langle u, \hat{f}_\Delta\rangle\!\rangle(p), \tag{3.54}$$

$$\langle\!\langle u, \hat{v}\rangle\!\rangle(p)\big(1 + a_k\langle\!\langle u, u_\Delta\rangle\!\rangle(p)\big) = \langle\!\langle u, \hat{f}_\Delta\rangle\!\rangle(p), \tag{3.55}$$

$$\langle\!\langle u, \hat{v}\rangle\!\rangle(p) = \big(1 + a_k\langle\!\langle u, u_\Delta\rangle\!\rangle(p)\big)^{-1}\langle\!\langle u, \hat{f}_\Delta\rangle\!\rangle(p). \tag{3.56}$$

Now we substitute (3.56) to (3.53), which gives a solution

$$\hat{v}(p) = \widehat{(G_k f)}(p) \;=\; \hat{f}_\Delta(p) - \frac{a_k u_\Delta(p)}{1 + a_k\langle\!\langle u, u_\Delta\rangle\!\rangle(p)}\langle\!\langle u, \hat{f}_\Delta\rangle\!\rangle(p). \tag{3.57}$$

From this solution we can conclude that G_k is a bounded operator. (The norm may depend on parameters of the problem s.t. η, $\bar{\mu}_k$, L).

Lemma 3.15. *G_k is a bounded operator on $\mathcal{L}^2(\eta\mathbb{Z}^d)$ for any $\bar{\mu}_k \geq 0$.*

Proof. Suppose first that $\bar{\mu}_k > 0$. Then $\Delta^\eta(p) \geq \bar{\mu}_k > 0$ and the first term on the r.h.s. of (3.57) satisfies $\|\hat{f}_\Delta\|_2 \leq \bar{\mu}_k^{-1}\|\hat{f}\|_2$. As for the second term, we first note that $\langle\!\langle u, u_\Delta\rangle\!\rangle(p) \geq 0$. Next, regarding that u is a bounded function (actually uniformly in η for $p \in [-\pi/\eta, \pi/\eta[^{\times d}$), we note that

$$\|\langle\!\langle u, \hat{f}_\Delta\rangle\!\rangle\|_2 \leq c\bar{\mu}_k^{-1}\|\hat{f}\|_2. \tag{3.58}$$

This concludes the analysis of the case of $\bar{\mu}_k > 0$.

Now suppose that $\bar{\mu}_k = 0$. In this case the only possible obstruction to boundedness are non-square-integrable singularities in (3.57). Let us justify that such singularities can only appear at $p = 0$ for $p \in [-\pi/\eta, \pi/\eta[^{\times d}$. This is obvious for the term $\hat{f}_\Delta(p)$ on the r.h.s. of (3.57). As for the second term, we first note that the denominator $1 + a_k\langle\!\langle u, u_\Delta\rangle\!\rangle(p)$ is irrelevant, since $\langle\!\langle u, u_\Delta\rangle\!\rangle(p) \geq 0$. The remaining expression $u_\Delta(p)\langle\!\langle u, \hat{f}_\Delta\rangle\!\rangle(p)$ might potentially also have non-square-integrable singularities for $p + 2\pi\ell'' = 0$, $-\frac{L^k-1}{2} \leq \ell''_\bullet \leq \frac{L^k-1}{2}$, $\ell'' \neq 0$, resulting from $\Delta^\eta(p) = \frac{1}{\eta^2}\sum_{\mu=0}^{d-1}|\mathrm{e}^{-\mathrm{i}p_\mu\eta} - 1|^2$ appearing in $\hat{f}_\Delta$. These are, however, cancelled by the factors $1 - \mathrm{e}^{-\mathrm{i}p_\mu} = 1 - \mathrm{e}^{-\mathrm{i}(p_\mu + 2\pi\ell''_\mu)}$ appearing in $u_\Delta(p)$.

Let us now exclude a non-square-integrable singularity at $p = 0$. For this purpose, we reformulate (3.57) as follows

$$\widehat{(G_k f)}(p) = \frac{\hat{f}(p)}{\Delta^\eta(p)} - \frac{a_k u_\Delta(p)}{1 + a_k\langle\!\langle u, u_\Delta\rangle\!\rangle(p)}\langle\!\langle u, \hat{f}_\Delta\rangle\!\rangle_{\ell''=0}(p) \tag{3.59}$$

$$- \frac{a_k u_\Delta(p)}{1 + a_k\langle\!\langle u, u_\Delta\rangle\!\rangle(p)}\langle\!\langle u, \hat{f}_\Delta\rangle\!\rangle_{\ell''\neq 0}(p), \tag{3.60}$$

where the subscripts $\ell'' = 0, \ell'' \neq 0$ pertain to the sum in (3.51). We note that (3.60) is square-integrable in the interval $p \in [-\pi, \pi]^{\times d}$: First, since $[-\pi, \pi]^{\times d} \ni p \mapsto \Delta^\eta(p + 2\pi\ell'')$ has no zeros for $\ell'' \neq 0$, we obtain

$$\|\langle\!\langle u, \hat{f}_\Delta\rangle\!\rangle_{\ell''\neq 0}\|_2 \leq C\|\hat{f}\|_2, \tag{3.61}$$

where C may depend on η. Furthermore, we have

$$\frac{|u_\Delta(p)|}{1 + a_k\langle\!\langle u, u_\Delta\rangle\!\rangle(p)} = \frac{|u(p)|}{\Delta^\eta(p) + |u(p)|^2 + a_k\langle\!\langle u, u_\Delta\rangle\!\rangle_{\ell''\neq 0}(p)} \leq \frac{1}{|u(p)|}, \tag{3.62}$$

where $|u(p)| \geq c > 0$. Now we consider (3.59). We rewrite it as follows:

$$\begin{aligned}
(3.59) &= \left(1 - \frac{a_k\langle\!\langle u, u_\Delta\rangle\!\rangle_{\ell''=0}(p)}{1 + a_k\langle\!\langle u, u_\Delta\rangle\!\rangle(p)}\right)\frac{\hat{f}(p)}{\Delta^\eta(p)} \\
&= \left(\frac{1 + a_k\langle\!\langle u, u_\Delta\rangle\!\rangle_{\ell''\neq 0}(p)}{1 + a_k\langle\!\langle u, u_\Delta\rangle\!\rangle_{\ell''=0}(p) + a_k\langle\!\langle u, u_\Delta\rangle\!\rangle_{\ell''\neq 0}(p)}\right)\frac{\hat{f}(p)}{\Delta^\eta(p)} \\
&= \left(\frac{1 + a_k\langle\!\langle u, u_\Delta\rangle\!\rangle_{\ell''\neq 0}(p)}{\Delta^\eta(p) + a_k|u(p)|^2 + a_k\langle\!\langle u, u_\Delta\rangle\!\rangle_{\ell''\neq 0}(p)\Delta^\eta(p)}\right)\hat{f}(p),
\end{aligned} \tag{3.63}$$

where the expression in bracket in (3.63) is manifestly regular near $p = 0$, since $|u(p)| \geq c > 0$. □

Now we state a rough estimate for the decay of the kernel of G_k. Its weakness is the dependence of the constants on k, hence on the lattice spacing $\eta = L^{-k}$. But it suffices to write formula (4.4) below.

Lemma 3.16. *The integral kernel of G_k satisfies*

$$|G_k(x, y)| \leq c_k \mathrm{e}^{-c_{1,k}|x-y|}, \tag{3.64}$$

where the constants c_k and $c_{1,k} > 0$ may depend on k.

Proof. We have

$$(G_k f)(x) \;=\; \frac{1}{(2\pi)^{d/2}} \int_{[-\pi/\eta,\pi/\eta[^{\times d}} dp\, \mathrm{e}^{\mathrm{i}p\cdot x}\widehat{(G_k f)}(p). \tag{3.65}$$

Now we set $f(x) = \delta_y^\eta(x)$, hence $\hat{f}(p) = (2\pi)^{-d/2}\mathrm{e}^{-\mathrm{i}p\cdot y}$ by (3.36). Thus we can write, using (3.57),

$$(G_k f)(x) \;=\; \frac{1}{(2\pi)} \int_{[-\pi/\eta,\pi/\eta[^{\times d}} dp\, \mathrm{e}^{\mathrm{i}p\cdot x}\Big(\hat{f}_\Delta(p) - \frac{a_k u_\Delta(p)}{(1 + a_k\langle\!\langle u, u_\Delta\rangle\!\rangle(p))}\langle\!\langle u, \hat{f}_\Delta\rangle\!\rangle(p)\Big). \tag{3.66}$$

We have

$$\hat{f}_\Delta(p) = \frac{1}{(2\pi)^{d/2}}\frac{\mathrm{e}^{-\mathrm{i}p\cdot y}}{\Delta^\eta(p)}, \quad \langle\!\langle u, \hat{f}_\Delta\rangle\!\rangle(p) = \sum_{\ell''_\bullet = -\frac{L^k-1}{2}}^{\frac{L^k-1}{2}} \overline{u(p+2\pi\ell'')}\frac{1}{(2\pi)^{d/2}}\frac{\mathrm{e}^{-\mathrm{i}(p+2\pi\ell'')\cdot y}}{\Delta^\eta(p+2\pi\ell'')}. \tag{3.67}$$

Consequently

$$(G_k f)(x) \;=\; \frac{1}{(2\pi)^{d/2}} \int_{[-\pi/\eta,\pi/\eta[^{\times d}} dp\, \mathrm{e}^{\mathrm{i}p\cdot(x-y)} g(p), \tag{3.68}$$

where g is real analytic (cf. the proof of Lemma 3.15) and periodic with period $2\pi/\eta$. Hence it remains periodic in the real direction after analytic continuation to $S_{\infty, c_{\mathrm{st}}^k} := \{\, z = p + \mathrm{i}q \in \mathbb{C}^d \,|\, p \in \mathbb{R}, |q| < c_{\mathrm{st}}^k\}$, where $0 < c_{\mathrm{st}}^k \leq 1$. (Such c_{st}^k exists, but may depend on k). Thus, by the Cauchy theorem for a rectangular contour C, we have for a fixed μ,

$$\begin{aligned}
0 \;&=\; \oint_C \mathrm{e}^{\mathrm{i}z_\mu(x-y)_\mu} g_\mu(z_\mu)\, dz_\mu \\
&=\; \int_{-\pi/\eta}^{\pi/\eta} \mathrm{e}^{\mathrm{i}p_\mu(x-y)_\mu} g_\mu(p_\mu) dp_\mu + \mathrm{i}\int_0^{q_{\mathrm{st},\mu}} \mathrm{e}^{\mathrm{i}(\pi/\eta + \mathrm{i}q_\mu)(x-y)_\mu} g_\mu(\pi/\eta + \mathrm{i}q) dq \\
&\quad + \int_{\pi/\eta}^{-\pi/\eta} \mathrm{e}^{\mathrm{i}(p_\mu + \mathrm{i}q_{\mathrm{st},\mu})(x-y)_\mu} g_\mu(p_\mu + \mathrm{i}q_{\mathrm{st},\mu}) dp_\mu + \mathrm{i}\int_{q_{\mathrm{st},\mu}}^0 \mathrm{e}^{\mathrm{i}(-\pi/\eta + \mathrm{i}q_\mu)(x-y)_\mu} g_\mu(-\pi/\eta + \mathrm{i}q) dq,
\end{aligned} \tag{3.69}$$

where $g_\mu(z_\mu) := g(z_0, \ldots, z_\mu, \ldots, z_{d-1})$ and $q_{\mathrm{st}} := \frac{1}{2}c_{\mathrm{st}}^k\frac{(x-y)}{|x-y|}$. We note that the second and fourth term on the r.h.s. of (3.69) cancel by periodicity and the remaining terms give

$$\int_{-\pi/\eta}^{\pi/\eta} \mathrm{e}^{\mathrm{i}p_\mu(x-y)_\mu} g_\mu(p_\mu) dp_\mu = \int_{-\pi/\eta}^{\pi/\eta} \mathrm{e}^{\mathrm{i}(p_\mu + \mathrm{i}q_{\mathrm{st},\mu})(x-y)_\mu} g_\mu(p_\mu + \mathrm{i}q_{\mathrm{st},\mu}) dp_\mu. \tag{3.70}$$

By iterating this argument, we obtain

$$\int_{[-\pi/\eta,\pi/\eta[^{\times d}} \mathrm{e}^{\mathrm{i}p\cdot(x-y)} g(p) dp = \int_{[-\pi/\eta,\pi/\eta[^{\times d}} \mathrm{e}^{\mathrm{i}(p+\mathrm{i}q_{\mathrm{st}})\cdot(x-y)} g(p + \mathrm{i}q_{\mathrm{st}}) dp_\mu \tag{3.71}$$

which gives (3.64). $\Box$

Now we will study the kernel of $G_k Q_k^*$. Now we substitute (3.44) to (3.57):

$$\begin{aligned}
(\widehat{G_k Q_k^* g})(p) \;&=\; u_\Delta(p)\hat{g}(p) - a_k u_\Delta(p)\big(1 + a_k\langle\!\langle u, u_\Delta\rangle\!\rangle(p)\big)^{-1}\langle\!\langle u, u_\Delta\hat{g}\rangle\!\rangle(p) \\
&=\; \big(1 - \big(1 + a_k\langle\!\langle u, u_\Delta\rangle\!\rangle(p)\big)^{-1} a_k\langle\!\langle u, u_\Delta\rangle\!\rangle(p)\big) u_\Delta(p)\hat{g}(p) \\
&=\; \frac{1}{1 + a_k\langle\!\langle u, u_\Delta\rangle\!\rangle(p)} u_\Delta(p)\hat{g}(p),
\end{aligned} \tag{3.72}$$

where we used that $\hat{g}$ has period 2π to pull it out of the bracket $\langle\!\langle \cdot\,, \cdot \rangle\!\rangle$. Next, we write

$$(G_k Q_k^* g)(x) = (2\pi)^{-d/2} \int_{[-\pi/\eta,\pi/\eta[^{\times d}} dp\, \mathrm{e}^{\mathrm{i}p\cdot x} \frac{1}{1 + a_k \langle\!\langle u, u_\Delta \rangle\!\rangle(p)} u_\Delta(p)\hat{g}(p). \tag{3.73}$$

Now we set $g = \delta_y^1$, which gives

$$(G_k Q_k^*)(x,y) = (2\pi)^{-d} \int_{[-\pi/\eta,\pi/\eta[^{\times d}} dp\, \mathrm{e}^{\mathrm{i}p\cdot(x-y)} \frac{1}{1 + a_k \langle\!\langle u, u_\Delta \rangle\!\rangle(p)} u_\Delta(p). \tag{3.74}$$

We note that $x \in \eta\mathbb{Z}^d$, whereas $y \in (L^k\eta)\mathbb{Z}^d = \mathbb{Z}^d$. Since $1/\eta = L^k$, we can write

$$(G_k Q_k^*)(x,y) = (2\pi)^{-d} \int_{[-\pi,\pi[^{\times d}} dp \sum_{\ell'_\bullet = -\frac{(L^k-1)}{2}}^{\frac{(L^k-1)}{2}} \mathrm{e}^{\mathrm{i}(p+2\pi\ell')\cdot(x-y)} \frac{1}{1 + a_k \langle\!\langle u, u_\Delta \rangle\!\rangle(p)} u_\Delta(p + 2\pi\ell'), \tag{3.75}$$

where we used (3.52). The main result of this subsection is:

Lemma 3.17. *There exist numerical constants c and $c_{\mathrm{st}} > 0$, s.t.*

$$|(G_k Q_k^*)(x,y)| \leq c\mathrm{e}^{-\frac{1}{2}c_{\mathrm{st}}|x-y|}. \tag{3.76}$$

Proof. The expression (3.75) has the form

$$(G_k Q_k^*)(x,y) \;=\; (2\pi)^{-d} \int_{[-\pi,\pi[^{\times d}} f(p)\, dp, \tag{3.77}$$

$$f(p) \;:=\; \sum_{\ell'_\bullet = -\frac{(L^k-1)}{2}}^{\frac{(L^k-1)}{2}} \mathrm{e}^{\mathrm{i}(p+2\pi\ell')\cdot(x-y)} \frac{1}{1 + a_k \langle\!\langle u, u_\Delta \rangle\!\rangle(p)} u_\Delta(p + 2\pi\ell'). \tag{3.78}$$

We know from Lemma 3.19 that f has an analytic continuation to $S_{\mathbb{R},c_{\mathrm{st}}} := \{\, z = p + \mathrm{i}q \in \mathbb{C}^d \mid p \in \mathbb{R}, |q| < c_{\mathrm{st}}\}$, where $0 < c_{\mathrm{st}} \leq 1$, and is bounded in this region as in (3.85). Thus, by the Cauchy theorem for a rectangular contour C, we have for a fixed μ

$$\begin{aligned} 0 = \oint_C f_\mu(z_\mu)\, dz_\mu \;&=\; \int_{-\pi}^{\pi} f_\mu(p_\mu) dp_\mu + \mathrm{i}\int_0^{q_{\mathrm{st},\mu}} f_\mu(\pi + \mathrm{i}q) dq \\ &+ \int_{\pi}^{-\pi} f_\mu(p_\mu + \mathrm{i}q_{\mathrm{st},\mu}) dp_\mu + i\int_{q_{\mathrm{st},\mu}}^{0} f(-\pi + \mathrm{i}q) dq, \end{aligned} \tag{3.79}$$

where $f_\mu(z_\mu) := f(z_0, \ldots, z_\mu, \ldots, z_{d-1})$ and $q_{\mathrm{st}} := \frac{1}{2}c_{\mathrm{st}}\frac{(x-y)}{|x-y|}$. We note that the second and fourth term on the r.h.s. of (3.79) cancel due to Lemma 3.18. Thus, by iteration, we obtain

$$\int_{[-\pi,\pi[^{\times d}} f(p)\, dp = \int_{[-\pi,\pi[^{\times d}} f(p + \mathrm{i}q_{\mathrm{st}})\, dp. \tag{3.80}$$

Thus we can write

$$\begin{aligned} (G_k Q_k^*)(x,y) &= (2\pi)^{-d} \int_{[-\pi,\pi[^{\times d}} f(p + \mathrm{i}q_{\mathrm{st}}) dp \\ &= (2\pi)^{-1} \int_{[-\pi,\pi[^{\times d}} \sum_{\ell'_\bullet = -\frac{L^k-1}{2}}^{\frac{L^k-1}{2}} \mathrm{e}^{\mathrm{i}(p+\mathrm{i}q_{\mathrm{st}}+2\pi\ell')\cdot(x-y)} \frac{1}{1 + a_k \langle\!\langle u, u_\Delta \rangle\!\rangle(p + \mathrm{i}q_{\mathrm{st}})} u_\Delta(p + \mathrm{i}q_{\mathrm{st}} + 2\pi\ell') dp. \end{aligned} \tag{3.81}$$

Now, making use of Lemma 3.19, we obtain (3.76). □

Lemma 3.18. *Suppose $f:\mathbb{R}^d\to\mathbb{C}$ satisfies $f(p+2\pi L^k)=f(p)$, $p\in\mathbb{R}$. Then*

$$F(p) := \sum_{\ell'=-\frac{L^k-1}{2}}^{\frac{L^k-1}{2}} f(p+2\pi\ell') \tag{3.82}$$

satisfies $F(p+2\pi)=F(p)$, $p\in\mathbb{R}$. If, in addition, f is analytic in a strip $S_{\infty,c_{\rm st}} := \{\, z:=p+\mathrm{i}q\in\mathbb{C}\,|\, p\in\mathbb{R}^d, |q|<c_{\rm st}\,\}$ then $f(z+2\pi L^k)=f(z)$ and $F(z+2\pi)=F(z)$ for all $z\in S_{\infty,c_{\rm st}}$.

Proof. We note the following

$$F(p+2\pi) = \sum_{\ell'=-\frac{L^k-1}{2}}^{\frac{L^k-1}{2}} f(p+2\pi(\ell'+1)), \tag{3.83}$$

thus the shift amounts to relabelling the series, apart from the boundaries of summation. Hence, we have

$$F(p+2\pi)-F(p) = f(p+2\pi(\frac{(L^k-1)}{2}+1)) - f(p+2\pi(-\frac{(L^k-1)}{2})) = 0, \tag{3.84}$$

where the last equality follows from the periodicity of f with period $2\pi L^k$. Regarding the last statement, it suffices to note that $z\mapsto f(z)$ and $z\mapsto f(z+2\pi L^k)$ are two analytic functions on $S_{\infty,c_{\rm st}}$ which, by assumption, coincide on the real line. The same argument applies to F. □

Lemma 3.19. *Let $0<c\le a_k\le 1$. We have for $-\frac{(L^k-1)}{2}\le \ell'_\bullet\le \frac{(L^k-1)}{2}$, $p\in[-\pi,\pi[^{\times d}$,*

$$\left|\frac{1}{1+a_k\langle\langle u,u_\Delta\rangle\rangle(p)}u_\Delta(p+2\pi\ell')\right| \le \frac{c}{\prod_{\mu=0}^{d-1}(1+|\ell'_\mu|)^{1+2/d}}. \tag{3.85}$$

The bound remains true also for the analytic continuation of the function under the modulus on the l.h.s. of (3.85) to the strip $S_{\pi,c_{\rm st}} := \{\, z=p+\mathrm{i}q\in\mathbb{C}^d\,|\, p\in\,]-\pi,\pi[, |q|<c_{\rm st}\}$, where $0<c_{\rm st}\le 1$ is a constant depending only on d.

Proof. See Appendix F. □

The most tedious part of the proof is to ensure that $c_{\rm st}$ can be chosen uniformly in the lattice spacing (and other parameters). This aspect is left to the reader in [Ba83, (2.44)-(2.51)]. Some hints can be found in [BOS89] in a different context. We give a lengthy but self-contained proof in Appendix F. Here we explain only the basic idea in the case $\bar\mu_0=0$. To start with, we write

$$\frac{1}{1+a_k\langle\langle u,u_\Delta\rangle\rangle(p)}u_\Delta(p+2\pi\ell') = \frac{1}{\Delta^\eta(p)+a_k\langle\langle u,u_\Delta\rangle\rangle(p)\Delta^\eta(p)}\frac{\Delta^\eta(p)}{\Delta^\eta(p+2\pi\ell')}u(p+2\pi\ell'). \tag{3.86}$$

This regularizes $\Delta^\eta(p+2\pi\ell')$ for $\ell'=0$ at $p=0$. More importantly, it facilitates the analysis of the first denominator, which we call D. We have

$$D(p) := \Delta^\eta(p)+a_k\langle\langle u,u_\Delta\rangle\rangle(p)\Delta^\eta(p) = \Delta^\eta(p)+a_k\langle\langle u,u_\Delta\rangle\rangle_{\ell''\neq 0}(p)\Delta^\eta(p)+a_k|u(p)|^2, \tag{3.87}$$

where $\langle\langle u,u_\Delta\rangle\rangle_{\ell''\neq 0}$ denotes the omission of $\ell''=0$ term. It appears explicitly as $a_k|u(p)|^2$ and we note that for $p\in[-\pi,\pi[$ we have the bound $a_k|u(p)|\ge c>0$. By positivity of the remaining terms, we conclude that $D(p)\ge a_k|u(p)|^2\ge c>0$. Thus, by the Taylor theorem

$$|D(p+\mathrm{i}q)| = |D(p)+\mathrm{i}q\cdot\nabla D(p+\mathrm{i}q')| \ge c-|q||\nabla D(p+\mathrm{i}q')|. \tag{3.88}$$

So it suffices to show that $|\nabla D(z)|\le c'$, for z in a strip as in the statement of Lemma 3.19, to ensure that the denominator does not vanish there. The dependence of the r.h.s. of (3.85) on ℓ' requires a more careful analysis.

4 The method of images

In this section we decode formula (2.43) of [Ba83] and prove estimate (2.122). We will relate the propagator $G_k(\Omega)$ with Neumann boundary conditions to the propagator G_k with free boundary conditions using the method of images. To this end, we need some information about the behaviour of the averaging operators under projections:

Lemma 4.1. *The following properties hold*

$$P_\mu Q_k^* Q_k P_\mu = Q_k^* Q_k, \quad \overline{P}_\mu Q_k^* Q_k \overline{P}_\mu = Q_k^* Q_k, \quad T(z) Q_k^* Q_k T(z)^* = Q_k^* Q_k, \quad z \in \mathbb{Z}, \tag{4.1}$$

where the reflections $P_\mu, \overline{P}_\mu$, $\mu = 1, \ldots, d-1$, are defined in (3.6), (3.26) and translations $T(z)$ in (3.30). Consequently,

$$G_k(P_\mu x, P_\mu y) = G_k(\overline{P}_\mu x, \overline{P}_\mu y) = G_k(x+z, y+z) = G_k(x,y), \quad \mu = 0, \ldots, d-1. \tag{4.2}$$

Proof. We recall that Ω is a union of unit boxes $\triangle_y$, $y \in \Omega_k$, since we set $n-1 = L^m$, $m \geq k$. Thus P_μ, $\overline{P}_\mu$, defined in (3.6), (3.26), preserve the pattern of unit boxes $\triangle_y$, $y \in \mathbb{Z}^d$. We recall from Remark 2.8 that $Q_k^* Q_k$ is the orthogonal projection on the subspace of functions in $\mathcal{L}^2(\eta\mathbb{Z}^d)$ which are block-constant (i.e. constant on blocks $\triangle_y$, $y \in \mathbb{Z}^d$). By assumption, the reflections P_μ and $\overline{P}_\mu$ leave this subspace invariant. Since they are self-adjoint, they also leave the orthogonal complement invariant. Thus we can write

$$P_\mu Q_k^* Q_k P_\mu f = P_\mu Q_k^* Q_k P_\mu (f_{\mathrm{bc}} + f_{\mathrm{bc}}^\perp) = P_\mu P_\mu f_{\mathrm{bc}} = f_{\mathrm{bc}} = Q_k^* Q_k f, \tag{4.3}$$

where we decomposed f into the block-constant part and its orthogonal complement. The argument regarding $\overline{P}$ and $T(z)$ is analogous. □

Define a set of image points $\mathrm{Img} := \{y_j\}_{j \in \mathbb{N}}$, by the following two requirements (cf. [GJ][Section 7.4])

- $y \in \mathrm{Img}$,
- The set Img is invariant under the reflections $P_\bullet$ and $\overline{P}_\bullet$ defined in (3.26).

This set is depicted in Figure 1. The following relation between the Green functions with free and Neumann boundary conditions holds true:

Lemma 4.2. *For $x, y \in \Omega$ the following identity holds*

$$G_k(\Omega)(x,y) = \sum_{y_j \in \mathrm{Img}} G_k(x, y_j). \tag{4.4}$$

Proof. Given Lemma 3.16 and the distribution of points y_j, as in the figure, we obtain that the sum in (4.4) is convergent. It suffices to check that

$$(-\Delta_\Omega^\eta + a_k Q_{\Omega,k}^* Q_{\Omega,k})(\text{r.h.s. of (4.4)}) \ = 1. \tag{4.5}$$

Let $\tilde{I} := \eta[-1, 0, \ldots, n-1, n]$ and $\tilde{\Omega} := \tilde{I}^{\times d}$ be a slightly larger box than Ω. We will check that for each $y \in \Omega$ and $x \in \eta\mathbb{Z}^d$ the expression

$$F_y(x) := \chi_{\tilde{\Omega}}(x) \sum_{y_j \in \mathrm{Img}} G_k(x, y_j) \tag{4.6}$$

is an element of $D_\Omega \subset \mathcal{L}^2(\eta\mathbb{Z}^d)$ (cf. Definition 3.8). We will use the criterion from Lemma 3.10. Let $\tilde{P}_\bullet$ denote the reflections $P_\bullet$ or $\overline{P}_\bullet$. We consider $x \in \partial\Omega$, so $\chi_{\tilde{\Omega}}(x) = \chi_{\tilde{\Omega}}(\tilde{P}_\bullet x) = 1$. (We need $\chi_{\tilde{\Omega}}$ in (4.6) only to ensure that $x \mapsto F_y(x)$ is in $\mathcal{L}^2(\eta\mathbb{Z}^d)$). Then

$$F_y(\tilde{P}x) = \sum_{y_j \in \mathrm{Img}} G_k(\tilde{P}x, y_j) = \sum_{y_j \in \mathrm{Img}} G_k(\tilde{P}x, \tilde{P}y_j) = \sum_{y_j \in \mathrm{Img}} G_k(x, y_j) = F_y(x), \tag{4.7}$$

where in the second step we used the invariance of the set Img under the reflections and then we used $G_k(\tilde{P}x, \tilde{P}y) = G_k(x, y)$, from Lemma 4.1. Thus $F_y \in D_\Omega$ and we obtain from Lemma 3.9 and consistency of the free and Neumann averaging operators that, for $x \in \Omega$, $F_{y,\Omega} := F_y|_\Omega$,

$$
\begin{aligned}
&((-\Delta^\eta_\Omega + a_k Q^*_{\Omega,k} Q_{\Omega,k}) F_{y,\Omega})(x) = ((-\Delta^\eta + a_k Q^*_k Q_k) F_y)(x) \\
&= \sum_{y_j \in \mathrm{Img}, x' \in \eta\mathbb{Z}^d} \langle \delta^\eta_x, (-\Delta^\eta + a_k Q^*_k Q_k)\delta^\eta_{x'}\rangle \langle \delta^\eta_{x'}, \frac{1}{-\Delta^\eta + a_k Q^*_k Q_k} \delta^\eta_{y_j}\rangle = \sum_{y_j \in \mathrm{Img}} \langle \delta^\eta_x, \delta^\eta_{y_j}\rangle = \delta^\eta_y(x).
\end{aligned}
\tag{4.8}
$$

Here in the second step we used that for $x \in \Omega$ we have $\langle \delta^\eta_x, (-\Delta^\eta + a_k Q^*_k Q_k)\delta^\eta_{x'}\rangle = 0$ unless $x' \in \tilde{\Omega}$ (so we could get rid of the function $\chi_{\tilde{\Omega}}$) and in the last step we made use of the fact that y is the only element of Img inside Ω. Since $(y, x) \mapsto \delta^\eta_y(x)$ is the kernel of the identity operator on $\mathcal{L}^2(\Omega)$, we conclude that $F_{y,\Omega}(x) = G(\Omega)(x, y)$. $\square$

Now we can state and prove the main theorem of this section which confirms (2.122).

Theorem 4.3. *For $x \in \Omega$ and $y \in \Omega_k$ we have the following bound*

$$
|(G_k(\Omega) Q^*_{\Omega,k})(x, y)| \le c\mathrm{e}^{-c_1|x-y|}, \tag{4.9}
$$

where c, c_1, $c_1 > 0$, are constants depending only on d, in particular independent of the size of Ω.

Proof. We compute the kernel we are interested in using Lemma 4.2. We first note that, for any $g \in \mathcal{L}^2(\Omega)$,

$$
(G_k(\Omega) g)(x) = \eta^d \sum_j \sum_{w \in \Omega} G_k(x, w_j) g(w), \tag{4.10}
$$

where the first sum is over the image points and we remember that each w_j is a function of w. Now suppose that $g = Q^*_{\Omega,k} f$, $f \in \mathcal{L}^2(\Omega_k)$. That is

$$
g(w) = (Q^*_{\Omega,k} f)(w) = f(z_w), \tag{4.11}
$$

where $z_w \in \Omega_k$ is the element of the coarse lattice defined by $w \in B_k(z_w)$ (cf. (2.43)).

Thus (4.10), (4.11) give

$$
(G_k(\Omega) g)(x) = (G_k(\Omega) Q^*_{\Omega,k} f)(x) = \eta^d \sum_j \sum_{w \in \Omega} G_k(x, w_j) f(z_w). \tag{4.12}
$$

Now the kernel has the form

$$
\begin{aligned}
(G_k(\Omega) Q^*_{\Omega,k})(x, y) &= (G_k(\Omega) Q^*_{\Omega,k} \delta^1_y)(x) = \eta^d \sum_j \sum_{w \in \Omega} G_k(x, w_j) \delta^1_y(z_w) \\
&= \eta^d \sum_j \sum_{w \in \Omega} G_k(x, w_j) \delta_{z_w, y} = \eta^d \sum_j \sum_{w \in B_k(y)} G_k(x, w_j),
\end{aligned}
\tag{4.13}
$$

where $B_k(y)$ is the unit box with label y. We observe that, with $j = (j_0, \ldots, j_{d-1}) \in \mathbb{Z}^d$, $w_{j,\mu} = (L^m\eta)(j_\mu + 1) - w_\mu$ for j_μ odd and $w_{j,\mu} = (L^m\eta) j_\mu + w_\mu$ for j_μ even, $\mu = 0, 1, \ldots, d-1$ (cf. figure and recall that $L^m\eta$ is the linear size of Ω). We can thus write, using Lemma 4.1

$$
(G_k(\Omega) Q^*_{\Omega,k})(x, y) = \eta^d \sum_j \sum_{w \in B_k(y)} G_k((\ldots, (L^m\eta)(j_\mu + 1) - x_\mu, \ldots, x_\nu - (L^m\eta) j_\nu, \ldots), w), \tag{4.14}
$$

where j_μ is odd and j_ν is even in (4.14).

Now we compute the kernel

$$
\begin{aligned}
(G_k Q_k^*)(x,y) &= \eta^d \sum_{w\in\eta\mathbb{Z}^d} G_k(x,w) Q_k^* \delta_y^1(w) \\
&= \eta^d \sum_{w\in\eta\mathbb{Z}^d} G_k(x,w)\delta_{z_w,y} = \eta^d \sum_{w\in B_k(y)} G_k(x,w), \qquad (4.15)
\end{aligned}
$$

whose exponential decay we know from Lemma 3.17. (In the last step we used $x\in\Omega, y\in\Omega_k$). Hence, (4.14) gives

$$
(G_k(\Omega)Q_{\Omega,k}^*)(x,y) = \sum_j (G_k Q_k^*)((\ldots, (L^m\eta)(j_\mu+1) - x_\mu, \ldots, x_\nu - (L^m\eta)j_\nu, \ldots), y). \qquad (4.16)
$$

From this formula and Lemma 3.17 the statement of the theorem is relatively clear. Thus we postpone further details to Appendix G. □

A Proof of Lemma 2.12

In the proof we will write $\partial_\mu^\eta := \partial_{\mu,\Omega}^\eta$, $\Delta^{\eta,\bar{\mu}_k} := \Delta_\Omega^{\eta,\bar{\mu}_k}$ and $Q_k := Q_{k,\Omega}$ for brevity. Let us start with some preparations. Define an operator e_q by its action on functions $f\in\mathcal{L}^2(\Omega)$ as

$$
(\mathrm{e}_q f)(x) = \mathrm{e}^{q\cdot x} f(x), \qquad (A.1)
$$

where $q\in\mathbb{R}^2$ is a vector s.t. $|q|\le 1$. Then we compute, using Lemma 2.1,

$$
\begin{aligned}
((\partial_\mu^\eta)_q f)(x) := \mathrm{e}_{-q}\{\partial_\mu^\eta(f\mathrm{e}_q)\}(x) &= \mathrm{e}_{-q}(x)(\partial_\mu^\eta f)(x)\mathrm{e}_q(x+\eta e_\mu) + \mathrm{e}_{-q}(x)(\partial_\mu^\eta \mathrm{e}_q)(x) f(x) \\
&= \frac{\mathrm{e}^{q_\mu\eta}-1}{\eta} f(x) + \mathrm{e}^{\eta q_\mu}(\partial_\mu^\eta f)(x) \\
&= q_\mu E_{q_\mu} f(x) + \mathrm{e}^{\eta q_\mu}(\partial_\mu^\eta f)(x), \qquad (A.2)
\end{aligned}
$$

where $E_{q_\mu} := \int_0^1 ds\, \mathrm{e}^{sq_\mu\eta}$ is independent of x. Since $|q|\le 1$, we have $|E_{q_\mu}|\le \mathrm{e}$. (We note that it is helpful for estimate (A.10) below to have the shift $(\,\cdot\,+\eta e_\mu)$ in e_q and not in f). Recall that for $y\in\Omega_k$

$$
(Q_k f)(y) = \frac{1}{L^{kd}} \sum_{x\in B_k(y)} f(x). \qquad (A.3)
$$

Define

$$
(Q_{k,q} f)(y) := (\mathrm{e}_{-q} Q_k \mathrm{e}_q f)(y) = \frac{1}{L^{kd}} \sum_{x\in B_k(y)} \mathrm{e}^{q\cdot(x-y)} f(x). \qquad (A.4)
$$

We will also need that $(Q_k^* f)(x) = f(y)$ if $x\in B_k(y)$, hence

$$
(Q_{k,-q}^* f)(x) = (\mathrm{e}_{-q} Q_k^* \mathrm{e}_q f)(x) = \mathrm{e}^{q\cdot(y-x)} f(y), \qquad (A.5)
$$

where in the second line we use (A.4).

After this preparation we move on to the proof of Lemma 2.12. Let

$$
\mathrm{D}_q := \mathrm{e}_{-q}\big[-\Delta^{\eta,\bar{\mu}_k} + a_k Q_k^* Q_k\big]\mathrm{e}_q. \qquad (A.6)
$$

We claim that there exists a constant c' such that for $f\in\mathcal{L}^2(\Omega)$

$$
|\langle f, [\mathrm{D}_q - \mathrm{D}_0] f\rangle| \le c'|q|\langle f, (-\Delta^\eta + 1) f\rangle. \qquad (A.7)
$$

First consider the term $\langle f, [\mathrm{e}_{-q} Q_k^* Q_k \mathrm{e}_q - Q_k^* Q_k] f\rangle$. From (A.4) and (A.5) this can be written as $\langle f, [Q_{k,-q}^* Q_{k,q} - Q_k^* Q_k] f\rangle$. We estimate

$$\begin{aligned}
\|(Q_{k,q}-Q_k)f\|_2^2 &= \sum_{y\in\Omega_k} |\frac{1}{L^{kd}} \sum_{x\in B_k(y)} [\mathrm{e}^{q\cdot(x-y)}-1] f(x)|^2 \\
&\le cq^2 \sum_{y\in\Omega_k} \frac{1}{L^{kd}} \sum_{x\in B_k(y)} \frac{1}{L^{kd}} \sum_{x'\in B_k(y)} (f(x)^2 + f(x')^2) \\
&\le cq^2 \frac{1}{L^{kd}} \sum_{x\in\Omega} f(x)^2 \le cq^2 \|f\|_2^2, \qquad (A.8)
\end{aligned}$$

where the sum over y and the sums over the boxes combined to the sum over the entire lattice. We used that $|x-y| \le \sqrt{d}$, if x in the unit box with label y. Estimate (A.8) gives

$$|\langle f, [\mathrm{e}_{-q} Q_k^* Q_k \mathrm{e}_q - Q_k^* Q_k] f\rangle| \le c|q| \|f\|_2^2, \qquad (A.9)$$

where we applied the Cauchy-Schwarz inequality.

The remaining part is estimated as follows

$$\begin{aligned}
&|\langle f, [\mathrm{e}_{-q}(-\Delta^{\eta,\bar{\mu}_k})\mathrm{e}_q - (-\Delta^{\eta,\bar{\mu}_k})] f\rangle| \\
&\quad = |\langle \mathrm{e}_q \partial_\mu^\eta (\mathrm{e}_{-q} f), \mathrm{e}_{-q} \partial_\mu^\eta (\mathrm{e}_q f)\rangle - \langle \partial_\mu^\eta f, \partial_\mu^\eta f\rangle| \\
&\quad = |\langle (-q_\mu) E_{(-q_\mu)} f + \mathrm{e}^{-\eta q_\mu} (\partial_\mu^\eta f), q_\mu E_{q_\mu} f + \mathrm{e}^{\eta q_\mu} (\partial_\mu^\eta f)\rangle - \langle \partial_\mu^\eta f, \partial_\mu^\eta f\rangle| \\
&\quad \le c|q| \langle f, (-\Delta^\eta + 1) f\rangle|, \qquad (A.10)
\end{aligned}$$

where summation over μ is understood. We used here (2.13), (A.2), $|q|^2 \le |q|$ and $|\langle \partial_\mu^\eta f, f\rangle| \le \frac{1}{2}\langle f, (-\Delta^\eta + 1) f\rangle|$. Using (A.10) and (A.9) we check (A.7).

Recall from Lemma 2.10 that $\langle f, \mathrm{D}_0 f\rangle \ge c_0 \langle f, (-\Delta^\eta + 1) f\rangle$, where $\mathrm{D}_0 := (-\Delta^{\eta,\bar{\mu}_k} + a_k Q_k^* Q_k)$. Thus we can write, choosing q s.t. $c'|q| \le c_0/2$, where c' appeared in (A.7),

$$\begin{aligned}
|\langle f, \mathrm{D}_q f\rangle| &\ge \langle f, \mathrm{D}_0 f\rangle - |\langle f, (\mathrm{D}_q - \mathrm{D}_0) f\rangle| \\
&\ge c_0 \ |\langle f, (-\Delta^\eta + 1) f\rangle| - \frac{c_0}{2} |\langle f, (-\Delta^\eta + 1) f\rangle| \\
&= \frac{1}{2} c_0 |\langle f, (-\Delta^\eta + 1) f\rangle| \ge \frac{1}{2} c_0 \|f\|_2^2. \qquad (A.11)
\end{aligned}$$

We note that D_q is invertible as a composition of invertible mappings. Substituting $f = \mathrm{D}_q^{-1} h$ to (A.11) we get

$$\frac{1}{2} c_0 \|\mathrm{D}_q^{-1} h\|_2^2 \le \langle \mathrm{D}_q^{-1} h, h\rangle \le \|h\|_2 \|\mathrm{D}_q^{-1} h\|_2 \quad \Rightarrow \quad \|\mathrm{D}_q^{-1} h\|_2 \le 2c_0^{-1} \|h\|_2. \qquad (A.12)$$

Since $\mathrm{D}_q^{-1} = \mathrm{e}_{-q} G_k(\Omega) \mathrm{e}_q$ this reads

$$\|\mathrm{e}_{-q} G_k(\Omega) \mathrm{e}_q h\|_2 \le c\|h\|_2. \qquad (A.13)$$

Now let $c_1 := \min\{\frac{1}{2}(c')^{-1} c_0, 1\}$. Then, for $|q| \le c_0/(2c')$ as specified above (A.11), and $\mathrm{supp}(f) \subset \triangle_y$, $\mathrm{supp}(f') \subset \triangle_{y'}$

$$\begin{aligned}
|\langle f, G_k(\Omega) f'\rangle| &= |\langle \mathrm{e}_q f, (\mathrm{e}_{-q} G_k(\Omega) \mathrm{e}_q) \mathrm{e}_{-q} f'\rangle| \\
&\le c\|\mathrm{e}_q f\|_2 \, \|\mathrm{e}_{-q} f'\|_2 \\
&\le c'' \mathrm{e}^{q\cdot(y-y')} \|f\|_2 \|f'\|_2. \qquad (A.14)
\end{aligned}$$

Taking $q = c_1 \frac{-(y-y')}{|y-y'|}$ gives the result.

B Proof of Lemma 2.16

The computation is analogous as in the proof of [Di13, Lemma 2]. The statement follows from the fact that a convolution of two Gaussian functions is again a Gaussian. It suffices to check $T^{L^j\eta}_{a,L}T^{\eta}_{a_j,L^j} = T^{\eta}_{a_{j+1},L^{j+1}}$ as (2.77) follows by iteration. We write

$$
\begin{aligned}
(T^{L^j\eta}_{a,L}T^{\eta}_{a_j,L^j})[\Omega,\rho](\psi') &= \left(\frac{b_1^{L^j\eta}}{2\pi}\right)^{\frac{|\Omega_{j+1}|}{2}} \int d\psi\, \mathrm{e}^{-\frac{1}{2}b_1^{L^j\eta}\sum_{y'\in\Omega_{j+1}}|\psi'(y')-(Q_{\Omega_j}\psi)(y')|^2}(T^{\eta}_{a_j,L^j}\rho)(\psi)\\
&= \left(\frac{b_1^{L^j\eta}}{2\pi}\right)^{\frac{|\Omega_{j+1}|}{2}} \int d\psi\, \mathrm{e}^{-\frac{1}{2}b_1^{L^j\eta}\sum_{y'\in\Omega_{j+1}}|\psi'(y')-(Q_{\Omega_j}\psi)(y')|^2}\\
&\quad\times\left(\frac{b_j^{\eta}}{2\pi}\right)^{\frac{|\Omega_j|}{2}} \int d\phi\, \mathrm{e}^{-\frac{1}{2}b_j^{\eta}\sum_{y\in\Omega_j}|\psi(y)-(Q_{\Omega,j}\phi)(y)|^2}\rho(\phi).
\end{aligned}
\tag{B.1}
$$

The problem boils down to computing the integral

$$
\int d\psi\, \mathrm{e}^{-\frac{1}{2}b_1^{L^j\eta}\sum_{y'\in\Omega_{j+1}}|\psi'(y')-(Q_{\Omega_j}\psi)(y')|^2}\mathrm{e}^{-\frac{1}{2}b_j^{\eta}\sum_{y\in\Omega_j}|\psi(y)-(Q_{\Omega,j}\phi)(y)|^2}. \tag{B.2}
$$

We denote the expression in the exponential by

$$
F(\psi) := \frac{1}{2}b_1^{L^j\eta}\sum_{y'\in\Omega_{j+1}}|\psi'(y')-(Q_{\Omega_j}\psi)(y')|^2 + \frac{1}{2}b_j^{\eta}\sum_{y\in\Omega_j}|\psi(y)-(Q_{\Omega,j}\phi)(y)|^2. \tag{B.3}
$$

We will find the minimum ψ_0 and then expand $\psi = \psi_0 + \tilde{\psi}$. Since the linear terms vanish, we get

$$
F(\psi) = F(\psi_0) + \frac{1}{2}\sum_{\tilde{y},\tilde{y}'}F''(\psi_0)(\tilde{y},\tilde{y}')\tilde{\psi}(\tilde{y})\tilde{\psi}(\tilde{y}'), \quad F''(\psi_0)(\tilde{y},\tilde{y}') := \partial_{\psi(\tilde{y})}\partial_{\psi(\tilde{y}')}F(\psi)|_{\psi=\psi_0}. \tag{B.4}
$$

Then we will obtain, referring to (D.1),

$$
(B.2) = \int d\psi\, \mathrm{e}^{-F(\psi)} = \mathrm{e}^{-F(\psi_0)}\int d\tilde{\psi}\, \mathrm{e}^{-\frac{1}{2}\sum_{\tilde{y},\tilde{y}'}F''(\psi_0)(\tilde{y},\tilde{y}')\tilde{\psi}(\tilde{y})\tilde{\psi}(\tilde{y}')} = \mathrm{e}^{-F(\psi_0)}\sqrt{\frac{(2\pi)^{|\Omega_j|}}{\det(F'')}}, \tag{B.5}
$$

provided that $F''(\psi_0)$ is invertible. (The latter property can can be read off from (B.8) below).

We compute the derivative of F

$$
\frac{\partial}{\partial\psi(\tilde{y})}(Q_{\Omega_j}\psi)(y') = \frac{\partial}{\partial\psi(\tilde{y})}\frac{1}{L^d}\sum_{y'_\bullet\le z_\bullet<y'_\bullet+L(L^j\eta)}\psi(z) = \frac{1}{L^d}\sum_{y'_\bullet\le z_\bullet<y'_\bullet+L(L^j\eta)}\delta_{z,\tilde{y}} = \frac{1}{L^d}\mathbb{1}_{B_{j+1}(y')}(\tilde{y}), \tag{B.6}
$$

where $\mathbb{1}_{B_{j+1}(y')}$ is the characteristic function of the L-box $B_{j+1}(y')$ in the lattice Ω_j. Hence,

$$
\begin{aligned}
\frac{\partial}{\partial\psi(\tilde{y})}F(\psi) &= -b_1^{L^j\eta}\sum_{y'\in\Omega_{j+1}}\big(\psi'(y')-(Q_{\Omega_j}\psi)(y')\big)\cdot\frac{1}{L^d}\mathbb{1}_{B_{j+1}(y')}(\tilde{y}) + b_j^{\eta}\sum_{y\in\Omega_j}\big(\psi(y)-(Q_{\Omega,j}\phi)(y)\big)\cdot\delta_{y,\tilde{y}}\\
&= -\frac{b_1^{L^j\eta}}{L^d}\big(\psi'(y'_{\tilde{y}})-(Q_{\Omega_j}\psi)(y'_{\tilde{y}})\big) + b_j^{\eta}\big(\psi(\tilde{y})-(Q_{\Omega,j}\phi)(\tilde{y})\big).
\end{aligned}
\tag{B.7}
$$

Thus at the minimum ψ_0 we have, using (2.43),

$$
-\frac{b_1^{L^j\eta}(b_j^{\eta})^{-1}}{L^d}\big(Q^*\psi' - Q^*Q\psi_0\big) + \psi_0 - Q_j\phi = 0, \tag{B.8}
$$

$$
\big(1 + \frac{b_1^{L^j\eta}(b_j^{\eta})^{-1}}{L^d}Q^*Q\big)\psi_0 = \frac{b_1^{L^j\eta}(b_j^{\eta})^{-1}}{L^d}Q^*\psi' + Q_j\phi. \tag{B.9}
$$

We note that for any projection P we have, for $|b|<1$,

$$\frac{1}{1+bP} = ((1+b)^{-1}-1)P+1 = -\frac{b}{1+b}P+1, \tag{B.10}$$

cf. (D.10) below. (In our case $b := \frac{b_1^{L^j\eta}(b_j^\eta)^{-1}}{L^d} < 1$ holds, because $L>1$). Thus we get

$$\begin{aligned}\psi_0 &= \big(-\frac{b^2}{1+b}+b\big)Q^*\psi' - \frac{b}{1+b}Q^*Q_{j+1}\phi + Q_j\phi \\ &= \frac{b}{1+b}Q^*\psi' - \frac{b}{1+b}Q^*Q_{j+1}\phi + Q_j\phi.\end{aligned} \tag{B.11}$$

We compute

$$\begin{aligned}\|\psi' - Q\psi_0\|^2_{2,\Omega_{j+1}} &= \|\psi' - \frac{b}{1+b}\psi' + \frac{b}{1+b}Q_{j+1}\phi - Q_{j+1}\phi\|^2_{2,\Omega_{j+1}} \\ &= \frac{1}{(1+b)^2}\|\psi' - Q_{j+1}\phi\|^2_{2,\Omega_{j+1}},\end{aligned} \tag{B.12}$$

$$\|\psi_0 - Q_j\phi\|^2_{2,\Omega_j} = \frac{b^2}{(1+b)^2}\|Q^*\psi' - Q^*Q_{j+1}\phi\|^2_{2,\Omega_j} = \frac{b^2}{(1+b)^2}\|\psi' - Q_{j+1}\phi\|^2_{2,\Omega_{j+1}}. \tag{B.13}$$

Now we write, using (B.12), (B.13),

$$\begin{aligned}F(\psi_0) &= \frac{1}{2}b_1^{L^j\eta}(L^{j+1}\eta)^{-d}\|\psi' - Q\psi_0\|^2_{2,\Omega_{j+1}} + \frac{1}{2}b_j^\eta(L^j\eta)^{-d}\|\psi_0 - Q_j\phi\|^2_{2,\Omega_j} \\ &= \frac{1}{2}\frac{1}{(1+b)^2}(L^j\eta)^{-d}\big(b_1^{L^j\eta}L^{-d} + b_j^\eta b^2\big)\|\psi' - Q_{j+1}\phi\|^2_{2,\Omega_{j+1}} \\ &= \frac{1}{2}\frac{1}{(1+b)^2}\big(b_1^{L^j\eta} + L^d b_j^\eta b^2\big)\sum_{y'\in\Omega_{j+1}}|\psi'(y') - Q_{j+1}\phi(y')|^2.\end{aligned} \tag{B.14}$$

We have that, with $b := \frac{b_1^{L^j\eta}(b_j^\eta)^{-1}}{L^d}$, $b_1^{L^j\eta} := a_1(L^{j+1}\eta)^{d-2}$, $b_j^\eta := a_j(L^j\eta)^{d-2}$

$$\begin{aligned}\frac{1}{2}\frac{1}{(1+b)^2}\big(b_1^{L^j\eta} + L^d b_j^\eta b^2\big) &= \frac{1}{2}\frac{1}{(1+b)^2}b_1^{L^j\eta}(1+b) = \frac{1}{2}\frac{1}{b_j^\eta + \frac{b_1^{L^j\eta}}{L^d}}b_j^\eta b_1^{L^j\eta} \\ &= \frac{1}{2}\frac{a_1a_j(L^{j+1}\eta)^{d-2}(L^j\eta)^{d-2}}{a_j(L^j\eta)^{d-2} + a_1(L^{j+1}\eta)^{d-2}/L^d} \\ &= \frac{1}{2}(L^{j+1}\eta)^{d-2}\frac{a_1a_j}{a_j + a_1/L^2} = \frac{1}{2}(L^{j+1}\eta)^{d-2}a_{j+1}.\end{aligned} \tag{B.15}$$

This, together with (B.14), (B.5) concludes the proof.

C Proof of Lemma 2.17.

We write

$$\begin{aligned}\rho_j(\psi) &= T^\eta_{a_j,L^j}[\Omega,\rho_0](\psi) \\ &= \left(\frac{b_j^\eta}{2\pi}\right)^{\frac{|\Omega_j|}{2}}\int d\phi\, \mathrm{e}^{-\frac{1}{2}b_j^\eta\sum_{y\in\Omega_j}|\psi(y)-(Q_{\Omega,j}\phi)(y)|^2}\rho_0(\phi) \\ &= \left(\frac{b_j^\eta}{2\pi}\right)^{\frac{|\Omega_j|}{2}}\int d\phi\, \mathrm{e}^{-\frac{1}{2}b_j^\eta\sum_{y\in\Omega_j}|\psi(y)-(Q_{\Omega,j}\phi)(y)|^2}\mathrm{e}^{-\frac{1}{2}\langle\phi,\Delta^{(0),\eta}(\Omega)\phi\rangle}\mathrm{e}^{\mathrm{i}\langle J,\phi\rangle}.\end{aligned} \tag{C.1}$$

We define the function

$$F(\phi) := \frac{1}{2} b_j^\eta \sum_{y \in \Omega_j} |\psi(y) - (Q_{\Omega,j}\phi)(y)|^2 + \frac{1}{2}\eta^d \sum_{x\in\Omega} \phi(x)(\Delta^{(0),\eta}(\Omega)\phi)(x) \tag{C.2}$$

and compute the derivatives (cf. (B.7))

$$\begin{aligned} \frac{\partial}{\partial\phi(\tilde{x})} F(\phi) &= -\frac{b_j^\eta}{L^{jd}}((Q^*_{\Omega,j}\psi)(\tilde{x}) - (Q^*_{\Omega,j}Q_{\Omega,j}\phi)(\tilde{x})) + \eta^d(\Delta^{(0),\eta}(\Omega)\phi)(\tilde{x}), & \text{(C.3)}\\ \frac{\partial}{\partial\phi(\tilde{x})}\frac{\partial}{\partial\phi(\tilde{x}')} F(\phi) &= [\frac{b_j^\eta}{L^{jd}}\eta^d Q^*_{\Omega,j}Q_{\Omega,j} + \eta^{2d}\Delta^{(0),\eta}(\Omega)](\tilde{x},\tilde{x}') = \eta^{2d}[(G_j^\eta(\Omega))^{-1}](\tilde{x},\tilde{x}'), & \text{(C.4)}\end{aligned}$$

where we made use of (2.2). Thus the first derivative of F vanishes at

$$\phi_0 = \frac{b_j^\eta}{(L^j\eta)^d} G_j^\eta(\Omega) Q^*_{\Omega,j}\psi = \frac{a_j}{(L^j\eta)^2} G_j^\eta(\Omega) Q^*_{\Omega,j}\psi =: \mathcal{H}_j^\eta(\Omega)\psi. \tag{C.5}$$

We can write $\phi = \phi_0 + \tilde{\phi}$, which gives

$$F(\phi) = F(\phi_0) + \frac{1}{2}\langle\tilde{\phi}, (G_j^\eta(\Omega))^{-1}\tilde{\phi}\rangle \tag{C.6}$$

so that, referring to (D.1),

$$\begin{aligned}\int d\phi\, \mathrm{e}^{-F(\phi)}\mathrm{e}^{\mathrm{i}\langle J,\phi\rangle} &= \mathrm{e}^{-F(\phi_0)}\mathrm{e}^{\mathrm{i}\langle J,\phi_0\rangle}\int d\tilde{\phi}\, \mathrm{e}^{-\frac{1}{2}\langle\tilde{\phi},(G_j^\eta(\Omega))^{-1}\tilde{\phi}\rangle}\mathrm{e}^{\mathrm{i}\langle J,\tilde{\phi}\rangle}\\ &= \mathrm{e}^{-F(\phi_0)}\mathrm{e}^{\mathrm{i}\langle J,\phi_0\rangle}\sqrt{(2\pi)^{|\Omega|}\det(G_j^\eta(\Omega))}\mathrm{e}^{-\frac{1}{2}\langle J, G_j^\eta(\Omega)J\rangle}. \end{aligned}\tag{C.7}$$

Now to determine $\Delta^{(j),L^j\eta}(\Omega)$ it suffices to compute for $\phi_0 = \frac{a_j}{(L^j\eta)^2}G_j^\eta(\Omega)Q^*_{\Omega,j}\psi$

$$\begin{aligned} F(\phi_0) &= \frac{1}{2}a_j(L^j\eta)^{-2}\|\psi - Q_{\Omega,j}\phi_0\|^2_{2,\Omega_j} + \frac{1}{2}\langle\phi_0, \Delta^{(0),\eta}(\Omega)\phi_0\rangle_\Omega\\ &= \frac{1}{2}a_j(L^j\eta)^{-2}\|\psi - \frac{a_j}{(L^j\eta)^2}Q_{\Omega,j}G_j^\eta(\Omega)Q^*_{\Omega,j}\psi\|^2_{2,\Omega_j}\\ &\quad + \frac{1}{2}\frac{a_j^2}{(L^j\eta)^4}\langle G_j^\eta(\Omega)Q^*_{\Omega,j}\psi, \Delta^{(0),\eta}(\Omega)G_j^\eta(\Omega)Q^*_{\Omega,j}\psi\rangle_\Omega\\ &= \frac{1}{2}a_j(L^j\eta)^{-2}\|\psi\|^2_{2,\Omega_j} - \left(\frac{a_j}{(L^j\eta)^2}\right)^2\langle\psi, Q_{\Omega,j}G_j^\eta(\Omega)Q^*_{\Omega,j}\psi\rangle_{\Omega_j}\\ &\quad + \frac{1}{2}\big(a_j(L^j\eta)^{-2}\big)^3\|Q_{\Omega,j}G_j^\eta(\Omega)Q^*_{\Omega,j}\psi\|^2_{2,\Omega_j}\\ &\quad + \frac{1}{2}\left(\frac{a_j}{(L^j\eta)^2}\right)^2\langle G_j^\eta(\Omega)Q^*_{\Omega,j}\psi, \Delta^{(0),\eta}(\Omega)G_j^\eta(\Omega)Q^*_{\Omega,j}\psi\rangle_\Omega, \end{aligned}\tag{C.8}$$

where we refer to (2.78) and (2.82) for definitions of $\Delta^{(0),\eta}(\Omega)$ and $G_j^\eta(\Omega)$.

Now in the last expression we write $\Delta^{(0),\eta}(\Omega) = G_j^\eta(\Omega)^{-1} - a_j(L^j\eta)^{-2}Q^*_{\Omega,j}Q_{\Omega,j}$ which leads to

$$F(\phi_0) = \frac{1}{2}a_j(L^j\eta)^{-2}\|\psi\|^2_{2,\Omega_j} - \frac{1}{2}\left(\frac{a_j}{(L^j\eta)^2}\right)^2\langle\psi, Q_{\Omega,j}G_j^\eta(\Omega)Q^*_{\Omega,j}\psi\rangle_{\Omega_j} = \frac{1}{2}\langle\psi, \Delta^{(j),L^j\eta}(\Omega)\psi\rangle_{\Omega_j}, \tag{C.9}$$

which concludes the proof.

D A formula for $C^{(j),L^j\eta}(\Omega)$

We repeat here the discussion from [Di13, Appendix C] slightly adapted to our situation. We start by recalling the standard Gaussian integral formula

$$\int_{\mathbb{R}^M} d\phi\, \mathrm{e}^{-\frac{1}{2}\langle\phi,C\phi\rangle+\langle J,\phi\rangle} = \sqrt{\frac{(2\pi)^M}{\det C}}\mathrm{e}^{\frac{1}{2}\langle\overline{J},C^{-1}J\rangle} \tag{D.1}$$

valid for any positive definite $M\times M$ matrix C and a complex-valued vector J. We use the short-hand notations valid only for this appendix

$$G_j^\eta := G_j^\eta(\Omega), \quad C^{(j)} := C^{(j),L^j\eta}(\Omega), \quad Q_j := Q_{\Omega,j}, \quad \Delta^{(j)} := \Delta^{(j),L^j\eta}(\Omega), \quad \tilde{a}_{j,\ell} := a_j(L^\ell\eta)^{-2}. \tag{D.2}$$

We recall that

$$\Delta^{(j)} := [\tilde{a}_{j,j} - \tilde{a}_{j,j}^2 Q_j G_j^\eta Q_j^*]_{\Omega_j}, \quad G_j^\eta := [-\Delta^\eta + \bar{\mu}_k + \tilde{a}_{j,j}Q_j^*Q_j]_\Omega^{-1}. \tag{D.3}$$

Lemma D.1 below gives a formula for

$$C^{(j)} := [\Delta^{(j)} + \frac{\tilde{a}_{1,j}}{L^2}Q^*Q]_{\Omega_j}^{-1} = \big[\tilde{a}_{j,j} - \tilde{a}_{j,j}^2 Q_j G_j Q_j^* + \frac{\tilde{a}_{1,j}}{L^2}Q^*Q\big]_{\Omega_j}^{-1} \tag{D.4}$$

which is used in the proof of Theorem 2.18.

Lemma D.1. *We have*

$$C^{(j)} = A_j + \tilde{a}_{j,j}^2 A_j Q_j G_{j+1}^\eta Q_j^* A_j, \tag{D.5}$$

where

$$A_j := [\tilde{a}_{j,j} + \tilde{a}_{1,j}L^{-2}Q^*Q]_{\Omega_j}^{-1}, \quad G_{j+1}^\eta := \big[-\Delta^\eta + \bar{\mu}_k + \frac{\tilde{a}_{j+1,j}}{L^2}Q_{j+1}^*Q_{j+1}\big]_\Omega^{-1}. \tag{D.6}$$

Proof. Using definition (D.4) and the Gaussian integration formula (D.1), we obtain for a suitable normalization constant Z_j:

$$\exp\left(\frac{1}{2}\langle f, C^{(j)}f\rangle\right) = Z_j\int d\Phi\,\exp\left(\langle\Phi,f\rangle - \frac{\tilde{a}_{1,j}}{2L^2}\|Q\Phi\|_2^2 - \frac{1}{2}\langle\Phi,\Delta^{(j)}\Phi\rangle\right). \tag{D.7}$$

Apart from this, we checked in Lemma 2.17 that

$$\exp(-\frac{1}{2}\langle\Phi,\Delta^{(j)}\Phi\rangle) = Z_j\int d\phi\exp\left(-\frac{\tilde{a}_{j,j}}{2}\|\Phi - Q_j\phi\|_2^2 - \frac{1}{2}\langle\phi,(-\Delta_\Omega^\eta+\bar{\mu}_k)\phi\rangle\right). \tag{D.8}$$

(We note that this expression is proportional to $\rho_j^{J=0}(\Phi)$). Inserting this in (D.7), we obtain

$$\begin{aligned}
&\exp\left(\frac{1}{2}\langle f, C^{(j)}f\rangle\right)\\
&= Z_j\int d\phi\,d\Phi\,\exp\left(\langle\Phi,f\rangle - \frac{\tilde{a}_{1,j}}{2L^2}\|Q\Phi\|_2^2 - \frac{\tilde{a}_{j,j}}{2}\|\Phi-Q_j\phi\|_2^2 - \frac{1}{2}\langle\phi,(-\Delta_\Omega^\eta+\bar{\mu}_k)\phi\rangle\right)\\
&= Z_j\int d\phi\,d\Phi\,\exp\bigg(\langle\Phi, f+\tilde{a}_{j,j}Q_j\phi\rangle - \frac{1}{2}\langle\Phi,\big(\tilde{a}_{j,j}+\tilde{a}_{1,j}L^{-2}Q^*Q\big)\Phi\rangle\\
&\qquad\qquad - \frac{\tilde{a}_{j,j}}{2}\|Q_j\phi\|_2^2 - \frac{1}{2}\langle\phi,(-\Delta_\Omega^\eta+\bar{\mu}_k)\phi\rangle\bigg)\\
&= Z_j'\int d\phi\exp\left(\frac{1}{2}\langle(f+\tilde{a}_{j,j}Q_j\phi),A_j(f+\tilde{a}_{j,j}Q_j\phi)\rangle - \frac{\tilde{a}_{j,j}}{2}\|Q_j\phi\|_2^2 - \frac{1}{2}\langle\phi,(-\Delta_\Omega^\eta+\bar{\mu}_k)\phi\rangle\right),
\end{aligned} \tag{D.9}$$

where Z'_j is a new normalization constant. Let us now compute $A_j := (\tilde{a}_{j,j} + \tilde{a}_{1,j}L^{-2}Q^*Q)^{-1}$. We note that if P is a projection, $a, b > 0$, $a^{-1}b < 1$, then

$$\begin{aligned}(a+bP)^{-1} &= a^{-1}(1+a^{-1}bP)^{-1} = a^{-1}\sum_{\ell=0}^{\infty}(a^{-1}b)^{\ell}P^{\ell} \\ &= a^{-1}I + a^{-1}\Big(\frac{1}{1+a^{-1}b} - 1\Big)P = a^{-1}(I-P) + \frac{1}{a+b}P.\end{aligned} \tag{D.10}$$

For $a = \tilde{a}_{j,j}$, $b = \tilde{a}_{1,j}L^{-2}$ we have $a^{-1}b = \tilde{a}_{j,j}^{-1}\tilde{a}_{1,j}L^{-2} < 1$ by (D.2), (2.75). Thus (D.10) gives

$$A_j = \frac{1}{\tilde{a}_{j,j}} + \Big(\frac{1}{\tilde{a}_{j,j} + \tilde{a}_{1,j}L^{-2}} - \frac{1}{\tilde{a}_{j,j}}\Big)Q^*Q = \frac{1}{\tilde{a}_{j,j}} - \frac{\tilde{a}_{j+1,j}L^{-2}}{\tilde{a}_{j,j}^2}Q^*Q. \tag{D.11}$$

Consequently

$$\begin{aligned}&\exp\Big(\frac{1}{2}\langle f, C^{(j)}f\rangle\Big) \\ &= Z'_j\int d\phi\,\exp\Big(\frac{1}{2}\langle(f+\tilde{a}_{j,j}Q_j\phi), A_j(f+\tilde{a}_{j,j}Q_j\phi)\rangle - \frac{\tilde{a}_{j,j}}{2}\|Q_j\phi\|_2^2 - \frac{1}{2}\langle\phi,(-\Delta_\Omega^\eta+\bar{\mu}_k)\phi\rangle\Big) \\ &= Z'_j\exp\Big(\frac{1}{2}\langle f, A_jf\rangle\Big)\int d\phi\,\exp\Big(\frac{1}{2}\langle\tilde{a}_{j,j}Q_jA_jf,\phi\rangle\Big)\exp\Big(-\frac{1}{2}\langle\phi,(-\Delta_\Omega^\eta+\bar{\mu}_k+\frac{\tilde{a}_{j+1,j}}{L^2}Q_{j+1}^*Q_{j+1})\phi\rangle\Big) \\ &= Z''_j\exp\Big(\frac{1}{2}\langle f, A_jf\rangle + \frac{\tilde{a}_{j,j}^2}{2}\langle f, A_jQ_jG_{j+1}^\eta Q_j^*A_jf\rangle\Big),\end{aligned} \tag{D.12}$$

where we noted that, given (D.11),

$$\big(-\Delta_\Omega^\eta + \bar{\mu}_k + \tilde{a}_{j,j}Q_j^*Q_j - \tilde{a}_{j,j}^2Q_j^*A_jQ_j\big) = \big(-\Delta_\Omega^\eta + \bar{\mu}_k + \frac{\tilde{a}_{j+1,j}}{L^2}Q_{j+1}^*Q_{j+1}\big) = (G_{j+1}^\eta)^{-1}. \tag{D.13}$$

Setting $f = 0$ in (D.12), we obtain $Z''_j = 1$. Then (D.5) follows. □

E Proof of Lemma 3.13

We compute the Fourier transform of (2.45) first for $f \in \mathcal{L}^1(\eta\mathbb{Z}^d)$:

$$\begin{aligned}(\widehat{Q_k^*Q_kf})(p) &= \frac{1}{(2\pi)^{d/2}L^{kd}}\sum_{x\in\eta\mathbb{Z}^d}\eta^d e^{-ip\cdot x}\sum_{[x_\bullet]\le x'_\bullet<[x_\bullet]+1}f(x') \\ &= \frac{1}{(2\pi)^{d/2}L^{kd}}\sum_{\ell_\bullet=0}^{L^k-1}\sum_{x\in\eta\mathbb{Z}^d}\eta^d e^{-ip\cdot x}f([x_\bullet]+\ell_\bullet\eta) \\ &= \frac{1}{(2\pi)^{d/2}L^{kd}}\sum_{\ell_\bullet=0}^{L^k-1}\sum_{\ell'_\bullet=0}^{L^k-1}\sum_{[x_\bullet]\in\mathbb{Z}}\eta^d e^{-ip_\nu([x_\nu]+\ell'_\nu\eta)}f([x_\bullet]+\ell_\bullet\eta) \\ &= \frac{1}{(2\pi)^{d/2}L^{kd}}\sum_{\ell_\bullet=0}^{L^k-1}\sum_{\ell'_\bullet=0}^{L^k-1}\sum_{\mathrm{m}\in\mathbb{Z}^d}\eta^d e^{-ip_\nu(\mathrm{m}_\nu+\ell'_\nu\eta)}f(\mathrm{m}+\ell\eta) \\ &= \frac{1}{(2\pi)^{d/2}L^{kd}}\sum_{\ell_\bullet=0}^{L^k-1}\sum_{\ell'_\bullet=0}^{L^k-1}e^{-ip_\nu\ell'_\nu\eta}\sum_{\mathrm{m}\in\mathbb{Z}^d}\eta^d e^{-ip_\nu\mathrm{m}_\nu}f(\mathrm{m}+\ell\eta) \\ &= \frac{\eta^d}{(2\pi)^{d/2}L^{kd}}\Big(\prod_{\nu=0}^{d-1}\frac{1-e^{-ip_\nu}}{1-e^{-ip_\nu\eta}}\Big)\sum_{\ell_\bullet=0}^{L^k-1}\sum_{\mathrm{m}\in\mathbb{Z}^d}e^{-ip_\nu\mathrm{m}_\nu}f(\mathrm{m}+\ell\eta).\end{aligned} \tag{E.1}$$

Thus, by Lemma E.1 below, we obtain

$$(\widehat{Q_k^* Q_k f})(p) \;=\; \frac{1}{L^{2kd}}\Big(\prod_{\mu=0}^{d-1}\frac{1-\mathrm{e}^{-\mathrm{i}p_\mu}}{1-\mathrm{e}^{-\mathrm{i}p_\mu\eta}}\Big)\sum_{\ell''_\bullet=-\frac{L^k-1}{2}}^{\frac{L^k-1}{2}}\Big(\prod_{\nu=0}^{d-1}\frac{1-\mathrm{e}^{\mathrm{i}(p_\nu+2\pi\ell''_\nu)}}{1-\mathrm{e}^{\mathrm{i}(p_\nu+2\pi\ell''_\nu)\eta}}\Big)\hat{f}(p+2\pi\ell''). \tag{E.2}$$

Finally, this formula is extended to all $f\in\mathcal{L}^2(\eta\mathbb{Z}^d)$ using unitarity of the Fourier transform, boundedness of $Q_k^*Q_k$, and boundedness of the multiplication operators and shifts by $2\pi\ell''$ on the r.h.s. of (E.2).

Lemma E.1. *The following relation holds for $f\in\mathcal{L}^1(\eta\mathbb{Z}^d)$*

$$(2\pi)^{-d/2}\sum_{\ell_\bullet=0}^{L^k-1}\sum_{\mathrm{m}\in\mathbb{Z}^d}\mathrm{e}^{-\mathrm{i}p\cdot\mathrm{m}}f(\mathrm{m}+\ell\eta)=\sum_{\ell''_\bullet=-\frac{L^k-1}{2}}^{\frac{L^k-1}{2}}\Big(\prod_{\nu=0}^{d-1}\frac{1-\mathrm{e}^{\mathrm{i}(p_\nu+2\pi\ell''_\nu)}}{1-\mathrm{e}^{\mathrm{i}(p_\nu+2\pi\ell''_\nu)\eta}}\Big)\hat{f}(p+2\pi\ell''), \tag{E.3}$$

where we extended $\hat{f}$ from $[-\pi/\eta,\pi/\eta[^{\times d}$ to a function on $\mathbb{R}^d$ by periodicity.

Proof. We note that for any $x\in\eta\mathbb{Z}^d$,

$$\frac{1}{L^{kd}}\sum_{\ell''_\bullet=-\frac{L^k-1}{2}}^{\frac{L^k-1}{2}}\mathrm{e}^{-\mathrm{i}2\pi\ell''x} \;=\; \mathrm{e}^{-\mathrm{i}\pi(L^k-1)x}\frac{1}{L^{kd}}\sum_{\ell''_\bullet=0}^{L^k-1}\mathrm{e}^{-\mathrm{i}2\pi\ell''x} = \begin{cases}1 & \text{if } x\in\mathbb{Z}^d,\\ 0 & \text{otherwise,}\end{cases} \tag{E.4}$$

where we used that L is odd. Thus we can write

$$\begin{aligned}(2\pi)^{-d/2}\sum_{\mathrm{m}\in\mathbb{Z}^d}\mathrm{e}^{-\mathrm{i}p\cdot\mathrm{m}}f(\mathrm{m}+\ell\eta) \;&=\; \frac{1}{L^{kd}}\sum_{\ell''_\bullet=-\frac{L^k-1}{2}}^{\frac{L^k-1}{2}}(2\pi)^{-d/2}\sum_{x\in\eta\mathbb{Z}^d}\mathrm{e}^{-\mathrm{i}(p+2\pi\ell'')\cdot x}f(x+\ell\eta)\\ &=\; \frac{1}{L^{kd}}\sum_{\ell''_\bullet=-\frac{L^k-1}{2}}^{\frac{L^k-1}{2}}(2\pi)^{-d/2}\sum_{x\in\eta\mathbb{Z}^d}\mathrm{e}^{-\mathrm{i}(p+2\pi\ell'')\cdot(x-\ell\eta)}f(x)\\ &=\; \sum_{\ell''_\bullet=-\frac{L^k-1}{2}}^{\frac{L^k-1}{2}}\mathrm{e}^{\mathrm{i}(p+2\pi\ell'')\cdot\ell\eta}\hat{f}(p+2\pi\ell'').\end{aligned} \tag{E.5}$$

Now the summation over ℓ gives the claim. □

F Proof of Lemma 3.19

Recall that

$$\Delta^\eta(p) \;:=\; \frac{1}{\eta^2}\sum_{\mu=0}^{d-1}|1-\mathrm{e}^{-\mathrm{i}p_\mu\eta}|^2+\bar{\mu}_k = \frac{2}{\eta^2}\sum_{\mu=0}^{d-1}(1-\cos(p_\mu\eta))+\bar{\mu}_k = \frac{4}{\eta^2}\sideset{}{^*}\sum_{\mu=0}^{d-1}\sin^2\big(\frac{p_\mu\eta}{2}\big), \tag{F.1}$$

where the star over the sum means addition of $\frac{\eta^2}{4}\bar{\mu}_k = \frac{1}{4}\bar{\mu}_0$. Now we list other relevant definitions:

$$u(p) := \eta^d \prod_{\nu=0}^{d-1} \left(\frac{1 - \mathrm{e}^{-\mathrm{i}p_\nu}}{1 - \mathrm{e}^{-\mathrm{i}p_\nu \eta}} \right) = \eta^d \prod_{\nu=0}^{d-1} \left(\frac{\mathrm{e}^{-\mathrm{i}\frac{p_\nu}{2}}}{\mathrm{e}^{-\mathrm{i}\frac{p_\nu \eta}{2}}} \frac{\sin(\frac{p_\nu}{2})}{\sin(\frac{p_\nu \eta}{2})} \right), \tag{F.2}$$

$$u_\Delta(p) := \frac{\eta^{d+2}}{4} \left(\prod_{\alpha=0}^{d-1} \frac{\mathrm{e}^{-\mathrm{i}\frac{p_\alpha}{2}}}{\mathrm{e}^{-\mathrm{i}\frac{p_\alpha \eta}{2}}} \right) \frac{1}{\sum^{*}{}_{\mu=0}^{d-1} \sin^2(\frac{p_\mu \eta}{2})} \left(\prod_{\nu=0}^{d-1} \frac{\sin(\frac{p_\nu}{2})}{\sin(\frac{p_\nu \eta}{2})} \right), \tag{F.3}$$

$$u_\Delta(p + 2\pi\ell') := \frac{\eta^{d+2}}{4} \left(\prod_{\alpha=0}^{d-1} \frac{\mathrm{e}^{-\mathrm{i}\frac{p_\alpha}{2}}}{\mathrm{e}^{-\mathrm{i}\frac{(p_\alpha + 2\pi\ell'_\alpha)\eta}{2}}} \right) \frac{1}{\sum^{*}{}_{\mu=0}^{d-1} \sin^2\left(\frac{p_\mu \eta}{2} + \pi\ell'_\mu \eta\right)} \left(\prod_{\nu=0}^{d-1} \frac{\sin(\frac{p_\nu}{2})}{\sin\left(\frac{p_\nu \eta}{2} + \pi\ell'_\nu \eta\right)} \right), \tag{F.4}$$

$$\begin{aligned} \langle\!\langle u, u_\Delta \rangle\!\rangle(p) &:= \sum_{\ell''_\bullet = -\frac{L^k-1}{2}}^{\frac{L^k-1}{2}} \frac{|u(p + 2\pi\ell'')|^2}{\Delta^\eta(p + 2\pi\ell'')} \\ &= \frac{\eta^{2d+2}}{4} \sum_{\ell''_\bullet = -\frac{L^k-1}{2}}^{\frac{L^k-1}{2}} \frac{1}{\sum^{*}{}_{\mu=0}^{d-1} \sin^2\left(\frac{p_\mu \eta}{2} + \pi\ell''_\mu \eta\right)} \prod_{\nu=0}^{d-1} \frac{\sin^2(\frac{p_\nu}{2})}{\sin^2(\frac{p_\nu \eta}{2} + \pi\ell'' \eta)}, \end{aligned} \tag{F.5}$$

where we used in (F.4) that $p_\alpha \mapsto \mathrm{e}^{-\mathrm{i}\frac{p_\alpha}{2}} \sin(\frac{p_\alpha}{2})$ has period 2π and in the last line that $p_\alpha \mapsto \sin^2(p_\alpha/2)$ has period 2π.

The function appearing under the modulus in (3.85) has the form (where we set $a := a_k$ for brevity)

$$\begin{aligned} H(p) &:= \frac{1}{1 + a\langle\!\langle u, u_\Delta \rangle\!\rangle(p)} u_\Delta(p + 2\pi\ell') \\ &= \frac{\eta^{d+2}}{4} \left(\prod_{\alpha=0}^{d-1} \frac{\mathrm{e}^{-\mathrm{i}\frac{p_\alpha}{2}}}{\mathrm{e}^{-\mathrm{i}\frac{(p_\alpha + 2\pi\ell'_\alpha)\eta}{2}}} \right) && \text{(F.6)} \\ &\times \frac{1}{1 + a\frac{\eta^{2d+2}}{4} \sum_{\ell''_\bullet = -\frac{L^k-1}{2}}^{\frac{L^k-1}{2}} \frac{1}{\sum^{*}{}_{\mu=0}^{d-1} \sin^2\left(\frac{p_\mu \eta}{2} + \pi\ell''_\mu \eta\right)} \prod_{\nu=0}^{d-1} \frac{\sin^2(\frac{p_\nu}{2})}{\sin^2(\frac{p_\nu \eta}{2} + \pi\ell''_\nu \eta)}} && \text{(F.7)} \\ &\times \frac{1}{\sum^{*}{}_{\mu=0}^{d-1} \sin^2\left(\frac{p_\mu \eta}{2} + \pi\ell'_\mu \eta\right)} \left(\prod_{\nu=0}^{d-1} \frac{\sin(\frac{p_\nu}{2})}{\sin\left(\frac{p_\nu \eta}{2} + \pi\ell'_\nu \eta\right)} \right). && \text{(F.8)} \end{aligned}$$

In the following two subsections we will study the case of large mass and of small mass, respectively. For this purpose we fix a constant $c_* > 0$ depending only on d and establish Lemma 3.19 first under the assumption $\frac{1}{4}\bar{\mu}_0 \geq c_* \eta^2$ and then under the complementary assumption $\frac{1}{4}\bar{\mu}_0 < c_* \eta^2$.

F.1 The case of large mass: $\frac{1}{4}\bar{\mu}_0 \geq c_* \eta^2$, $c_* > 0$

The function in the bracket in (F.6) is clearly entire analytic, thus bounded in any compact set. By its definition, for any such set the bound can be chosen uniformly in $\eta \leq 1/2$. In other words, the function

$$H_1(z) := \frac{\eta^{d+2}}{4} \left(\prod_{\alpha=0}^{d-1} \frac{\mathrm{e}^{-\mathrm{i}\frac{z_\alpha}{2}}}{\mathrm{e}^{-\mathrm{i}\frac{(z_\alpha + 2\pi\ell'_\alpha)\eta}{2}}} \right) \tag{F.9}$$

satisfies $|H_1(z)| \leq c\eta^{d+2}$ for z in the closure of $S_{\pi,1} := \{\, p + \mathrm{i}q \in \mathbb{C}^d \,|\, p \in]-\pi, \pi[, |q| < 1 \,\}$. Let us now analyse the function $\tilde{H}_3$ defined by (F.8):

$$\tilde{H}_3(z) := \frac{1}{\sum_{\mu=0}^{d-1} \sin^2\left(\frac{z_\mu \eta}{2} + \pi\ell'_\mu \eta\right) + \frac{1}{4}\bar{\mu}_0} \left(\prod_{\nu=0}^{d-1} \frac{\sin(\frac{z_\nu}{2})}{\sin\left(\frac{z_\nu \eta}{2} + \pi\ell'_\nu \eta\right)} \right). \tag{F.10}$$

First, we note that for $\ell' = 0$ we have by Lemmas F.5, F.4 and the bound $\frac{1}{4}\bar{\mu}_0 \geq c_* \eta^2$

$$|\tilde{H}_3(z)| \leq \frac{c}{\eta^{2+d} c_*}. \tag{F.11}$$

Now let us assume $\ell' \neq 0$. We have, by Lemmas F.4, F.5, F.7,

$$\begin{aligned} |\tilde{H}_3(z)| &\leq \frac{c}{\eta^{2+d}} \frac{1}{\sum_{\mu=0}^{d-1} |z_\mu + 2\pi \ell'_\mu|^2} \frac{1}{\prod_{\nu=0}^{d-1}(1+|\ell'_\nu|)} \\ &\leq \frac{c}{\eta^{2+d}} \frac{1}{\sum_{\mu,\ell'_\mu \neq 0}(1+\ell'_\mu)^2} \frac{1}{\prod_{\nu=0}^{d-1}(1+|\ell'_\nu|)} \\ &\leq \frac{c}{\eta^{2+d}} \frac{1}{\prod_{\nu=0}^{d-1}(1+|\ell'_\nu|)^{1+2/d}}, \end{aligned} \tag{F.12}$$

where in the last step we used that the geometric mean is smaller than the arithmetic mean. Considering (F.11), we see that (F.12) holds for all $\ell' \in \mathbb{Z}^d$.

The function in the middle factor (F.7) will be called $\tilde{H}_2 = 1/\tilde{F}$. We will find a strip $S_{\pi, c_{\mathrm{st}}} = \{ p + \mathrm{i}q \,|\, p \in]-\pi, \pi[^{\times d}, |q| < c_{\mathrm{st}} \}$ s.t. $\tilde{F}$ has no zeros there. We have

$$\tilde{F}(z) := 1 + a\frac{\eta^{2d+2}}{4} \sum_{\ell''_\bullet = -\frac{L^k-1}{2}}^{\frac{L^k-1}{2}} \frac{1}{\sum^{*\,d-1}_{\mu=0} \sin^2\left(\frac{z_\mu \eta}{2} + \pi \ell''_\mu \eta\right)} \prod_{\nu=0}^{d-1} \frac{\sin^2(\frac{z_\nu}{2})}{\sin^2(\frac{z_\nu \eta}{2} + \pi \ell''_\nu \eta)}. \tag{F.13}$$

By the Taylor theorem, we can write

$$\tilde{F}(p + \mathrm{i}q) = \tilde{F}(p) + \mathrm{i}q \cdot (\nabla \tilde{F})(p + \mathrm{i}sq), \tag{F.14}$$

for some $0 \leq s \leq 1$. (Due to this restriction, possible dependence of s on parameters of the problem does not cause any complications). Using Lemma F.1 below, we obtain

$$|\tilde{F}(p + \mathrm{i}q)| \geq |\tilde{F}(p)| - |q||(\nabla \tilde{F})(p + \mathrm{i}sq)| \geq 1 - |q|ca \tag{F.15}$$

for some $c \geq 0$. Thus there exists a numerical constant $c_{\mathrm{st}} > 0$ s.t. the function H satisfies in the strip $p \in [-\pi, \pi[^{\times d}$, $|q| < c_{\mathrm{st}}$ the bound

$$|H(z)| \leq \frac{c}{\prod_{\mu=0}^{d-1}(1+|\ell'_\mu|)^{1+2/d}}. \tag{F.16}$$

Thus we have proven Lemma 3.19 under the assumption $\frac{1}{4}\bar{\mu}_0 \geq c_* \eta^2$, $c_* > 0$. In the next subsection we will treat the case $\frac{1}{4}\bar{\mu}_0 < c_* \eta^2$.

Lemma F.1. *For $\tilde{F}$ defined in (F.13), there holds, for $p \in [-\pi, \pi]^{\times d}$, $|q_\bullet| \leq 1/d$,*

$$\tilde{F}(p) \geq 1, \tag{F.17}$$

$$|\nabla \tilde{F}(p + \mathrm{i}q)| \leq ca. \tag{F.18}$$

Proof. Inequality (F.17) is obvious from (F.13). Now we write

$$\tilde{F}(z) := 1 + a\frac{\eta^{2d+2}}{4} \sum_{\ell''_\bullet = -\frac{L^k-1}{2}}^{\frac{L^k-1}{2}} \underbrace{\frac{1}{\sum^{*\,d-1}_{\mu=0} \sin^2\left(\frac{z_\mu \eta}{2} + \pi \ell''_\mu \eta\right)}}_{F^{(1)}_{\ell''}(z)} \underbrace{\prod_{\nu=0}^{d-1} \frac{\sin^2(\frac{z_\nu}{2})}{\sin^2(\frac{z_\nu \eta}{2} + \pi \ell''_\nu \eta)}}_{F^{(2)}_{\ell''}(z)}. \tag{F.19}$$

To estimate $\nabla\tilde{F}$, we first consider $\ell''=0$. We have, by Lemma F.5 and the assumption $\frac{1}{4}\bar{\mu}_0 \geq c_*\eta^2$,

$$|F^{(1)}_{\ell''=0}(z)| = \left|\frac{1}{\sum^{*\,d-1}_{\mu=0}\sin^2\left(\frac{z_\mu\eta}{2}\right)}\right| \leq \frac{1}{c_*\eta^2}, \tag{F.20}$$

$$|\partial_{z_\alpha}F^{(1)}_{\ell''=0}(z)| = \left|\frac{\frac{\eta}{2}\sin\left(z_\alpha\eta\right)}{(\sum^{*\,d-1}_{\mu=0}\sin^2\left(\frac{z_\mu\eta}{2}\right))^2}\right| \leq \frac{c\eta^2}{c_*^2\eta^4}. \tag{F.21}$$

Furthermore, Lemma F.3 gives,

$$|F^{(2)}_{\ell''=0}(z)|, |\partial_{z_\alpha}F^{(2)}_{\ell''=0}(z)| \leq \frac{c}{\eta^{2d}}. \tag{F.22}$$

Hence,

$$|\partial_{z_\alpha}(F^{(1)}_{\ell''=0}F^{(2)}_{\ell''=0})(z)| \leq \frac{c}{\eta^{2d+2}}. \tag{F.23}$$

Now let us now assume $\ell'' \neq 0$. We have, by Lemma F.4,

$$\begin{aligned}\left|F^{(1)}_{\ell''}(z)\right| &\leq \left|\frac{1}{\sum^{*\,d-1}_{\mu=0}\sin^2\left(\frac{z_\mu\eta}{2}+\pi\ell''_\mu\eta\right)}\right| \leq \frac{c}{\sum^{d-1}_{\mu=0}|\frac{z_\mu\eta}{2}+\pi\ell''_\mu\eta|^2}\\ &\leq \frac{c}{\eta^2}\frac{1}{\sum_{\mu,\ell''_\mu\neq 0}(1+|\ell''_\mu|)^2} \leq \frac{c}{\eta^2}\frac{1}{\prod^{d-1}_{\mu=0}(1+|\ell''_\mu|)^{2/d}},\end{aligned} \tag{F.24}$$

where in the third step we used Lemma F.6 and in the last step the fact that the geometric mean is smaller than the arithmetic mean. Regarding the derivative, we write

$$\begin{aligned}\left|\partial_{z_\alpha}F^{(1)}_{\ell''}(z)\right| &= \left|\frac{\frac{\eta}{2}\sin\left(z_\alpha\eta+2\pi\ell''_\alpha\eta\right)}{\left(\sum^{*\,d-1}_{\mu=0}\sin^2\left(\frac{z_\mu\eta}{2}+\pi\ell''_\mu\eta\right)\right)^2}\right| \leq c\frac{\frac{\eta}{2}|z_\alpha\eta+2\pi\ell''_\alpha\eta|}{\left(\sum^{d-1}_{\mu=0}|\frac{z_\mu\eta}{2}+\pi\ell''_\mu\eta|^2\right)^2}\\ &\leq c\frac{\eta^2(1+|\ell''_\alpha|)}{\eta^4\left(\sum_{\mu,\ell''_\mu\neq 0}(1+|\ell''_\mu|)^2\right)^2} \leq \frac{c}{\eta^2}\frac{\prod^{d-1}_{\alpha=0}(1+|\ell''_\alpha|)}{\prod^{d-1}_{\mu=0}(1+|\ell''_\mu|)^{4/d}}\\ &\leq \frac{c}{\eta^2}\left(\prod^{d-1}_{\alpha'=0}(1+|\ell''_{\alpha'}|)\right)^{1-4/d},\end{aligned} \tag{F.25}$$

where in the second step we used Lemmas F.4, F.5, in the third step $|z_\alpha| \leq c$ and Lemma F.6 and in the fourth step the fact that the geometric mean is smaller than the arithmetic mean.

On the other hand, Lemma F.3 gives

$$|F^{(2)}_{\ell''}(z)|, |\partial_{z_\alpha}F^{(2)}_{\ell''}(z)| \leq \frac{c}{\eta^{2d}}\frac{1}{\prod^{d-1}_{\nu=0}(1+|\ell''_\nu|)^2}. \tag{F.26}$$

Thus, altogether,

$$|\partial_{z_\alpha}(F^{(1)}_{\ell''}F^{(2)}_{\ell''})(z)| \leq \frac{c}{\eta^{2d+2}}\frac{1}{\prod^{d-1}_{\nu=0}(1+|\ell''_\nu|)^{1+4/d}}, \tag{F.27}$$

which holds for all $\ell'' \in \mathbb{Z}^d$ by (F.23). Substituting (F.27) to (F.19) and extending the region of summation in ℓ'' to whole $\mathbb{Z}^d$ we obtain (F.18). □

F.2 The case of small mass: $\frac{1}{4}\bar{\mu}_0 < c_*\eta^2$

We obtain from (F.6)–(F.8)

$$H(p) := \frac{1}{1+a\langle\!\langle u, u_\Delta\rangle\!\rangle(p)} u_\Delta(p+2\pi\ell')$$

$$= \frac{\eta^{d+2}}{4}\Bigg(\prod_{\alpha=0}^{d-1}\frac{\mathrm{e}^{-\mathrm{i}\frac{p_\alpha}{2}}}{\mathrm{e}^{-\mathrm{i}\frac{(p_\alpha+2\pi\ell'_\alpha)\eta}{2}}}\Bigg) \tag{F.28}$$

$$\times \frac{1}{\sum^{*\,d-1}_{\mu'=0}\sin^2\left(\frac{p_{\mu'}\eta}{2}\right) + a\frac{\eta^{2d+2}}{4}\sum_{\ell''_\bullet=-\frac{L^k-1}{2}}^{\frac{L^k-1}{2}}\frac{\sum^{*\,d-1}_{\mu'=0}\sin^2\left(\frac{p_{\mu'}\eta}{2}\right)}{\sum^{*\,d-1}_{\mu=0}\sin^2\left(\frac{p_\mu\eta}{2}+\pi\ell''_\mu\eta\right)}\prod_{\nu=0}^{d-1}\frac{\sin^2(\frac{p_\nu}{2})}{\sin^2(\frac{p_\nu\eta}{2}+\pi\ell''_\nu\eta)}} \tag{F.29}$$

$$\times \frac{\sum^{*\,d-1}_{\mu'=0}\sin^2\left(\frac{p_{\mu'}\eta}{2}\right)}{\sum^{*\,d-1}_{\mu=0}\sin^2\left(\frac{p_\mu\eta}{2}+\pi\ell'_\mu\eta\right)}\Bigg(\prod_{\nu=0}^{d-1}\frac{\sin(\frac{p_\nu}{2})}{\sin\left(\frac{p_\nu\eta}{2}+\pi\ell'_\nu\eta\right)}\Bigg), \tag{F.30}$$

where we divided and multiplied by $\sum^{*\,d-1}_{\mu'=0}\sin^2\left(\frac{p_{\mu'}\eta}{2}\right)$. The function appearing in (F.28) we treated already in (F.9). Now we consider the function appearing in (F.30). We will show that it is bounded in the closure of $S_{\pi,1}$, and thus (by inspection) analytic in $S_{\pi,1}$. We rewrite it as follows:

$$H_3(z) := \frac{\sum_{\mu'=0}^{d-1}\sin^2\left(\frac{z_{\mu'}\eta}{2}\right)+\bar{\mu}_0/4}{\sum_{\mu=0}^{d-1}\sin^2\left(\frac{z_\mu\eta}{2}+\pi\ell'_\mu\eta\right)+\bar{\mu}_0/4}\Bigg(\prod_{\nu=0}^{d-1}\frac{\sin(\frac{z_\nu}{2})}{\sin\left(\frac{z_\nu\eta}{2}+\pi\ell'_\nu\eta\right)}\Bigg). \tag{F.31}$$

For $\ell' \neq 0$, we have, by Lemmas F.5, F.4

$$\begin{aligned}
|H_3(p)| &\leq c\frac{\sum_{\mu'=0}^{d-1}|\frac{z_{\mu'}\eta}{2}|^2+c_*\eta^2}{\sum_{\mu=0}^{d-1}|\frac{z_\mu\eta}{2}+\pi\ell'_\mu\eta|^2}\Bigg(\prod_{\nu=0}^{d-1}\frac{|\frac{z_\nu}{2}|}{|\frac{z_\nu\eta}{2}+\pi\ell'_\nu\eta|}\Bigg)\\
&\leq \frac{c}{\eta^d}\frac{1}{\sum_{\mu=0}^{d-1}|\frac{z_\mu}{2}+\pi\ell'_\mu|^2}\Bigg(\prod_{\nu=0}^{d-1}\frac{|\frac{z_\nu}{2}|}{|\frac{z_\nu}{2}+\pi\ell'_\nu|}\Bigg)\\
&\leq \frac{c}{\eta^d}\frac{1}{\sum_{\mu,\ell'_\mu\neq 0}(1+\ell'_\mu)^2}\Bigg(\prod_{\nu=0}^{d-1}\frac{1}{(1+|\ell'_\nu|)}\Bigg)\\
&\leq \frac{c}{\eta^d}\frac{1}{(1+|\ell'_0|)^{1+2/d}\dots(1+|\ell'_{d-1}|)^{1+2/d}},
\end{aligned} \tag{F.32}$$

where in the second step we used the fact that $z_{\mu'}$ belong to compact sets, in the third step we applied Lemmas F.6, F.7 and in the last step we use that the geometric mean is always smaller that the arithmetic mean. By Lemma F.5, the bound remains true for $\ell' = 0$.

The function in the middle factor (F.29) will be called $H_2 = 1/F$. We will find a strip $S_{\pi,c_{\rm st}} = \{\, p+\mathrm{i}q \,|\, p \in$

$]-\pi,\pi[^{\times d}$, $|q|<c_{\rm st}\}$ s.t. F has no zeros there. We have

$$F(z) := \sideset{}{^*}\sum_{\mu'=0}^{d-1}\sin^2(\frac{z_{\mu'}\eta}{2}) + a\frac{\eta^{2d+2}}{4}\sum_{\ell''_\bullet=-\frac{L^k-1}{2}}^{\frac{L^k-1}{2}} \frac{\sideset{}{^*}\sum_{\mu'=0}^{d-1}\sin^2\left(\frac{z_{\mu'}\eta}{2}\right)}{\sideset{}{^*}\sum_{\mu=0}^{d-1}\sin^2\left(\frac{z_\mu\eta}{2}+\pi\ell''_\mu\eta\right)}\prod_{\nu=0}^{d-1}\frac{\sin^2(\frac{z_\nu}{2})}{\sin^2(\frac{z_\nu\eta}{2}+\pi\ell''_\nu\eta)}$$

$$= a\frac{\eta^{2d+2}}{4}\prod_{\nu=0}^{d-1}\frac{\sin^2(\frac{z_\nu}{2})}{\sin^2(\frac{z_\nu\eta}{2})} \tag{F.33}$$

$$+a\frac{\eta^{2d+2}}{4}\sum_{\ell''\neq 0,\,\ell''_\bullet=-\frac{L^k-1}{2}}^{\frac{L^k-1}{2}} \frac{\sideset{}{^*}\sum_{\mu'=0}^{d-1}\sin^2\left(\frac{z_{\mu'}\eta}{2}\right)}{\sideset{}{^*}\sum_{\mu=0}^{d-1}\sin^2\left(\frac{z_\mu\eta}{2}+\pi\ell''_\mu\eta\right)}\prod_{\nu=0}^{d-1}\frac{\sin^2(\frac{z_\nu}{2})}{\sin^2(\frac{z_\nu\eta}{2}+\pi\ell''_\nu\eta)} \tag{F.34}$$

$$+\sideset{}{^*}\sum_{\mu'=0}^{d-1}\sin^2(\frac{z_{\mu'}\eta}{2}), \tag{F.35}$$

where (F.33) is the $\ell''=0$ term. By the Taylor theorem, we can write

$$F(p+\mathrm{i}q) = F(p) + \mathrm{i}q\cdot(\nabla F)(p+\mathrm{i}sq), \tag{F.36}$$

for some $0\le s\le 1$. Using Lemma F.2 below, we obtain

$$|F(p+\mathrm{i}q)| \ge |F(p)| - |q||(\nabla F)(p+\mathrm{i}sq)| \ge c_1 a\eta^2 - |q|c_2 a\eta^2 \tag{F.37}$$

for some $c_1, c_2 > 0$.

Thus there exists a numerical constant $c_{\rm st}>0$ s.t. the function H satisfies in the strip $p\in[-\pi,\pi[^{\times d}$, $|q|<c_{\rm st}$ the bound

$$|H(z)| \le \frac{c}{\prod_{\mu=0}^{d-1}(1+|\ell'_\mu|)^{1+2/d}}. \tag{F.38}$$

We used that η^{-d-2} coming from (F.37), (F.32) is cancelled by η^{d+2} appearing in (F.28). This completes the proof of Lemma 3.19.

Lemma F.2. *For F defined in (F.33), there holds, for $p\in[-\pi,\pi]^{\times d}$, $|q_\bullet|\le 1/d$,*

$$F(p) \ge ca\eta^2 > 0, \tag{F.39}$$

$$|\nabla F(p+\mathrm{i}q)| \le ca\eta^2. \tag{F.40}$$

Proof. We denote the three terms in (F.33), (F.34), (F.35) by F_1, F_2, F_3, respectively. We recall the notation $\mathrm{sinc}(z) := \frac{\sin(z)}{z}$. We start with the lower bound in (F.39): We have, by Lemma F.5,

$$F_1(p) = a\frac{\eta^{2d+2}}{4}\prod_{\nu=0}^{d-1}\frac{\sin^2(\frac{p_\nu}{2})}{\sin^2(\frac{p_\nu\eta}{2})} = a\frac{\eta^{2d+2}}{4\eta^{2d}}\prod_{\nu=0}^{d-1}\frac{\mathrm{sinc}^2(\frac{p_\nu}{2})}{\mathrm{sinc}^2(\frac{p_\nu\eta}{2})} \ge ac_+\eta^2, \tag{F.41}$$

where $c_+>0$. Since $F_2(p), F_3(p)\ge 0$, we obtain (F.39).

Now we compute and estimate the derivatives: By Lemma F.3,

$$|\partial_{z_\mu}F_1(z)| = \frac{a\eta^{2d+2}}{4}\left|\partial_{z_\mu}\prod_{\nu=0}^{d-1}\left(\frac{\sin(\frac{z_\nu}{2})}{\sin(\frac{z_\nu\eta}{2})}\right)^2\right| \le ca\eta^2. \tag{F.42}$$

Now we consider, for $\ell''\neq 0$

$$F_{2,\ell''}(z) := a\frac{\eta^{2d+2}}{4}\underbrace{\frac{\sideset{}{^*}\sum_{\mu'=0}^{d-1}\sin^2\left(\frac{z_{\mu'}\eta}{2}\right)}{\sideset{}{^*}\sum_{\mu=0}^{d-1}\sin^2\left(\frac{z_\mu\eta}{2}+\pi\ell''_\mu\eta\right)}}_{F^{(1)}_{2,\ell''}(z)}\underbrace{\prod_{\nu=0}^{d-1}\frac{\sin^2(\frac{z_\nu}{2})}{\sin^2(\frac{z_\nu\eta}{2}+\pi\ell''_\nu\eta)}}_{F^{(2)}_{2,\ell''}(z)}. \tag{F.43}$$

We first estimate the auxiliary functions $F^{(1)}_{2,\ell''}, F^{(2)}_{2,\ell''}$ and their derivatives. We have by Lemma F.4

$$|F^{(1)}_{2,\ell''}(z)| \leq c\frac{\sum_{\mu'=0}^{d-1}|z_{\mu'}|^2 + c_*}{\sum_{\mu=0}^{d-1}|z_\mu + 2\pi\ell''_\mu|^2} \leq \frac{c}{\sum_{\mu,\ell''_\mu\neq 0}(1+|\ell''_\mu|)^2} \leq c\frac{1}{\prod_{\mu=0}^{d-1}(1+|\ell''_\mu|)^{2/d}}, \tag{F.44}$$

where we made use in the second step of $\sum_{\mu'=0}^{d-1}|z_{\mu'}|^2 \leq c$, of Lemma F.6 and in the last step of the fact that the geometric mean is smaller than the arithmetic mean. Furthermore, setting $z^\nu := z_\nu + 2\pi\ell''_\nu$ and applying Lemma F.3 we get

$$\left|F^{(2)}_{2,\ell''}(z)\right| \leq \frac{c}{\eta^{2d}}\frac{1}{\prod_{\nu=0}^{d-1}(1+|\ell''_\nu|)^2}. \tag{F.45}$$

Next, we consider the derivatives. We compute:

$$\begin{aligned}
&|\partial_{z_\alpha}F^{(1)}_{2,\ell''}(z)| = \left|\partial_{z_\alpha}\frac{\overset{*}{\sum}_{\mu'=0}^{d-1}\sin^2\left(\frac{z_{\mu'}\eta}{2}\right)}{\overset{*}{\sum}_{\mu=0}^{d-1}\sin^2\left(\frac{z^{\mu}\eta}{2}\right)}\right| \\
&= \eta\left|\frac{\sin\left(\frac{z_\alpha\eta}{2}\right)\cos(\frac{z_\alpha\eta}{2})\overset{*}{\sum}_{\mu=0}^{d-1}\sin^2\left(\frac{z^\mu\eta}{2}\right) - \left(\overset{*}{\sum}_{\mu'=0}^{d-1}\sin^2\left(\frac{z_{\mu'}\eta}{2}\right)\right)\sin\left(\frac{z^\alpha\eta}{2}\right)\cos(\frac{z^\alpha\eta}{2})}{\left(\overset{*}{\sum}_{\mu=0}^{d-1}\sin^2\left(\frac{z^\mu\eta}{2}\right)\right)^2}\right| \\
&\leq c\frac{|z_\alpha|(\sum_{\mu=0}^{d-1}|z^\mu|^2 + c_*) + \left(\sum_{\mu'=0}^{d-1}|z_{\mu'}|^2 + c_*\right)|z^\alpha|}{\left(\sum_{\mu=0}^{d-1}|z^\mu|^2\right)^2} \leq c\frac{1}{\left(\sum_{\mu=0}^{d-1}|z^\mu|^2\right)} + c\frac{|z^\alpha|}{\left(\sum_{\mu=0}^{d-1}|z^\mu|^2\right)^2} \\
&\leq c\frac{1}{\sum_{\mu,\ell''_\mu\neq 0}(1+|\ell''_\mu|)^2} + c\frac{(1+|\ell''_\alpha|)}{\left(\sum_{\mu,\ell''_\mu\neq 0}(1+|\ell''_\mu|)^2\right)^2} \leq c\frac{1}{\sum_{\mu,\ell''_\mu\neq 0}(1+|\ell''_\mu|)^2} \leq c\frac{1}{\prod_{\mu=0}^{d-1}(1+|\ell''_\mu|)^{2/d}},
\end{aligned} \tag{F.46}$$

where, in the next to the last line we used Lemmas F.5, F.4 and in the last line Lemma F.6 and the fact that the geometric mean is smaller that the arithmetic mean. Now we move on to the derivative of $F^{(2)}_{2,\ell''}$ and use Lemma F.3

$$\left|\partial_{z_\alpha}F^{(2)}_{2,\ell''}(z)\right| \leq \frac{c}{\eta^{2d}}\frac{1}{\prod_{\nu=0}^{d-1}(1+|\ell''_\nu|)^2}. \tag{F.47}$$

Thus, altogether, we get

$$|\partial_{z_\alpha}F_{2,\ell''}(z)| \leq ca\eta^2\frac{1}{\prod_{\nu=0}^{d-1}(1+|\ell_\nu|)^{2+2/d}}. \tag{F.48}$$

Similarly for

$$F_2(z) := \sum_{\ell''\neq 0,\, \ell''_\bullet = -\frac{L^k-1}{2}}^{\frac{L^k-1}{2}} F_{2,\ell''}(z) \quad \text{we have} \quad |\partial_{z_\alpha}F_2(z)| \leq ca\eta^2 \tag{F.49}$$

by extending the region of summation over ℓ'' to $\mathbb{Z}^d$.

Finally, we note the obvious relation regarding $F_3(z) = \sum_{\mu'=0}^{d-1}\sin^2(\frac{z_{\mu'}\eta}{2})$:

$$|\partial_{z_\alpha}F_3(z)| = |\eta(\frac{z_\alpha\eta}{2})\mathrm{sinc}(\frac{z_\alpha\eta}{2})\cos(\frac{z_\alpha\eta}{2})| \leq c\eta^2, \tag{F.50}$$

where we used Lemma F.5. □

F.3 Technical lemmas

Lemma F.3. *For $z = p + \mathrm{i}q$, $p \in [-\pi, \pi]^{\times d}$, $|q| \le 1$*

$$\Big|\partial_{z_\mu}^\alpha\Big(\prod_{\nu=0}^{d-1}\frac{\sin^2(\frac{z_\nu}{2})}{\sin^2(\frac{z_\nu\eta}{2}+\pi\ell''_\nu\eta)}\Big)\Big| \le \frac{c}{\eta^{2d}}\frac{1}{\prod_{\nu=0}^{d-1}(1+|\ell_\nu|)^2}, \quad |\alpha| = 0, 1. \tag{F.51}$$

Proof. We set $z^\nu := z_\nu + 2\pi\ell''_\nu$ and recall that $|\ell''_\nu| \le \frac{(L^k-1)}{2}$, so $|\mathrm{Re}(\frac{z^\nu\eta}{2})| \le \frac{\pi}{2}$. We compute

$$\Big|\prod_{\nu=0}^{d-1}\frac{\sin^2(\frac{z_\nu}{2})}{\sin^2(\frac{z^\nu\eta}{2})}\Big| = \frac{1}{\eta^{2d}}\Big|\prod_{\nu=0}^{d-1}\frac{\mathrm{sinc}^2(\frac{z_\nu}{2})z_\nu^2}{\mathrm{sinc}^2(\frac{z^\nu\eta}{2})(z^\nu)^2}\Big| \le \frac{c}{\eta^{2d}}\prod_{\nu=0}^{d-1}\Big|\frac{z_\nu}{z^\nu}\Big|^2 \le \frac{c}{\eta^{2d}}\frac{1}{\prod_{\nu=0}^{d-1}(1+|\ell''_\nu|)^2}, \tag{F.52}$$

where we applied Lemmas F.5,F.7. Now we move on to the derivative:

$$\begin{aligned}
\Big|\partial_{z_\mu}\Big(\prod_{\nu=0}^{d-1}\frac{\sin^2(\frac{z_\nu}{2})}{\sin^2(\frac{z^\nu\eta}{2})}\Big)\Big| &= 2\Big|\Big(\frac{\sin(\frac{z_\mu}{2})}{\sin(\frac{z^\mu\eta}{2})}\Big)\partial_{z_\mu}\Big(\frac{\sin(\frac{z_\mu}{2})}{\sin(\frac{z^\mu\eta}{2})}\Big)\prod_{\nu\neq\mu}\Big(\frac{\sin(\frac{z_\nu}{2})}{\sin(\frac{z^\nu\eta}{2})}\Big)^2\Big| \\
&= 2\Big|\Big(\frac{\mathrm{sinc}(\frac{z_\mu}{2})}{\mathrm{sinc}(\frac{z^\mu\eta}{2})}\frac{z_\mu}{\eta z^\mu}\Big)\partial_{z_\mu}\Big(\frac{\sin(\frac{z_\mu}{2})}{\sin(\frac{z^\mu\eta}{2})}\Big)\prod_{\nu\neq\mu}\Big(\frac{\mathrm{sinc}(\frac{z_\nu}{2})}{\mathrm{sinc}(\frac{z^\nu\eta}{2})}\frac{z_\nu}{z^\nu\eta}\Big)^2\Big| \\
&\le \frac{c}{\eta^{2d-1}}\frac{1}{(1+|\ell''_\mu|)\prod_{\nu\neq\mu}(1+|\ell''_\nu|)^2}\Big|\frac{\frac{1}{2}\cos(\frac{z_\mu}{2})\sin(\frac{z^\mu\eta}{2})-\frac{\eta}{2}\cos(\frac{z^\mu\eta}{2})\sin(\frac{z_\mu}{2})}{\sin^2(\frac{z^\mu\eta}{2})}\Big|. \\
&= \frac{c}{\eta^{2d-1}}\frac{1}{(1+|\ell''_\mu|)\prod_{\nu\neq\mu}(1+|\ell''_\nu|)^2}\Big|\frac{\frac{1}{2}\cos(\frac{z_\mu}{2})\mathrm{sinc}(\frac{z^\mu\eta}{2})\frac{z^\mu\eta}{2}-\frac{\eta}{2}\cos(\frac{z^\mu\eta}{2})\sin(\frac{z_\mu}{2})}{\mathrm{sinc}^2(\frac{z^\mu\eta}{2})(\frac{z^\mu\eta}{2})^2}\Big| \\
&\le \frac{c}{\eta^{2d}}\frac{1}{\prod_{\nu=0}^{d-1}(1+|\ell''_\nu|)^2},
\end{aligned} \tag{F.53}$$

where in the third step we used Lemmas F.5, F.7 and in the last step Lemmas F.5, F.6, provided that $\ell''_\mu \neq 0$ as required in this latter lemma. In the case $\ell''_\mu = 0$ we have, trivially, $(1+|\ell''_\mu|) = (1+|\ell''_\mu|)^2$ and the numerator under the absolute value above can be rewritten and estimated as follows

$$\big|\frac{z_\mu\eta}{4}\big|\big|\cos(\frac{z_\mu}{2})\mathrm{sinc}(\frac{z_\mu\eta}{2}) - \cos(\frac{z_\mu\eta}{2})\mathrm{sinc}(\frac{z_\mu}{2})\big| \le c|z_\mu|^3\eta \tag{F.54}$$

by inspection of Taylor expansions of cos and sinc. This compensates $(\frac{z^\mu\eta}{2})^2$ in the denominator (up to one inverse power of η which is taken into account) and concludes the proof. □

Lemma F.4. *The following bounds hold true for $\ell \neq 0$, $z_\bullet = p_\bullet + \mathrm{i}q_\bullet$, $p_\bullet \in [-\pi, \pi]$, $|q_\bullet| \le 1/d$ and any $\delta \ge 0$*

$$\sum_{\mu=0}^{d-1}\big|\frac{z_\mu\eta}{2}+\pi\ell_\mu\eta\big|^2 \le c\big|\sum_{\mu=0}^{d-1}\sin^2\big(\frac{z_\mu\eta}{2}+\pi\ell_\mu\eta\big)+\delta\big|, \tag{F.55}$$

$$\delta \le c\big|\sum_{\mu=0}^{d-1}\sin^2\big(\frac{z_\mu\eta}{2}+\pi\ell_\mu\eta\big)+\delta\big|. \tag{F.56}$$

Proof. Let us compute for arbitrary $w = x + \mathrm{i}y \in \mathbb{C}$

$$\begin{aligned}
\sin^2\big(\frac{w}{2}\big) &= \frac{1}{2}\big(1-\cos(x+\mathrm{i}y)\big)\\
&= \frac{1}{2}\big(1-\cos(x)\cos(\mathrm{i}y)+\sin(x)\sin(\mathrm{i}y)\big)\\
&= \frac{1}{2}\big(1-\cos(x)\mathrm{ch}(y)+\mathrm{i}\sin(x)\mathrm{sh}(y)\big)\\
&= \frac{1}{2}\big(1-\mathrm{ch}(y)+\mathrm{ch}(y)-\cos(x)\mathrm{ch}(y)+\mathrm{i}\sin(x)\mathrm{sh}(y)\big)\\
&= \frac{1}{2}\big(1-\mathrm{ch}(y)\big)+\frac{1}{2}\mathrm{ch}(y)(1-\cos(x))+\frac{\mathrm{i}}{2}\mathrm{sh}(y)\sin(x)\\
&= -\mathrm{sh}^2\big(\frac{y}{2}\big)+\mathrm{ch}(y)\sin^2\big(\frac{x}{2}\big)+\frac{\mathrm{i}}{2}\mathrm{sh}(y)\sin(x). \qquad \text{(F.57)}
\end{aligned}$$

Hence, setting $z^\mu := z_\mu + 2\pi\ell_\mu$

$$\begin{aligned}
\Big|\sum_{\mu=0}^{d-1}\sin^2\big(\frac{z^\mu\eta}{2}\big)+\delta\Big| &\geq -\sum_{\mu=0}^{d-1}\mathrm{sh}^2\big(\frac{q_\mu\eta}{2}\big)+\sum_{\mu=0}^{d-1}\mathrm{ch}(q_\mu\eta)\sin^2\big(\frac{p^\mu\eta}{2}\big)+\delta\\
&\geq -\sum_{\mu=0}^{d-1}\mathrm{sh}^2\big(\frac{q_\mu\eta}{2}\big)+\sum_{\mu=0}^{d-1}\sin^2\big(\frac{p^\mu\eta}{2}\big)+\delta. \qquad \text{(F.58)}
\end{aligned}$$

By assumption, there is an index $\hat{\mu}$ for which $\ell_{\hat{\mu}} \neq 0$ and suppose, for definiteness, $\ell_{\hat{\mu}} > 0$. For $p^{\hat{\mu}} := p_{\hat{\mu}} + 2\pi\ell_{\hat{\mu}}$ and $p_{\hat{\mu}} \in [-\pi, \pi]$ we have

$$\frac{p^{\hat{\mu}}\eta}{2} \geq \frac{\pi\eta}{2} \quad \Rightarrow \quad \sin^2\big(\frac{p^\mu\eta}{2}\big) \geq \sin^2\big(\frac{\pi\eta}{2}\big). \qquad \text{(F.59)}$$

In fact, since $k \in \mathbb{N}$, $L > 1$, we have $\eta \leq 1/2$. Thus $\frac{p_{\hat{\mu}}\eta}{2} \in [-\pi\eta/2, \pi\eta/2]$. By shifting this interval by $\pi\eta \leq \pi\ell_{\hat{\mu}}\eta \leq \frac{\pi}{2}$, we stay away of zeros of $\sin^2(\cdot)$. (We used here that $\ell_{\hat{\mu}} \leq (L^k - 1)/2$ and $\eta = L^{-k}$). Now for $x \in [0, 1/2]$ we can write

$$\mathrm{sh}(x/d) = \frac{\mathrm{sh}(x/d)}{\sin(\pi x)}\sin(\pi x) = \frac{1}{\pi d}\frac{\mathrm{shc}(x/d)}{\mathrm{sinc}(\pi x)}\sin(\pi x) \leq \frac{3}{\pi d}\sin(\pi x), \qquad \text{(F.60)}$$

where $\mathrm{shc}(x) := \frac{\mathrm{sh}(x)}{x}, \mathrm{sinc}(x) := \frac{\sin(x)}{x}$ and we used that $\mathrm{sinc}(\pi x) \geq 1/2$ and $\mathrm{shc}(x/d) \leq 3/2$ for $x \in [0, 1/2]$ as can be read off from their graphs. (We also stress for future reference that $3/\pi < 1$). Hence, using (F.59), (F.60)

$$\mathrm{sh}^2\big(\frac{(q_\mu d)\eta}{2d}\big) \leq \frac{1}{d^2}\left(\frac{3}{\pi}\right)^2\sin^2\big(\pi\frac{(q_\mu d)\eta}{2}\big) \leq \frac{1}{d^2}\left(\frac{3}{\pi}\right)^2\sin^2\big(\pi\frac{\eta}{2}\big) \leq \frac{1}{d^2}\left(\frac{3}{\pi}\right)^2\sin^2\big(\frac{p^{\hat{\mu}}\eta}{2}\big). \qquad \text{(F.61)}$$

Consequently,

$$\sum_{\mu=0}^{d-1}\mathrm{sh}^2\big(\frac{(q_\mu d)\eta}{2d}\big) \leq \frac{1}{d}\left(\frac{3}{\pi}\right)^2\sin^2\big(\frac{p^{\hat{\mu}}\eta}{2}\big). \qquad \text{(F.62)}$$

Thus coming back to (F.58), we can write for some constant $c > 0$ depending only on d

$$\Big|\sum_{\mu=0}^{d-1}\sin^2\big(\frac{z^\mu\eta}{2}\big)+\delta\Big| \geq c\sum_{\mu=0}^{d-1}\sin^2\big(\frac{p^\mu\eta}{2}\big)+\delta \geq c\sum_{\mu=0}^{d-1}\Big|\frac{p^\mu\eta}{2}\Big|^2+\delta, \qquad \text{(F.63)}$$

which proves (F.56). Finally, arguing like in (F.59) and using $|q_\mu| \le 1/d$,

$$\left|\frac{(p_{\hat\mu} + 2\pi\ell_{\hat\mu})\eta}{2}\right|^2 \ge \left(\frac{\pi\eta}{2}\right)^2 \ge \sum_{\mu=0}^{d-1}\left(\frac{q_\mu\eta}{2}\right)^2, \tag{F.64}$$

hence

$$\left|\frac{(p_{\hat\mu} + 2\pi\ell_{\hat\mu})\eta}{2}\right|^2 \ge \frac{1}{2}\left|\frac{(p_{\hat\mu} + 2\pi\ell_{\hat\mu})\eta}{2}\right|^2 + \frac{1}{2}\sum_{\mu=0}^{d-1}\left|\frac{q_\mu\eta}{2}\right|^2. \tag{F.65}$$

which, combined with (F.63), gives (F.55). □

Lemma F.5. *We have, for* $\mathrm{Re}(z) \in [-\pi/2, \pi/2]$, $|\mathrm{Im}(z)| \le 1$

$$0 < c_- \le |\frac{\sin(z)}{z}| \le c_+, \tag{F.66}$$

$$|\partial_z(\frac{\sin(z)}{z})| \le c_0, \tag{F.67}$$

where c_0, c_-, c_+ *are independent of* z *within the above restrictions.*

Proof. Except for the lower bound in (F.66), all the bounds simply follow from the fact that entire analytic functions are bounded on compact sets. Regarding the lower bound, we compute

$$\begin{aligned}\left|\frac{\sin(z)}{z}\right|^2 &= \frac{\left|\mathrm{e}^{\mathrm{i}p}\mathrm{e}^{-q} - \mathrm{e}^{-\mathrm{i}p}e^{q}\right|^2}{p^2+q^2} = \frac{\mathrm{e}^{-2q} - \mathrm{e}^{2\mathrm{i}p} - \mathrm{e}^{-2\mathrm{i}p} + \mathrm{e}^{2q}}{p^2+q^2} \\ &= 4\frac{(\frac{\mathrm{e}^{q}-\mathrm{e}^{-q}}{2})^2 + (\frac{\mathrm{e}^{\mathrm{i}p}-\mathrm{e}^{-\mathrm{i}p}}{2\mathrm{i}})^2}{p^2+q^2} = 4\frac{\sin^2(p) + \sinh^2(q)}{p^2+q^2} \\ &= 4\frac{\frac{\sin^2(p)}{p^2}p^2 + \frac{\sinh^2(q)}{q^2}q^2}{p^2+q^2}.\end{aligned} \tag{F.68}$$

The last expression can be estimated by

$$0 < c_1 \le 4\frac{\frac{\sin^2(p)}{p^2}p^2 + \frac{\sinh^2(q)}{q^2}q^2}{p^2+q^2} \le c_2, \tag{F.69}$$

where we used that $0 < c \le \frac{\sin^2(p)}{p^2}, \frac{\sinh^2(q)}{q^2}$ in the relevant region. □

Lemma F.6. *The following bound holds true for* $z \in [-\pi, \pi] + \mathrm{i}[-1, 1]$ *and* $\ell \in \mathbb{Z}$, $\ell \neq 0$,

$$|z - 2\pi\ell| \ge c_+(1 + |\ell|) \tag{F.70}$$

for some $c_+ > 0$.

Proof. We write

$$|z - 2\pi\ell| \ge |p - 2\pi\ell| \ge 2\pi|\ell| - \pi \ge \frac{\pi}{2}(1 + |\ell|). \quad \square \tag{F.71}$$

Lemma F.7. *The following bound holds true for* $z \in [-\pi, \pi[+ \mathrm{i}[-1, 1[$ *and* $\ell \in \mathbb{Z}$

$$\left|1 + \frac{2\pi\ell}{z}\right| \ge \frac{1 + |\ell|}{6}. \tag{F.72}$$

Proof. We write

$$\left|1+\frac{2\pi\ell}{z}\right| \geq \begin{cases} 1 & \text{for } \ell = 0, \\ \frac{2\pi|\ell|}{|z|} - 1 & \text{for } \ell \neq 0. \end{cases} \tag{F.73}$$

It is clear that the claimed bound holds for $\ell = 0$. As for the remaining case, we note that $|z| \leq \pi + 1 \leq \frac{3}{2}\pi$ hence

$$\frac{2\pi|\ell|}{|z|} - 1 \geq \frac{4}{3}|\ell| - 1 \geq \frac{1+|\ell|}{6}, \tag{F.74}$$

which concludes the proof. □

G Completion of the proof of Theorem 4.3

Making use of Lemma 3.17, we obtain from (4.16)

$$\begin{aligned} |(G_k(\Omega)Q^*_{\Omega,k})(x,y)| \; &\leq \; c\sum_j \Big(\prod_{j_\mu \text{ even}} \mathrm{e}^{-\frac{1}{2}c_{\mathrm{st}}|x_\mu - y_\mu - (L^m\eta)j_\mu|}\Big)\Big(\prod_{j_\nu \text{ odd}} \mathrm{e}^{-\frac{1}{2}c_{\mathrm{st}}|(L^m\eta)(j_\nu+1) - y_\nu - x_\nu|}\Big), && \text{(G.1)} \\ &= \sum_{\sigma_0,\dots\sigma_{d-1}\in\{\text{even/odd}\}} c\prod_{\alpha=0}^{d-1}\Big(\sum_{j_\alpha\in\mathbb{Z}_{\sigma_\alpha}} h_{\sigma_\alpha,j_\alpha}(x_\alpha,y_\alpha)\Big), && \text{(G.2)} \\ h_{\mathrm{even},j_\mu}(x_\mu,y_\mu) \; &:= \; \mathrm{e}^{-\frac{1}{2}c_{\mathrm{st}}|x_\mu - y_\mu - (L^m\eta)j_\mu|}, \quad h_{\mathrm{odd},j_\nu}(x_\nu,y_\nu) := \mathrm{e}^{-\frac{1}{2}c_{\mathrm{st}}|(L^m\eta)(j_\nu+1)-y_\nu-x_\nu|} && \text{(G.3)} \end{aligned}$$

and $\mathbb{Z}_{\sigma_\alpha}$ denotes the set of even or odd integers, respectively. Since the sum over σ has 2^d terms, it suffices to control the sums over j_α.

• Let us first analyse the sum over j_μ even. To simplify notation in the analysis below we write $x := x_\mu, y := y_\mu$ and $j := j_\mu$. Suppose first that $j \geq 2$. Then

$$|x-y-(L^m\eta)j| = (L^m\eta)(j-2) + 2(L^m\eta) - (x-y) \geq (L^m\eta)(j-2) + |x-y|. \tag{G.4}$$

Here the last inequality boils down for $x \geq y$ (resp. $x \leq y$) to $L^m\eta \geq (x-y)$ (resp. $L^m\eta \geq 0$) which hold true since $L^m\eta$ is the linear size of Ω. For $j = 0$ we obviously have

$$|x-y-(L^m\eta)j| = |x-y|. \tag{G.5}$$

Now for $j \leq -2$ we can write

$$\begin{aligned} |x-y-(L^m\eta)j| = (L^m\eta)(-j) + x - y \; &= \; (L^m\eta)(-j-2) + 2(L^m\eta) + x - y \\ &\geq \; (L^m\eta)(-j-2) + |x-y|. \end{aligned} \tag{G.6}$$

Here the last inequality boils down to for $x \geq y$ (resp. $x \leq y$) to $L^m\eta \geq 0$ (resp. $L^m\eta \geq y - x$).

Coming back to (G.2), we have

$$\begin{aligned} \sum_{j \text{ even}} c\mathrm{e}^{-\frac{1}{2}c_{\mathrm{st}}|x-y-(L^m\eta)j|} \leq c\mathrm{e}^{-\frac{1}{2}c_{\mathrm{st}}|x-y|} \; &+ \sum_{j\geq 2,\, j \text{ even}} c\mathrm{e}^{-\frac{1}{2}c_{\mathrm{st}}((L^m\eta)(j-2)+|x-y|)} \\ &+ \sum_{j\leq -2,\, j\text{ even}} c\mathrm{e}^{-\frac{1}{2}c_{\mathrm{st}}((L^m\eta)(-j-2)+|x-y|)} \leq c\mathrm{e}^{-\frac{1}{2}c_{\mathrm{st}}|x-y|}, \end{aligned} \tag{G.7}$$

where we used that $L^m\eta \geq 1$ (since $\eta = L^{-k}$, $m \geq k$) to avoid any dependence of constants on η and then summed up the geometric series.

• Let us now analyse the sum in (G.2) over odd j_ν. To simplify notation in the analysis below we write $x := x_\nu$, $y := y_\nu$ and $j := j_\nu$. Suppose first that $j \geq 3$. Then

$$\begin{aligned}|(L^m\eta)(j+1) - y - x| &= (L^m\eta)(j+1) - y - x \\ &= (L^m\eta)(j-1) + 2L^m\eta - y - x \geq (L^m\eta)(j-1) + |y-x|,\end{aligned} \tag{G.8}$$

where the last equality boils down for $y \geq x$ (resp. $y \leq x$) to $L^m\eta \geq y$ (resp. $L^m\eta \geq x$). Let us now consider $j = 1$. We have

$$|2(L^m\eta) - y - x| = 2(L^m\eta) - y - x \geq |y-x|. \tag{G.9}$$

In fact, for $y \geq x$ (resp. $y \leq x$) this inequality boils down to $y \leq L^m\eta$ (resp. $x \leq L^m\eta$).

Next, we take $j = -1$. Then

$$|(L^m\eta)(j+1) - y - x| = |-y-x| = y + x \geq |y-x|, \tag{G.10}$$

where we used that $x, y \geq 0$. Now suppose $j \leq -3$. Then

$$\begin{aligned}|(L^m\eta)(j+1) - y - x| &= -(L^m\eta)(j+1) + y + x = -(L^m\eta)(j+3) + 2(L^m\eta) + y + x \\ &\geq (L^m\eta)|j+3| + |y-x|.\end{aligned} \tag{G.11}$$

where we used again that $x, y \geq 0$.

Coming back to (G.2), we can write

$$\begin{aligned}\sum_{j>0,\, j \text{ odd}} c\mathrm{e}^{-\frac{1}{2}c_{\mathrm{st}}|(L^m\eta)(j+1)-y-x|} &\leq \sum_{j>0,\, j \text{ odd}} c\mathrm{e}^{-\frac{1}{2}c_{\mathrm{st}}((L^m\eta)(j-1)+|y-x|)} \\ &= c \sum_{j>0,\, j \text{ odd}} \mathrm{e}^{-\frac{1}{2}c_{\mathrm{st}}(L^m\eta)(j-1)} \mathrm{e}^{-\frac{1}{2}c_{\mathrm{st}}|y-x|} \\ &\leq c \sum_{j>0,\, j \text{ odd}} \mathrm{e}^{-\frac{1}{2}c_{\mathrm{st}}(j-1)} \mathrm{e}^{-\frac{1}{2}c_{\mathrm{st}}|y-x|} \\ &= c' \mathrm{e}^{-\frac{1}{2}c_{\mathrm{st}}|y-x|},\end{aligned} \tag{G.12}$$

where we noted that $L^m\eta \geq 1$. The sum of $j < 0$ is treated analogously, using (G.11).

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