Unitary modules and conformal nets associated with the $\mathcal{W}_3$-algebra with $c \geq 2$

Yoh Tanimoto

University of Rome "Tor Vergata"

IWOTA Lancaster 2020 Special Session
Quantum Groups and Algebraic Quantum Field Theory
18 August 2021

VOA and conformal nets (and quantum algebras)

- Two formulations of chiral components of two-dimensional conformal field theory: vertex operator algebras (VOA, algebraic), conformal net (analytic). For most examples there is a direct relation between them, Carpi-Kawahigashi-Longo-Weiner.
- Such a relation exists only if **unitary**.
- Representation theory. Cf. talk by Runkel, Kazhdan-Lusztig-Finkelberg, Carpi-Ciamprone-Giannone-Pinzari, Gui, Tener.

Main results

- The $\mathcal{W}_3$-algebra: a VOA extension of the Virasoro algebra (the stress-energy tensor)
- Unitarity of certain lowest weight representations, including the vacuum representation.
- Construction of the corresponding conformal nets (**new chiral CFT**) and some representations.

Vertex Operator Algebras

Formal definition: vector space V, field $Y(a,z) \in \mathrm{End}(V)[[z^{\pm 1}]]$, where $a \in V$, the vacuum vector $\Omega \in V$ satisfying certain axioms...

One can take "primary fields" $\phi_j(z) = Y(a_j, z)$, and these are diffeomorphism covariant **local fields** on S^1, that is, $\phi_j(z)$ are operator-valued distributions and $\phi_j(z)$ and $\phi_k(w)$ commute for $z \neq w$ in the sense of distributions: chiral components of a **two-dimensional conformal field theory**.

Conversely, from a family of diffeomorphism covariant local fields satisfying technical conditions, one can construct a VOA (Carpi-Kawahigashi-Longo-Weiner '18, cf. Frenkel-Kac-Radul-Wang '95 "the existence theorem")

Examples: Heisenberg algebra (the $\mathrm{U}(1)$-current $[a(z), a(w)] = \partial_w \delta(z-w)$), the Virasoro algebra (the stress-energy tensor), the WZW models...

The Virasoro algebra

The Virasoro algebra is generated by $\{L_n : n \in \mathbb{Z}, C\}$ with

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{C}{12}m(m^2-1)\delta_{m+n,0},$$

where C is a central element.

This is an infinite-dimensional Lie algebra, the central extension of the polynomial vector fields on S^1. One can construct **lowest weight representations (modules)** parametrized by $c, h \in \mathbb{R}$, where there is a vector Ω such that $L_n\Omega = 0$ for $n > 0, L_0\Omega = h\Omega, C\Omega = c\Omega$, and spanned by vectors of the form $L_{-n_1}\cdots L_{-n_k}\Omega$, $n_j > 0$. This is equipped with an invariant sesquilinear form $\langle\cdot,\cdot\rangle$, with respect to which $L_n^* = L_{-n}$.

One considers the field $L(z) = \sum_n L_n z^{-n-2}$ in the "vacuum representation" $h = 0, c \in \mathbb{R}$. Then this generates a VOA (modulo a certain quotient).

The Virasoro algebra: question of unitarity

Unitarity: the invariant sesquilinear form is positive semi-definite.

Unitary lowest weight representations are

- discrete series $c = 1 - \frac{6}{m(m+1)}, m = 2, 3, 4, \cdots,$
  $h = \frac{((m+1)r-ms)^2-1}{4m(m+1)}, r = 1, 2, \cdots, m-1, s = 1, 2, \cdots, r$
- continuous region $c \geq 1, h \geq 0$.

This is proven by

- constructing concrete unitary representations, by embedding the Virasoro algebra into some larger algebra (Goddard-Kent-Olive)
- computing the determinant of the Gram matrix on each subspace spanned by $L_{-n_1} \cdots L_{-n_k}\Omega$ with fixed $n = \sum n_j$, (Kac determinant formula, Feigin-Fuchs)
- for $c \geq 1, h \geq 0$, it is enough that there is one unitary representation
- proving that other values of c, h give non-unitary representations (Friedan-Qiu-Shenker)

The $\mathcal{W}_3$-algebra

A **non-Lie algebraic** extension of the Virasoro algebra:

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{C}{12}m(m^2-1)\delta_{m+n,0}, \\
[L_m, W_n] &= (2m-n)W_{m+n}, \\
[W_m, W_n] &= \frac{C}{3\cdot 5!}(m^2-4)(m^2-1)m\delta_{m+n,0} \\
&\quad + b^2(m-n)\Lambda_{m+n} + \frac{1}{20}(m-n)(2m^2-mn+2n^2-8)L_{m+n},
\end{aligned}$$

where $\Lambda_n = \sum_{k>-2} L_{n-k}L_k + \sum_{k\le -2} L_k L_{n-k} - \frac{3}{10}(n+2)(n+3)L_n$ and $b^2 = \frac{16}{22+5C}$.

The lowest weight representations ($L_n\Omega = W_n\Omega = 0$ for $n > 0$, $L_0\Omega = h\Omega, W_0\Omega = w\Omega, C\Omega = c\Omega$) are parametrized by $(c, h, w) \in \mathbb{R}$. If $h = w = 0$, a VOA can be constructed from $L(z) = \sum_n L_n z^{-n-2}$ and $W(z) = \sum_n W_n z^{-n-3}$.

The $\mathcal{W}_3$-field

As fields, they satisfy

$$\begin{aligned}
[L(z), L(\zeta)] &= \delta(z-\zeta)\partial_\zeta L(\zeta) + 2\partial_\zeta \delta(z-\zeta)L(\zeta) + \frac{c}{12}\partial_\zeta^3 \delta(z-\zeta), \\
[L(z), W(\zeta)] &= 3\partial_\zeta \delta(z-\zeta)W(\zeta) + \delta(z-\zeta)\partial_\zeta W(\zeta), \\
[W(z), W(\zeta)] &= \frac{c}{3\cdot 5!}\partial_\zeta^5 \delta(z-\zeta) + \frac{1}{3}\partial_\zeta^3 \delta(z-\zeta)L(\zeta) + \frac{1}{2}\partial_\zeta^2 \delta(z-\zeta)\partial L(\zeta) \\
&\quad + \partial_\zeta \delta(z-\zeta)\left(\frac{3}{10}\partial_\zeta^2 L(\zeta) + 2b^2 \Lambda(\zeta)\right) \\
&\quad + \delta(z-\zeta)\left(\frac{1}{15}\partial_\zeta^3 L(\zeta) + b^2 \partial_\zeta \Lambda(\zeta)\right)
\end{aligned}$$

where $b^2 = \frac{16}{22+5c}$ and $\Lambda(z) = :L(z)^2: - \frac{3}{10}\partial_z^2 L(z)$.

The $\mathcal{W}_3$-algebra: unitarity of the vacuum representations

- For $c \geq 2$, the $\mathcal{W}_3$-algebra can be realized in the tensor product of two free fields, but unitary only for $\alpha_0 = 0$ (Fateev-Zamolodchikov):

$$\tilde{L}(z;\alpha_0) = \frac{1}{2} : a_{[1]}(z)^2 : + \frac{1}{2} : a_{[2]}(z)^2 : + \sqrt{2}\alpha_0 \partial a_{[1]}(z),$$

$$\begin{aligned}\tilde{W}(z;\alpha_0) = \frac{b}{12i}\big[& i2\sqrt{2} : a_{[2]}(z)^3 : - i6\sqrt{2} : a_{[1]}(z)^2 : a_{[2]}(z) \\ & - i6\alpha_0 \partial a_{[1]}(z) a_{[2]}(z) - i18\alpha_0 a_{[1]}(z) \partial a_{[2]}(z) \\ & - i6\sqrt{2}\alpha_0^2 \partial^2 a_{[2]}(z)\big],\end{aligned}$$

  where :: represents the normal product.
- It is not obvious that the lowest weight representations exist for all values (the $\mathcal{W}_3$-algebra is not a Lie algebra), but they do. Even if some lowest weight representations are unitary, one cannot consider their tensor product.
- An analogue of Kac determinant formula has been computed (Mizoguchi, Afkhami=Jeddi-Colville-Hartman-Maloney-Perlmutter).

The $\mathcal{W}_3$-algebra: restoring the unitarity

- The Fateev-Zamolodchikov representation is not unitary for $\alpha_0 \neq 0$:

$$(z^3 \tilde{W}(z;\alpha_0))^* = z^3 \tilde{W}(z;\alpha_0), \quad (z^2 \tilde{L}(z;\alpha_0))^* = z^2 \tilde{L}(z;\alpha_0)$$

  with respect to the scalar product coming from the Heisenberg algebra $a_{[1]}(z), a_{[2]}(z)$.
- Consider the automorphism
  $a_{[1]}(z) \mapsto a_{[1]}(z) + i\frac{\alpha_0(z-1)}{\sqrt{2}z(z+1)} + \frac{i\alpha_0}{\sqrt{2}z}, \quad a_{[2]}(z) \mapsto a_{[2]}(z)$
  (cf. Buchholz-Schulz=Mirbach).
- By composition, for $\alpha_0 \in \mathbb{R}$ we restore unitarity except the point $z = -1$.
- On the subspace generated from $\Omega_{[1]} \otimes \Omega_{[2]}$, unitarity holds.
- (unitarity is a purely algebraic property, but this is proven for the first time for $c = 2 + 24\alpha_0^2 \geq 2$ by these techniques inspired by QFT)

The $\mathcal{W}_3$-algebra: restoring the unitarity

Theorem (Carpi-T.-Weiner arXiv:1910.08334, to appear in Transform. Groups)

The lowest weight representations of the $\mathcal{W}_3$-algebra associated with the values $c \geq 2, h = w = 0$ are unitary.

By composing this with further automorphisms that preserve unitarity, and using the Kac determinant formula, we also have

Theorem

Let $c \geq 2$. By the above construction, the irreducible lowest weight representation of the $\mathcal{W}_3$-algebra $V_{c,h,w}^{\mathcal{W}_3}$ is unitary for

$$h \geq \frac{c-2}{24}, \quad |w| \leq \sqrt{\frac{8}{198+45c}}\left(2h - \frac{c-2}{12}\right)^{\frac{3}{2}} \quad (h, w \in \mathbb{R}).$$

The $\mathcal{W}_3$-conformal net

A conformal net is associated to a uitary VOA if the generating fields commute strongly (**strong locality**, Carpi-Kawahigashi-Longo-Weiner '18)

- $\mathcal{A}(I) = \{W(f), L(f) : \operatorname{supp} f \subset I\}''$, the von Neumann algebra generated by the polar decomposition, where $W(f) = \sum_n \overline{f}_n W_n$.
- W-field has conformal dimension 3, and does not satisfy the linear energy bound.
- W **satisfies a local energy bound**

$$W(f^2)(L(f) + (r(f) + \epsilon)I)^{-2}$$

for some $r(f) > 0$, where f is a test function $\Rightarrow$ strong locality.

Theorem (Carpi-T.-Weiner arXiv:2103.16475)

The $\mathcal{W}_3$-algebra for $c \geq 2$ has an associated unitary simple VOA which is strongly local. One can construct the corresponding conformal Haag-Kastler net on S^1.