

Towards lattice construction of QFT according to Bałaban-Dimock

Yoh Tanimoto

University of Rome “Tor Vergata”

Joint with W. Dybalski, A. Stottmeister

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Summary

- We want to construct QFT.
- We want to construct the ϕ_3^4 model (cf. Glimm-Jaffe, Feldman-Osterwalder, Magnen-Sénéor, Seiler-Simon, Park, Gubinelli-Hofmanová, Duch-Hairer-Yi-Zhao...), with **techniques potentially applicable** to more interesting models.
- Take a classical Lagrangian. Discretize. Split the small and large field regions. Determine the counterterms by the renormalization group (RG) method for the small field regions, while large field regions get tiny factors. Prove the existence of the limit.
- In ϕ_3^4 , there is a huge simplification (superrenormalizability).
- So we want to complete the program in this case (following Bałaban-Dimock), and then try to extend it to more difficult models.
- Partial results: convergence of the partition functions under small field conditions.

QFT as Schwinger functions

We take the Euclidean formulation of QFT.

A set of **Schwinger functions** (n -point correlation functions) consists of a set of distribution $\{S_n\}$ (defined on the noncoincident points in \mathbb{R}^{dn} , where the d is the dimension of the model) satisfying the **Osterwalder-Schrader axioms** [OS73](#), [OS75](#):

- Locality
- Euclidean invariance
- Reflection positivity
- Clustering

If we have such $\{S_n\}$, then we can *reconstruct* Gårding-Wightman quantum fields on a Hilbert space, by first inverse-Wick-rotating $\{S_n\}$ to the Minkowski space then reconstructing the operator-valued distributions.

Lattice construction of QFT

Symbolically we would like to have a measure μ on the space of distributions given by the weight, say for $V(\phi(x)) = \frac{1}{4}\lambda\phi(x)^4$,

$$\phi \mapsto \frac{e^{-\int dx(\frac{1}{2}\sum_{\nu}(\partial_{\nu}\phi(x))^2 + \frac{1}{2}\mu\phi(x)^2 + V(\phi(x)))}}{\int \mathcal{D}\phi e^{-\int dx(\frac{1}{2}\sum_{\nu}(\partial_{\nu}\phi(x))^2 + \frac{1}{2}\mu\phi(x)^2 + V(\phi(x)))}}$$

and define the correlation functions

$$S_n(f_1, \dots, f_n) = \int d\mu \phi(f_1) \cdots \phi(f_n).$$

But we need to make sense of $\mathcal{D}\phi$ and $\phi(x)^2$, $V(\phi(x)) = \frac{1}{4}\lambda\phi(x)^4$ for distributions.

\Rightarrow discretize the space, define correlation functions on lattices and take the continuum limit.

Need for renormalization

Say that we would like to have the continuum limit that behaves like the measure defined on the **unit lattice** by (with $\int dx = L^{-3N} \sum_{x \in \mathbb{T}_M^{-N}}$)

$$\rho^0(\phi) = \exp \left(-\frac{1}{2} \langle \phi, (-\Delta + \bar{\mu}) \phi \rangle - V^0(\phi) \right).$$

We make the **ansatz** that, on the **fine lattice** with lattice spacing L^{-N} , the model is given by

$$\begin{aligned} \rho^N(\phi) &= \exp \left(-\frac{1}{2} \langle \phi, (-\Delta + \bar{\mu}) \phi \rangle - V^N(\phi) \right) \\ V^N(\phi) &= \int dx \left(\varepsilon^N + \frac{1}{2} \mu^N \phi(x)^2 + \frac{1}{4} \lambda \phi(x)^4 \right), \end{aligned}$$

where ε^N, μ^N are *counterterms*.

We want to choose counterterms ε^N, μ^N in such a way that the model “looks like” ρ^0 when we average out the field in the unit lattice.

(exponential) Block averaging

We want to choose counterterms ε^N, μ^N in such a way that the model “looks like” ρ_0 when we average out the field in the unit lattice.

For a probability weight $\rho(\phi)$ where

- ϕ lives on the *fine* lattice with spacing L^{-N} ,

define for

- Φ_N on the *unit* lattice with spacing 1

the following probability weight for Φ_N :

$$\rho_N(\Phi_N) := \mathcal{N}_{a,N}^{-1} \int d\phi \exp\left(-\frac{a_N}{2} \|\Phi_N - Q_N\phi\|^2\right) \rho(\phi),$$

where $Q_N\phi(x)$ is the average of ϕ in the unit box containing x on the unit lattice.

We should choose ε^N, μ^N in such a way that $\rho_N(\Phi_N)$ looks similar to ρ_0 on the unit lattice (*renormalization*).

The ϕ_3^4 -model

Cf. Dimock '13 Rev. Math. Phys., J. Math. Phys., '14 Ann. Henri Poincaré "The Renormalization Group According to Balaban I, II, III"

- Fix $L \in \mathbb{N}, L > 0$ (fixed, large)
- For M, N , consider $\mathbb{T}_M^{-N} = L^{-N}\mathbb{Z}^3/L^M\mathbb{Z}^3$.
- L^M : lattice size, L^{-N} : lattice spacing
- For $\phi : \mathbb{T}_M^{-N} \rightarrow \mathbb{R}$, define (with $\int dx = L^{-3N} \sum_{x \in \mathbb{T}_M^{-N}}$)

$$\rho_0^N(\phi) = \exp\left(-\frac{1}{2}\langle\phi, (-\Delta + \bar{\mu})\phi\rangle - V^N(\phi)\right),$$

$$V^N(\phi) = \varepsilon^N \text{Vol}(\mathbb{T}_M) + \frac{1}{2}\mu^N \int \phi(x)^2 dx + \frac{1}{4}\lambda \int \phi(x)^4 dx.$$

- $\bar{\mu}, \lambda$ are parameters, fixed.
- We can define correlation functions $S_{N,n}$ as distributions.
- ε^N, μ^N are counterterms. We should choose them in such a way that the limit $N \rightarrow \infty$ of $S_{N,n}$ converges to a nontrivial model.

The block averaging

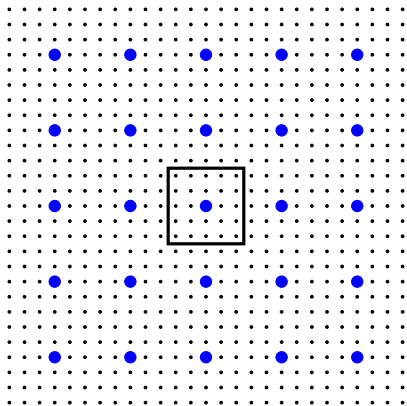
- (scale up to the unit lattice to consider the flow of coupling)
- We want to fix $\varepsilon_0^N := L^{-3N} \varepsilon^N$, $\mu_0^N := L^{-2N} \mu^N$ in such a way that the model seems to have the prescribed parameters $\bar{\mu}$.
- Block averaging: $(Q_N \phi)(y) = L^{-3N} \sum_{x \in B^N(y)} \phi(x)$, $B^N(y)$ is the unit box (containing L^{3N} vertices).
- For $a_N > 0$ (a certain bounded sequence), $\tilde{\Phi}_N : \mathbb{T}_M^N \rightarrow \mathbb{R}$,

$$\tilde{\rho}_N^N(\tilde{\Phi}_N) = \mathcal{N}_{a_N}^{-1} \int d\phi \exp\left(-\frac{a_N}{2} \|\tilde{\Phi}_N - Q_N \phi\|^2\right) \rho_0^N(\phi)$$

(then scale back to the unit lattice $\Phi_N : \mathbb{T}_{M-N}^0, \rho_N^N$. tilde will be omitted)

- ρ_0^N depends on ε_0^N, μ_0^N . We want to choose them in such a way that ρ_N^N “looks like”

$$\rho_N^N(\Phi_N) \sim \exp\left(-\frac{1}{2} \langle \Phi_N, (-\Delta + \bar{\mu}) \Phi_N \rangle - \frac{1}{4} \lambda \int \Phi_N(x)^4 dx\right).$$



Theorem (Dimock '14)

There are choices of ε_0^N, μ_0^N such that

$$Z_{M,N} := \int d\phi \rho_0^N(\phi), \quad Z_{M,N}(0) : \text{the value with } \lambda = 0,$$
$$\exp(-c \text{Vol } \mathbb{T}_M) \leq \frac{Z_{M,N}}{Z_{M,N}(0)} \leq \exp(c \text{Vol } \mathbb{T}_M),$$

that is, $\frac{Z_{M,N}}{Z_{M,N}(0)}$ is bounded as $N \rightarrow \infty$.

- How are the choices made?
- Does the sequence converge, not just for subsequence?
- Do correlation functions converge?

$$S_{N,n}(f_1, \dots, f_n) = \frac{\int d\phi \rho(\phi) \phi(f_1) \cdots \phi(f_n)}{Z_{M,N}}$$

Cluster expansion

One might want to use cluster expansion

$$\begin{aligned}\rho_N^N(\Phi_N) &= \mathcal{N}_{a_N}^{-1} \int d\phi \exp\left(-\frac{a_N}{2} \|\Phi_N - Q_N \phi\|^2\right) \rho_0^N(\phi) \\ &\stackrel{?}{=} \exp\left(\sum_X \underline{H}^\sharp(\Phi_N, X)\right)\end{aligned}$$

but we want to have detailed control over $\underline{H}^\sharp(\Phi_N, X)$:

- A main contribution should be of the classical form
- The rest should be estimated by $e^{-\kappa|X|}$
- The expansion converges (for some configurations Φ_N)

We achieve this by

- Doing N-times block averaging to L -box, only for “small fields”.
- Extracting the minimum of the quadratic part.
- The ϕ^4 term remains the same (superrenormalizability)

One step renormalization group

Let $Q\phi$ be the averaged field in the box of size L . We define

$$\begin{aligned}\rho_1^N(\Phi_1) &= \mathcal{N}_a^{-1} \int d\phi \exp \left(-\frac{a}{2} \|\Phi_1 - Q\phi\|^2 - \frac{1}{2} \langle \phi, (-\Delta + \bar{\mu})\phi \rangle - V^N(\phi) \right) \\ &= \mathcal{N}_a^{-1} \int d\phi \exp \left(-S_1(\Phi_1, \phi) - V^N(\phi) \right)\end{aligned}$$

and $S_1(\Phi_1, \phi)$ is a quadratic form with respect to ϕ . Let

$\phi_1(\Phi_1) = a_1 G_1 Q^T \Phi_1$ be its minimum, where

$$G_1 = (-\Delta + \bar{\mu} + aQ^T Q)^{-1}.$$

The lattice Green function G_1 is nonlocal but the kernel decays exponentially (Bałaban, [Dybalski-Stottmeister-T. '23](#)) so has the random walk expansion

$$G_1 = \sum_{n=0}^{\infty} \sum_{\omega_0, \dots, \omega_n} \underline{G}_1(\square_{\omega_0}) \cdots \underline{G}_1(\square_{\omega_n}).$$

One step renormalization group

Write $\phi = \phi_1 + Z = aG_1 Q^T \Phi_1 + Z$, then the quadratic part in Z takes the form $-\frac{1}{2}\langle Z, (-\Delta + \bar{\mu} + aL^{-2}Q^T Q)Z \rangle = -\langle Z, C_0^{-1}Z \rangle$. Extract the Z -independent part:

$$\begin{aligned}\rho_1^N(\Phi_1) &= \exp\left(-S_1^0(\Phi_1, \phi_1)\right) \int dZ \exp\left(-\langle Z, C_0^{-1}Z \rangle + E^+(\phi_1, Z)\right) \\ &= Z_1 \exp\left(-S_1^0(\Phi_1, \phi_1) + E^+(\phi_1, 0)\right) \int d\mu_l(W) \exp(\delta E^+(\phi_1, W)),\end{aligned}$$

where E^+ term contains $\int dx \phi(x)^4$ and

$\delta E^+(\phi_1, W) := E^+(\phi_1, W) - E^+(\phi_1, 0)$. We changed to $Z = W = C_0^{\frac{1}{2}} W$, where $d\mu_l(W)$ is ultralocal (the product measure with the identity covariance), while $C_0^{\frac{1}{2}}$ is nonlocal but has random walk expansion.

Small field conditions

We want to apply cluster expansion

$$\begin{aligned} & \rho_1^N(\Phi_1) \\ &= Z_1 \exp\left(-S_1^0(\Phi_1, \phi_1) + E^+(\phi_1, 0)\right) \int d\mu_l(W) \exp(\delta E^+(\phi_1, W)) \\ &\stackrel{?}{=} Z_1 \exp\left(-S_1^0(\Phi_1, \phi_1) - V_1(\phi_1) + E_1(\phi_1)\right) \end{aligned}$$

But we have to impose the **small field conditions**, for a fixed $p > 1$,

- $|\Phi_1 - Q\phi_1| \leq (N \log L - \log \lambda)^p$,
- $|\partial\phi_1| \leq (N \log L - \log \lambda)^p$,
- $|\phi_1| \leq \lambda^{-\frac{1}{4}} (N \log L - \log \lambda)^p$.

and take a *large block* M to apply random walk expansion.

If any of these conditions is violated *somewhere*, $\rho_1^N(\Phi_1)$ gets a factor $\sim \exp(-(N \log L - \log \lambda)^{2p})$ for each such violation.

RG flow and coupling constant

So actually we add some characteristic functions $\underline{\chi}_0^w, \chi_0$, then

$$\begin{aligned}\rho_1^N(\Phi_1) &= \mathcal{N}_a^{-1} \int d\phi \exp\left(-\frac{a}{2}\|\Phi_1 - Q\phi\|^2\right) \rho_0^N(\phi) \underline{\chi}_0^w(\Phi_1) \chi_0(\phi) \\ &= Z_1 \exp(-S_1(\Phi_1, \phi_1) - V_1(\phi_1) + E_1(\phi_1)).\end{aligned}$$

By repeating this k times under the small field condition, one has

$$\rho_k^N(\Phi_k) = Z_k \exp(-S_k(\Phi_k, \phi_k) - V_k(\phi_k) + E_k(\phi_k)).$$

We consider the maps

$$E_{k+1} = \mathcal{T}_E^N(E_k, \mu_k), \quad \mu_{k+1} = \mathcal{T}_\mu^N(E_k, \mu_k)$$

The sequence of maps $\mathcal{T}_\bullet^N(E_k, \mu_k)$, $k = 0, \dots, N-1$ is the **renormalization group flow**. [Dimock Part I, Theorems 14, 24]

We impose $\mu_N^N = 0, E_0^N = 0$. There is a solution for λ small.

Not all configurations satisfy the small field conditions...

$$\rho_k(\Phi_k) = Z_k \sum_{\mathbf{\Pi}} \int d\Phi_{k,\Omega^c} dW_{N,\mathbf{\Pi}} K_{k,\mathbf{\Pi}} C_{k,\mathbf{\Pi}} \chi_k \exp\left(-S_k^+(\Lambda_k) + E_N(\Lambda_k) + R_{k,\mathbf{\Pi}}(\Lambda_k) + B_{k,\pi}(\Lambda_k)\right)$$

where $\sum_{\mathbf{\Pi}}$ denotes the sum over all possible divisions of configurations into possible small field regions in $\mathbf{\Pi} = \Lambda_0 \supset \Omega_0 \supset \Lambda_1 \supset \cdots \supset \Omega_k$.

If there are more large field regions, the contribution gets more factors of $\exp(-p_k^2)$, $p_k = ((N - k) \log L - \log \lambda)^p$.

Convergence?

- UV stability: $Z_{M,N}/Z_{M,N}(0) = \int d\phi \rho^N(\phi)/Z_{M,N}(0)$ is bounded.
- UV convergence(?): $Z_{M,N}/Z_{M,N}(0) = \int d\phi \rho^N(\phi)/Z_{M,N}(0)$ is convergent.

Actually we want the convergence of n -point functions, which can be calculated through partition functions $Z_{M,N}(\lambda; J)$ with “source” J .

Having convergence is important for Euclidean invariance.

So we want to compare $Z_{M,N}$ and $Z_{M,N+1}$ at the k -th and $(k+1)$ -st step of their flows.

Let us look at the small field case:

$$Z_{M,N}^P = Z_k^N \int e^{-S_k^N(\Phi_k^N, \phi_k^N) - V_k^N(\phi_k^N) + E_k^N(\phi_k^N)} \underline{\chi}_k^w \chi_k d\Phi_k^N$$

$$Z_{M,N+1}^P = Z_{k+1}^{N+1} \int e^{-S_{k+1}^{N+1}(\Phi_{k+1}^{N+1}, \phi_{k+1}^{N+1}) - V_{k+1}^{N+1}(\phi_{k+1}^{N+1}) + E_{k+1}^{N+1}(\phi_{k+1}^{N+1})} \underline{\chi}_{k+1}^w \chi_{k+1} d\Phi_{k+1}^{N+1}$$

Comparing flows

Under the small field conditions, we define the flows

$$\text{N-th flow} \quad \mu_{k+1}^N = \mathcal{T}_\mu(\mu_k^N, E_k^N), E_{k+1}^N = \mathcal{T}_E(\mu_k^N, E_k^N)$$

$$\mu_N^N = 0, E_0^N = 0,$$

$$\text{(N + 1)-st flow} \quad \mu_{k+1}^{N+1} = \mathcal{T}_\mu(\mu_k^{N+1}, E_k^{N+1}), E_{k+1}^{N+1} = \mathcal{T}_E(\mu_k^{N+1}, E_k^{N+1})$$

$$\mu_{N+1}^{N+1} = 0, E_0^{N+1} = 0$$

from

$$\rho_{k+1}^{N'}(\Phi_{k+1}^N) = \int \exp\left(-\frac{a}{2}\|\Phi_{k+1}^N - Q\Phi_k^N\|\right) \rho_k^{N'}(\Phi_k^N) \underline{\chi}_k^{N,w}(\Phi_k^N) \chi_k^N(\Phi_k^N) d\Phi_k^N$$

where

$$\rho_k^{N'}(\Phi_k) = Z_k \exp(-S_k(\Phi_k, \phi_k) - V_k(\phi_k) + E_k(\phi_k)).$$

We need to estimate the difference between the RG flow for $N, N + 1$, e.g.,

$$E_k^N(\phi_k^N) - E_{k+1}^{N+1}(\phi_{k+1}^{N+1}), \quad \mu_k^N - \mu_{k+1}^{N+1},$$

Comparing flows

Look at the E -terms. We write

$$E_k^N(\phi_k^N) - E_k^N(Q\phi_{k+1}^{N+1}) + E_k^N(Q\phi_{k+1}^{N+1}) - E_{k+1}^{N+1}(\phi_{k+1}^{N+1}).$$

The first difference is the continuity of the activity E_k^N . This reduced to the analyticity of E_k^N (Dimock '13 Rev. Math. Phys.) and the estimate of $\phi_k^N - Q\phi_{k+1}^{N+1}$ (King '86 Comm. Math. Phys.)

In the second difference, put $QE_k^N(\phi_{k+1}^{N+1}) = E_k^N(Q\phi_{k+1}^{N+1})$, so

$$E_k^N(Q\phi_{k+1}^{N+1}) - E_{k+1}^{N+1}(\phi_{k+1}^{N+1}) = QE_k^N(\phi_{k+1}^{N+1}) - E_{k+1}^{N+1}(\phi_{k+1}^{N+1}),$$

as the difference between functionals in the flow $N + 1$.

Comparing flows

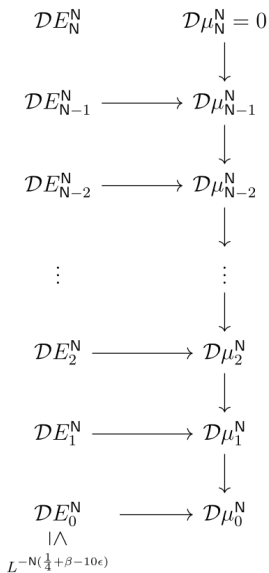
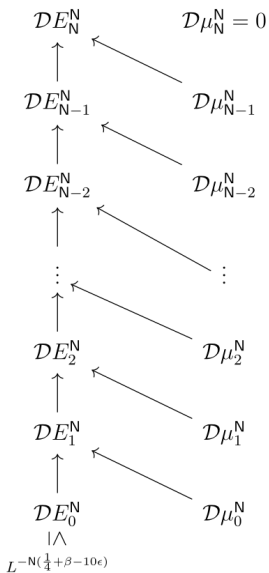
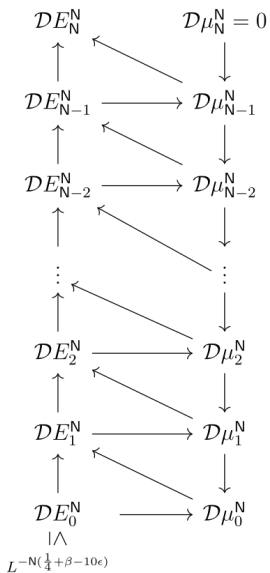
Assume that

$$E_k^N(Q\phi_{k+1}^{N+1}) - E_{k+1}^{N+1}(\phi_{k+1}^{N+1}) = QE_k^N(\phi_{k+1}^{N+1}) - E_{k+1}^{N+1}(\phi_{k+1}^{N+1}),$$

is small. At the next step, we have

$$\begin{aligned} & E_{k+1}^N(Q\phi_{k+2}^{N+1}) - E_{k+2}^{N+1}(\phi_{k+2}^{N+1}) \\ &= Q\mathcal{T}_E^N(E_k^N, \mu_k^N) - \mathcal{T}_E^{N+1}(QE_k^N, \mu_k^N)(\phi_{k+2}^{N+1}) \\ &\quad + \mathcal{T}_E^{N+1}(QE_k^N, \mu_k^N)(\phi_{k+2}^{N+1}) - \mathcal{T}_E^{N+1}(E_{k+2}^{N+1}, \mu_{k+1}^{N+1})(\phi_{k+2}^{N+1}). \end{aligned}$$

- The first term is the “commutator” between \mathcal{T} and Q , giving a factor $L^{-\gamma'k}$ using $\|G_{k+1}^{N+1} - QG_k^N Q^T\|_2 \leq CL^{-\gamma'k}$.
- The second term is the difference inside the $N + 1$ flows applied to QE_k^N and E_{k+1}^{N+1} . Inductive estimates. A factor of $L^{-\gamma'k}$.



Result (the difference between small field contributions)

Theorem (Dybalski-Stottmeister-T. work in progress)

For $0 \leq k \leq N$,

$$\begin{aligned} & \chi_{\cap-k'}^{\cap}(\Phi_{k'}) \\ & \times \left(S_k^N(\Phi_{k'}, \phi_k^N) + V_k^N(\phi_k^N) - E_k^N(\phi_k^N) \right. \\ & \quad \left. - S_{k+1}^{N+1}(\Phi_{k'}, \phi_{k+1}^{N+1}) - V_{k+1}^{N+1}(\phi_{k+1}^{N+1}) + E_{k+1}^N(\phi_{k+1}^{N+1}) \right) \\ & \leq C(L^{-\gamma_1 k} L^{3(N-k)} + L^{-\gamma_2(N-k)}) \text{vol} \mathbb{T}_M, \end{aligned}$$

where $k' = N - k$ and $\Phi_{k'} : \mathbb{T}_{M+k'}^0 \rightarrow \mathbb{R}$.

Take $k > \frac{3-\gamma_1}{3}$. The whole difference gets $L^{-\gamma''N}$, $\gamma'' > 0$.

Outlook/treatable models?

- ϕ_3^4 -model: full convergence, correlation functions, infinite volume limit
- large coupling constant/negative mass \Rightarrow Higgs mechanism?

(superrenormalizable)

- 3d Scalar QED (correlation functions using perturbation theory [King '86](#), existence of the flow [Dimock '15](#))
- 3d QED (UV stability [Dimock '22](#))
- YM_3 (UV stability [Bałaban '85](#))

(just renormalizable)

- YM_4 (UV stability up to a perturbative analysis? [Bałaban '89](#))
- 2d sigma models (solution of the variational problem [Dybalski-Stottmeister-T. 2025](#))