Operator-algebraic construction of integrable QFT

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Operator-algebraic QFT

Haag-Kastler nets

In the operator-algebraic approach, a model of **Quantum Field Theory** is a **family of von Neumann algebras** $\{\mathcal{A}(O)\}$ parametrized by open regions $\{O\}$ in the spacetime $\mathbb{R}^d$, which satisfies **locality**, covariance,...

- Not so many examples: Constructive QFT in $d = 2, 3$ (Glimm-Jaffe), Conformal QFT in $d = 1, 2$ (Kawahigashi, Longo, Rehren,...). **No nontrivial example in** $d = 4$.
- Traditional construction: take a "quantum field" ϕ (operator-valued distribution) and define $\mathcal{A}(O) := \{e^{i\phi(f)} : \mathrm{supp}\, f \subset O\}''$.

Recent construction methods

No **local** quantum field. Essential use of the Tomita-Takesaki theory.

- "**integrable**" models in $d = 2$ (Schroer, Lechner, T.,...)
- Models on the de Sitter spacetime $d \geq 2$? (Barata-Jäkel-Mund)

Half-sided modular inclusion

- $\mathcal{N} \subset \mathcal{M}$: von Neumann algebras,
- Ω: cyclic and separating for $\mathcal{M}, \mathcal{N}$
- Δ^{it}: the modular group of $\mathcal{M}$ with respect to Ω.

$\mathcal{N} \subset \mathcal{M}$ is a **Half-sided modular inclusion** if $\mathrm{Ad}\,\Delta^{it}(\mathcal{N}) \subset \mathcal{N}$ for $t \geq 0$.

Theorem (Wiesbrock, Araki-Zsido)

If $\mathcal{N} \subset \mathcal{M}$ is a half-sided modular inclusion with respect to Ω, then

$$P := \frac{1}{2\pi}(\log \Delta_{\mathcal{N}} - \log \Delta)$$

is self-adjoint and positive, where $\Delta_{\mathcal{N}}$ is the modular operator of $\mathcal{N}$. If we set $U(s) = e^{isP}$, then it holds that $\mathrm{Ad}\,\Delta^{it}(U(s)) = U(e^{-2\pi t}s)$ and $\mathrm{Ad}\,U(1)(\mathcal{M}) = \mathcal{N}$.

The theorem of Borchers

- $\mathcal{M}$: von Neumann algebras,
- Ω: cyclic and separating for $\mathcal{M}$
- Δ^{it}: the modular group of $\mathcal{M}$ with respect to Ω.

Theorem (Borchers)

Let $U(s)$ be a one-parameter group of unitaries with a positive generator such that $\mathrm{Ad}\, U(s)(\mathcal{M}) \subset \mathcal{M}$ for $s \geq 0$. Then it holds that
$\mathrm{Ad}\, \Delta^{it} U(s) = U(e^{-2\pi t} s)$

Half-sided modular inclusion and conformal nets ($d = 1$)

Conformal net in $d = 1$: a family of von Neumann algebras $\{\mathcal{A}(I)\}$ associated with intervals $\{I\}$ of the circle $S^1 = \mathbb{R} \cup \{\infty\}$.
Locality: $\mathcal{A}(I_1)$ and $\mathcal{A}(I_2)$ commute if $I_1 \cap I_2 = \emptyset$.

Theorem (Guido-Longo-Wiesbrock)

*If Ω is cyclic for $\mathcal{M} \cap \mathcal{N}'$ (**standardness**), then there is a conformal (Möbius-covariant) net $\mathcal{A}$ on S^1 such that*
$\mathcal{M} = \mathcal{A}(\mathbb{R}_+), \mathcal{N} = \mathcal{A}(\mathbb{R}_+ + 1), \mathcal{M} \cap \mathcal{N}' = \mathcal{A}(0, 1)$.
*There is a **one-to-one correspondence** between standard half-sided modular inclusions and conformal nets which are "strongly additive" ($\mathcal{A}(I_1) \vee \mathcal{A}(I_2) = \mathcal{A}(I)$, where $I \setminus \{p\} = I_1 \cup I_2$).*

Some problems:

- Can one construct half-sided modular inclusions "algebraically"?
- Is there half-sided modular inclusions which satisfy $\mathcal{M} \cap \mathcal{N}' = \mathbb{C}\mathbb{1}$?
- Find handy criteria which give standardness.

Wedge inclusion in $d = 2$

Goal: construct two-dimensional $d = 2$ nets $\{\mathcal{A}(O)\}$.
Idea: start with a **single** von Neumann algebra $\mathcal{A}(W_{\mathrm{R}})$,
$W_{\mathrm{R}} = \{(a_0, a_1) \in \mathbb{R}^2 : a_1 > |a_0|\}$.
A **Borchers triple** is a triple $(\mathcal{M}, U, \Omega)$ such that

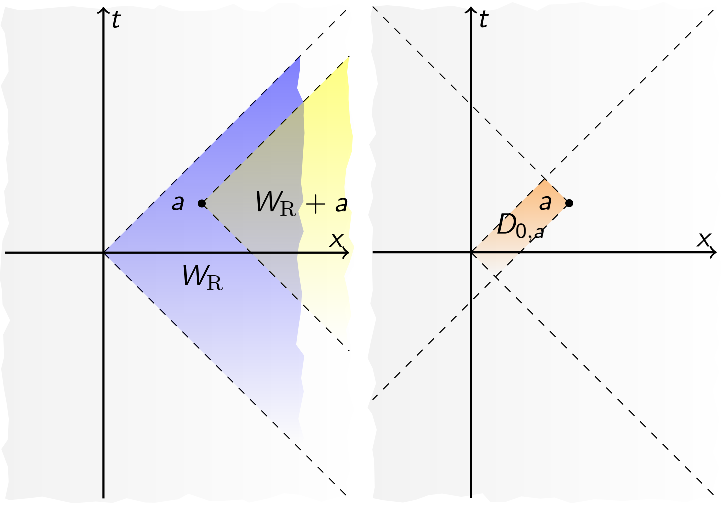
- $\mathcal{M}$: von Neumann algebra.
- U: unitary representation of $\mathbb{R}^2$ with "positive energy", $\mathrm{Ad}\, U(a)(\mathcal{M}) \subset \mathcal{M}$ for $a \in W_{\mathrm{R}}$.
- Ω: cyclic and separating for $\mathcal{M}$, $U(a)\Omega = \Omega$.

Lemma

*For a Borchers triple $(\mathcal{M}, U, \Omega)$ such that Ω is cyclic for $\mathcal{M} \cap \mathrm{Ad}\, U(a)(\mathcal{M}')$, $a \in W_{\mathrm{R}}$ (**strict locality**), one can define a net by $\mathcal{A}(D_{a,b}) := \mathrm{Ad}\, U(a)(\mathcal{M}) \cap \mathrm{Ad}\, U(b)(\mathcal{M}')$, $D_{a,b} = (W_{\mathrm{R}} + a) \cap (W_{\mathrm{L}} + b)$.*

Problem: Are there handy criteria for the cyclicity of Ω for the relative commutant $\mathcal{M} \cap \mathrm{Ad}\, U(a)(\mathcal{M}')$?

Standard wedge and double cone



Modular nuclearity

- $\mathcal{N} \subset \mathcal{M}$: inclusion of type III factors, Ω: cyclic and separating for both, Δ: the modular operator for $\mathcal{M}$.
- **Modular nuclearity** (Buchholz-D'antoni-Longo): if the map

$$\mathcal{N} \ni A \longmapsto \Delta^{\frac{1}{4}} A\Omega \in \mathcal{H}$$

  is nuclear, then there is a type I factor $\mathcal{R}$ s.t. $\mathcal{N} \subset \mathcal{R} \subset \mathcal{M}$ (**split**).
- Sketch of proof: By assumption, the map

$$\mathcal{N} \ni \mathcal{A} \longmapsto \langle JA\Omega, \cdot\, \Omega\rangle = \langle \Delta^{\frac{1}{2}} A^*\Omega, \cdot\, \Omega\rangle \in \mathcal{M}_*$$

  is nuclear. $\langle JBJ\Omega, A\Omega\rangle = \sum \varphi_{1,n}(A)\varphi_{2,n}(B)$ and one may assume that $\varphi_{k,n}$ are normal. This defines a normal state on $\mathcal{N} \otimes \mathcal{M}'$ which is equivalent to $\mathcal{N} \vee \mathcal{M}'$.
- **Bisognano-Wichmann property**: for $\mathcal{M} = \mathcal{A}(W_{\mathrm{R}})$ in a net $\mathcal{A}$, Δ^{it} is very often the Lorentz boost.

Internal symmetry

An **internal symmetry** on a Borchers triple $(\mathcal{M}, U, \Omega)$ is a unitary representation V of a group G such that $\mathrm{Ad}\, V(g)(\mathcal{M}) = \mathcal{M}$, $[V(g), U(a)] = 0$ and $V(g)\Omega = \Omega$.
We take an action V of S^1. $V(t) = e^{itQ}$. $\widetilde{V}(t) = e^{itQ\otimes Q}$.

Theorem (T.)

*Let $\widetilde{\mathcal{M}}_t = (\mathcal{M} \otimes \mathbb{1}) \vee \mathrm{Ad}\, \widetilde{V}(t)\, (\mathbb{1} \otimes \mathcal{M})$, $\widetilde{U}(a) = U(a) \otimes U(a)$, $\widetilde{\Omega} = \Omega \otimes \Omega$. Then $(\widetilde{\mathcal{M}}_t, \widetilde{U}, \widetilde{\Omega})$ is a Borchers triple inequivalent to the tensor product (**interacting**). If the starting triple satisfies modular nuclearity, so does also the new triple for $t \in 2\pi\mathbb{Q}$ ($\Longrightarrow$ new nets).*

Proof: $\widetilde{V}(t) = \sum_k V(kt) \otimes dE(k)$.
$\widetilde{\mathcal{M}}_t' = \mathrm{Ad}\, \widetilde{V}(t)\, (\mathcal{M}' \otimes \mathbb{1}) \vee (\mathbb{1} \otimes \mathcal{M}')$.

Lemma (Lechner): If there is a type I factor between $\mathrm{Ad}\, \widetilde{U}(a)(\widetilde{\mathcal{M}}_t)$ and $\widetilde{\mathcal{M}}_t$, $a \in W_{\mathrm{R}}$ (**wedge-split inclusion**), then strict locality follows.

Factorizing S-matrix models: Hilbert space

- **Input**: analytic function $S : \mathbb{R} + i(0, \pi) \to \mathbb{C}$,

$\overline{S(\theta)} = S(\theta)^{-1} = S(-\theta) = S(\theta + \pi i)$, $\theta \in \mathbb{R}$. e.g. $S(\theta) = \frac{\sinh\theta - i\sin b}{\sinh\theta + i\sin b}$

- S-symmetric Fock space: $\mathcal{H} = \bigoplus \mathcal{H}_n, \mathcal{H}_0 = \mathbb{C}\Omega, \mathcal{H}_1 = L^2(\mathbb{R}, d\theta)$, $\mathcal{H}_n = P_n\mathcal{H}_1^{\otimes n}$, where P_n is the projection onto S-**symmetric** functions:

$$\Psi_n(\theta_1, \cdots, \theta_n) = S(\theta_{k+1} - \theta_k)\Psi_n(\theta_1, \cdots, \theta_{k+1}, \theta_k, \cdots, \theta_n).$$

- Zamolodchikov-Faddeev algebra (deformed CCR): S-symmetrized creator/annihilator $z^\dagger(\xi) = Pa^\dagger(\xi)P, z(\xi) = Pa(\xi)P, P = \bigoplus_n P_n$, $(a(\xi)\Psi_n)(\theta_1, \cdot, \theta_n) = \int d\theta\overline{\xi(\theta)}\Psi_{n+1}(\theta, \theta_1, \cdots, \theta_n)$.
- **Wedge-local field**: $\phi(f) = z^\dagger(f^+) + z(J_1 f^-)$,

$$f^\pm(\theta) = \int dx e^{ix\cdot p(\theta)} f(x), \quad p(\theta) = (m\cosh\theta, m\cosh\theta),$$

$(J_1\xi)(\theta) = \overline{\xi(\theta)}$.

Factorizing S-matrix models: operators

$\phi'(g) = J\phi(g_j)J$, $g_j(x) = \overline{g(-x)}$,
$J = \sum J_n, (J_n\Psi_n)(\theta_1, \cdots, \theta_n) = \overline{\Psi_n(\theta_n, \cdots, \theta_1)}$.
If $\mathrm{supp}\, f \subset W_\mathrm{L}, \mathrm{supp}\, g \subset W_\mathrm{R}$, then

$$[e^{i\phi(f)}, e^{i\phi'(g)}] = 0.$$

- $\mathcal{M} := \{e^{i\phi'(g)} : \mathrm{supp}\, g \subset W_\mathrm{R}\}''$.
- $(U(a)\Psi)_n(\theta_1, \cdots, \theta_n) = e^{i\sum_k p(\theta_k)\cdot a}\Psi_n(\theta_1, \cdots, \theta_n)$.
- $\Omega \in \mathcal{H}_0$.

Theorem (Lechner)

$(\mathcal{M}, U, \Omega)$ is a Borchers triple. The modular group of $\mathcal{M}$ with respect to Ω is the Lorentz boost: $\Delta^{it}\Psi_n(\theta_1, \cdots, \theta_n) = \Psi_n(\theta_1 + 2\pi t, \cdots, \theta_n + 2\pi t)$.

Factorizing S-matrix models: modular nuclearity

Modular nuclearity for $\mathrm{Ad}\, U(a)(\mathcal{M}) \subset \mathcal{M}$: $\mathcal{M} \ni A \mapsto \Delta^{\frac{1}{4}} U(a) A\Omega \in \mathcal{H}$,

$$(\Delta^{\frac{1}{4}} U(a) A\Omega)_n(\theta_1, \cdots, \theta_n) = \\ e^{-ima_1 \sum_k \sinh(\theta_k - \frac{\pi i}{2})} (A\Omega)_n \left(\theta_1 - \frac{\pi i}{2}, \cdots, \theta_n - \frac{\pi i}{2}\right),$$

which contains a strongly damping factor $e^{-c \sum_k \cosh \theta_k}$.

- (1) Bounded analytic extension. (2) Cauchy integral.

$A \in \mathcal{M} \implies \langle [\phi(f), A]\Phi, \Psi\rangle$ is supported in $\mathbb{R}_- \implies$ analyticity of the Fourier transform of $\langle \Phi_n, A\Omega\rangle$ in each θ_k. $\implies$ represent the value at $(\theta_1 - \frac{\pi i}{2}, \cdots, \theta_n - \frac{\pi i}{2})$ as a Cauchy integral $\implies$ nuclearity at n.
For the whole map: need the condition $S(0) = -1$.

Theorem (Lechner)

For the Borchers triple $(\mathcal{M}, U, \Omega)$ associated to an S-matrix with a certain regularity condition, the inclusion $\mathrm{Ad}\, U(a)(\mathcal{M}) \subset \mathcal{M}$ satisfies the modular nuclearity for $a \in W_{\mathrm{R}}$ sufficiently large.

Factorizing S-matrix models: matrix-valued S

Models may have several species of particles. $\mathcal{H}_1 = L^2(\mathbb{R})^{\oplus D}$, S becomes a matrix-valued function and should satisfy the **Yang-Baxter equation**

$$S(\theta)_{12}S(\theta+\theta')_{23}S(\theta')_{12} = S(\theta')_{23}S(\theta+\theta')_{12}S(\theta)_{23},$$

to define a **representation** D_n of $\mathfrak{S}_n$ on $\mathcal{H}_1^{\otimes n}$. For $\tau_k = (k, k+1)$,

$$(D_n(\tau_k)\Psi_n)(\theta_1,\cdots,\theta_n) = S(\theta_{k+1}-\theta_k)\Psi_n(\theta_1,\cdots,\theta_{k+1},\theta_k,\cdots,\theta_n).$$

Theorem (Lechner-Schützenhofer)

One can construct a Borchers triple as before.

Lemma (Alazzawi-Lechner)

*The representation D_n is **unitarily equivalent** to the ferminionic representation $D_n^-(\tau_k)(\xi_1\otimes\cdots\otimes\xi_n) = (-1)\xi_1\otimes\cdots\otimes\xi_{k+1}\otimes\xi_k\otimes\cdots\otimes\xi_n$, intertwined by a **multiplication operator** in $L^\infty(\mathbb{R}^n, M_{D^{2n}}(\mathbb{C}))$.*

Problem: bounded analytic intertwiner?
($\Longrightarrow$ modular nuclearity)

S-matrix with poles

S: scalar S-matrix with poles at $\theta = \frac{\pi i}{3}, \frac{2\pi i}{3}$, $S(\theta) = S(\theta + \frac{\pi i}{3})S(\theta - \frac{\pi i}{3})$, $R = \mathrm{Res}_{\theta = \frac{2\pi i}{3}} S(\theta)$. P_n: S-symmetrization, $\mathcal{H} = \bigoplus P_n \mathcal{H}_1^{\otimes n}$, $\mathcal{H}_1 = L^2(\mathbb{R})$,

$$\mathrm{Dom}(\chi_1(f)) := H^2\left(-\tfrac{\pi}{3}, 0\right)$$
$$(\chi_1(f))\xi(\theta) := \sqrt{2\pi|R|} f^+\left(\theta + \tfrac{\pi i}{3}\right)\xi\left(\theta - \tfrac{\pi i}{3}\right),$$

where $H^2(\alpha, \beta)$ is the space of analytic functions in $\mathbb{R} + i(\alpha, \beta)$ such that $\xi(\cdot - \gamma i)$ is uniformly bounded in L^2-norm, $\gamma \in (\alpha, \beta)$, and f^+ is analytic.

$$\chi_n(f) = nP_n\left(\chi_1(f) \otimes I \otimes \cdots \otimes I\right)P_n,$$
$$\chi(f) := \bigoplus \chi_n(f),$$
$$(\chi_1'(g)\xi)(\theta) := (J_1\chi_1(g_j)J_1)(\theta) = \sqrt{2\pi|R|}g^+\left(\theta - \frac{\pi i}{3}\right)\xi\left(\theta + \frac{\pi i}{3}\right),$$
$$\chi_n'(g) := J_n\chi_n(g_j)J_n, \quad \chi'(g) := J\chi(g_j)J.$$

Wedge-local fields and weak commutativity

New field:

$$\widetilde{\phi}(f) := \phi(f) + \chi(f) = z^\dagger(f^+) + \chi(f) + z(J_1 f^-),$$
$$\widetilde{\phi}'(g) := J\widetilde{\phi}(g_j)J, \;\; \chi'(g) = J\chi(g_j)J.$$

Theorem (Cadamuro-T.)

For real $f, g, \mathrm{supp}\, f \subset W_{\mathrm{L}}, \mathrm{supp}\, g \subset W_{\mathrm{R}}$, *then*

$$\langle \widetilde{\phi}(f)\Phi, \widetilde{\phi}'(g)\Psi\rangle = \langle \widetilde{\phi}'(g)\Phi, \widetilde{\phi}(f)\Psi\rangle, \;\; \Phi, \Psi \in \mathrm{Dom}(\widetilde{\phi}(f)) \cap \mathrm{Dom}(\widetilde{\phi}'(g)).$$

Remark: $\chi_1(f)$ has a natural self-adjoint extension for nice f.
Problems:

- Is the operator $\chi_n(f) = nP_n(\chi_1(f) \otimes I \otimes \cdots \otimes I) P_n$ self-adjoint?
- Do the fields $\widetilde{\phi}(f), \widetilde{\phi}'(g)$ strongly commute? ($\Longrightarrow$ new nets)

Summary

- Operator algebras play a crucial role in QFT, not only in the classical analysis (Araki-Haag scattering theory, Doplicher-Haag-Roberts superselection theory), but in **construction of models**.
- **Tomita-Takesaki theory** is essential for covariance and locality.

Open problems

- Half-sided modular inclusions
- criteria for standard inclusion
- Yang-Baxter equation, analytic intertwiners
- strong commutativity of the fields for S with poles