Towards Haag-Kastler nets for integrable QFT with bound states

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Introduction

Mathematical formulations of **relativistic** quantum field theory:

- Wightman fields, Osterwalder-Schrader axioms
- **operator-algebraic approach** (Haag-Kastler nets)

Recent progress and **construction of 2d Haag-Kastler nets**:

- relations with Tomita-Takesaki theory (Bisognano-Wichmann '75, Borchers '92): Lorentz boosts = modular group
- factorizing scalar S-matrix models without bound states (Lechner '08)
- twisting by inner symmetry (T. '14)
- models on de Sitter spacetime (Barata-Jäkel-Mund, in progress)

Principal idea

- Construction of Haag-Kastler nets **not through** Wightman fields.
- **Observables in certain extended regions** are simpler. Compact regions can be obtained by intersection.

Quantum fields

Wightman axioms

- ϕ: operator-valued distribution on $\mathbb{R}^d$, $[\phi(x), \phi(y)] = 0$ if $x \perp y$ (**observable at** x)
- U: the spacetime symmetry, $U(g)\phi(x)U(g)^* = \phi(gx)$
- Ω the vacuum vector

Equivalently, one considers n-point functions (Wightman functions)

$$W(x_1, t_1, x_2, t_2, \cdots, x_n, t_n) = \langle \Omega, \phi(x_1, t_1)\phi(x_2, t_2) \cdots \phi(x_n, t_n)\Omega \rangle.$$

or their Wick-rotations $S(\cdots x_k, t_k \cdots) := W(\cdots x_k, it_k \cdots)$ (Schwinger functions, Osterwalder-Schrader axioms).

- examples in 2 and 3 dimensions: constructive QFT (Glimm, Jaffe, ...), conformal field theories.
- $\phi(f) = \int dx f(x)\phi(x)$ is an unbounded operator.

Net of observables

Haag-Kastler net

$\mathcal{A}(O)$: **von Neumann algebras** (weakly closed $*$-algebras of bounded operators on a Hilbert space $\mathcal{H}$) parametrized by open regions $O \subset \mathbb{R}^d$

- isotony: $O_1 \subset O_2 \Rightarrow \mathcal{A}(O_1) \subset \mathcal{A}(O_2)$
- locality: $O_1 \perp O_2 \Rightarrow [\mathcal{A}(O_1), \mathcal{A}(O_2)] = 0$
- Poincaré covariance: $\exists U$: positive energy rep of $\mathcal{P}_+^\uparrow$ such that $U(g)\mathcal{A}(O)U(g)^* = \mathcal{A}(gO)$
- vacuum: $\exists \Omega$ such that $U(g)\Omega = \Omega$ and cyclic for $\mathcal{A}(O)$

Correspondence: $\mathcal{A}(O) = \{e^{i\phi(f)} : \mathrm{supp} f \subset O\}''$
(**observables measurable in** O, $\mathcal{M}' = \{x : [x, y] = 0, y \in \mathcal{M}\}$)
One can extract **S-matrix** from a net (Araki-Haag scattering theory)
$\Longrightarrow$ Haag-Kastler net contains all the physical information of the model.

Observables in wedges

Interacting quantum fields are difficult to construct (c.f. form factor program: expand Wightman functions by matrix elements of the field. The problem of convergence remains for almost all models).

In terms of Haag-Kastler net: **Infinitely many** von Neumann algebras with consistency conditions $\Longrightarrow$ difficult to construct nets **directly**.

Consider observables in **wedge**-region

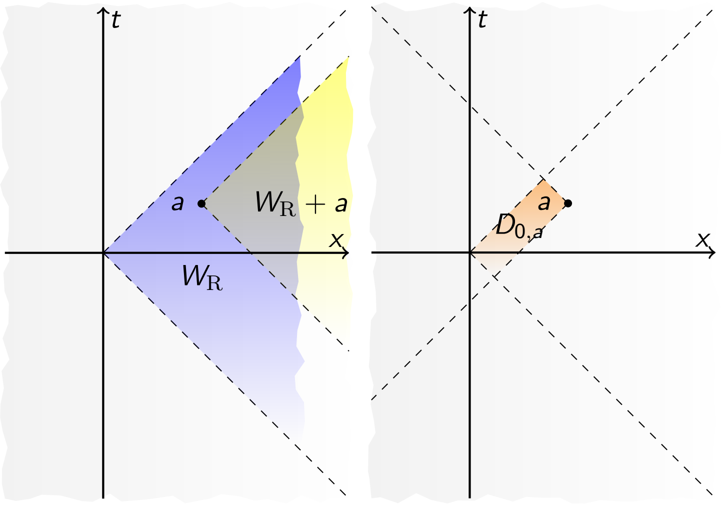
$$W_{\mathrm{R}} := \{a = (a_0, a_1) : |a_0| < a_1\}.$$

Extended regions contain more observables, hence more **tractable** ones.

In **two-dimensional** space time, the whole net $\mathcal{A}$ can be recovered from wedges (Borchers '92):

$$\mathcal{A}(D_{a,b}) = (U(a)\mathcal{A}(W_{\mathrm{R}})U(a)^*) \cap (U(b)\mathcal{A}(W_{\mathrm{R}})'U(b)^*).$$

Standard wedge and double cone



Borchers triples

Definition

$(\mathcal{M}, T, \Omega)$, where $\mathcal{M}$: vN algebra, T: positive-energy rep of $\mathbb{R}^2$, Ω: vector, is a Borchers triple if Ω is cyclic and separating for $\mathcal{M}$ and

- $\mathrm{Ad}\, T(a)(\mathcal{M}) \subset \mathcal{M}$ for $a \in W_{\mathrm{R}}$, $T(a)\Omega = \Omega$

Borchers triple $\Longrightarrow$ net

If one defines a "net" by $\mathcal{A}(D_{a,b}) := (U(a)\mathcal{M}U(a)^*) \cap (U(b)\mathcal{M}'U(b)^*)$, then T can be extended to a rep U of Poincaré group and (A, U, Ω) **satisfies all the axioms of local net except the cyclicity of vacuum**.

General construction scheme

- to construct new Borchers triples (wedge-local QFT)
- to show the cyclicity of vacuum (strict locality)

An example of wedge-construction

An **internal symmetry** on a Borchers triple $(\mathcal{M}, T, \Omega)$ is a unitary representation W of a group G such that $\mathrm{Ad}\, V(g)\mathcal{M} = \mathcal{M}$, $[V(g), T(a)] = 0$ and $V(g)\Omega = \Omega$.
We take an action V of S^1. $V(t) = e^{itQ}$. $\widetilde{V}(t) = e^{itQ\otimes Q}$.

Theorem (T. '14, arXiv:1301.6090, Forum of Mathematics, Sigma)

Let $\widetilde{\mathcal{M}}_t = (\mathcal{M} \otimes \mathbb{1}) \vee \mathrm{Ad}\widetilde{V}(t)\,(\mathbb{1} \otimes \mathcal{M})$, $\widetilde{T}(a) = T(a) \otimes T(a)$, $\widetilde{\Omega} = \Omega \otimes \Omega$. *Then* $(\widetilde{\mathcal{M}}_t, \widetilde{T}, \widetilde{\Omega})$ *is a Borchers triple.*

Proof: $\widetilde{V}(t) = \sum_k V(kt) \otimes dE(k)$.
$\widetilde{\mathcal{M}}'_t = \mathrm{Ad}\widetilde{V}(t)\,(\mathcal{M}' \otimes \mathbb{1}) \vee (\mathbb{1} \otimes \mathcal{M}')$.

Strict locality holds if the starting triple satisfies wedge-split property $\Longrightarrow$ new Haag-Kastler nets with **nontrivial S-matrix** $e^{itQ\otimes Q}$.
(complex free field $\Longrightarrow$ Federbush model? Wedge-split in $P(\phi)_2$ models?)

Free field

- Symmetric Fock space: $\mathcal{H}_1 = L^2(\mathbb{R}, d\theta)$, $\mathcal{H}_n = P_n \mathcal{H}_1^{\otimes n}$, where P_n is the projection onto **symmetric** functions:

$$\Psi_n(\theta_1, \cdots, \theta_n) = \Psi_n(\theta_1, \cdots, \theta_{k+1}, \theta_k, \cdots, \theta_n).$$

- Unsymmetrized annihilation operator:

$$(a(\xi)\Psi)_n(\theta_1, \cdots, \theta_n) = \int d\theta\, \overline{\xi(\theta)} \Psi_{n+1}(\theta, \theta_1, \cdots, \theta_n),$$

  and symmetrized annihilation and creation operators:
  $z^\dagger(\xi) = Pa^\dagger(\xi)P, z(\xi) = Pa(\xi)P, P = \bigoplus_n P_n$.

- **Free field**: $\phi(f) = z^\dagger(f^+) + z(J_1 f^-)$,

$$f^{\pm}(\theta) = \int dx e^{ix\cdot p(\theta)} f(x), \quad p(\theta) = (m\cosh\theta, m\cosh\theta),$$

  If $\mathrm{supp} f \perp \mathrm{supp} g$, then $[e^{i\phi(f)}, e^{i\phi(g)}] = 0$.

Factorizing S-matrix models (Lechner, Schroer)

- **Input**: analytic function $S : \mathbb{R} + i(0,\pi) \to \mathbb{C}$,

$$\overline{S(\theta)} = S(\theta)^{-1} = S(-\theta) = S(\theta + \pi i), \quad \theta \in \mathbb{R}.$$

- S-symmetric Fock space: $\mathcal{H}_1 = L^2(\mathbb{R}, d\theta)$, $\mathcal{H}_n = P_n \mathcal{H}_1^{\otimes n}$, where P_n is the projection onto S-**symmetric** functions:

$$\Psi_n(\theta_1, \cdots, \theta_n) = S(\theta_{k+1} - \theta_k)\Psi_n(\theta_1, \cdots, \theta_{k+1}, \theta_k, \cdots, \theta_n).$$

- Zamolodchikov-Faddeev algebra: S-symmetrized creation and annihilation operators $z^\dagger(\xi) = Pa^\dagger(\xi)P, z(\xi) = Pa(\xi)P, P = \bigoplus_n P_n$.
- **Wedge-local field**: $\phi(f) = z^\dagger(f^+) + z(J_1 f^-)$,

$$f^\pm(\theta) = \int dx e^{ix\cdot p(\theta)} f(x), \quad p(\theta) = (m\cosh\theta, m\cosh\theta),$$

J_1 is the one-particle CPT operator, $\phi'(g) = J\phi(g_j)J$, $g_j(x) = \overline{g(-x)}$. If $\mathrm{supp} f \subset W_\mathrm{L}, \mathrm{supp} g \subset W_\mathrm{R}$, then $[e^{i\phi(f)}, e^{i\phi'(g)}] = 0$.

Observables in wedges and double cones

One defines

$$\mathcal{M} = \{e^{\phi'(g)} : \mathrm{supp} g \subset W_{\mathrm{R}}\}''$$
$$(U(a)\Psi_n)(\theta_1, \cdots, \theta_n) = e^{ip(\theta)\cdot a}\Psi_n(\theta_1, \cdots, \theta_n),$$

then $(\mathcal{M}, U, \Omega)$ is a Borchers triple. U can be extended to the Poincaré group: the Lorentz boosts are given by the modular group Δ^{it}: $J\Delta^{\frac{1}{2}}x\Omega := x^*\Omega$.

One defines a "net" by $\mathcal{A}(D_{a,b}) := (U(a)\mathcal{M}U(a)^*) \cap (U(b)\mathcal{M}'U(b)^*)$

One can prove that $\mathcal{A}(D_{0,a})$ is sufficiently large by checking the **modular nuclearity** condition: let Δ^{it} be the Lorentz boosts (the modular group). The map

$$\mathcal{M} \ni x \longmapsto \Delta^{\frac{1}{4}}U(a)x\Omega$$

is nuclear $\Longrightarrow$ Haag-Kastler net with two-particle S-matrix S (Lechner '08).

S-matrix with poles

If S has a pole:

$$[\phi(f), \phi'(g)]\Psi_1(\theta_1) =$$
$$-\int d\theta \, (f^+(\theta)g^-(\theta)S(\theta_1 - \theta) - f^+(\theta + \pi i)g^-(\theta + \pi i)S(\theta_1 - \theta + \pi i))$$
$$\times \Psi_1(\theta_1)$$

obtains the **residues** of S and does not vanish.

- Example (Bullough-Dodd models): poles at $\theta = \frac{\pi i}{3}, \frac{2\pi i}{3}$, residues $-R, R$

$$S_B(\theta) = \frac{\tanh \frac{1}{2}\left(\theta + \frac{2\pi i}{3}\right)}{\tanh \frac{1}{2}\left(\theta - \frac{2\pi i}{3}\right)} \cdot \frac{\tanh \frac{1}{2}\left(\theta + \frac{(B-2)\pi i}{3}\right)}{\tanh \frac{1}{2}\left(\theta - \frac{(B-2)\pi i}{3}\right)} \frac{\tanh \frac{1}{2}\left(\theta - \frac{B\pi i}{3}\right)}{\tanh \frac{1}{2}\left(\theta + \frac{B\pi i}{3}\right)},$$

where $0 < B < 2, B \neq 1$. $S(\theta) = S\left(\theta + \frac{\pi i}{3}\right) S\left(\theta - \frac{\pi i}{3}\right)$.

New wedge-local field?

The bound state operator

S: two-particle S-matrix, P_n: S-symmetrization,
$\mathcal{H} = \bigoplus P_n \mathcal{H}_1^{\otimes n}$, $\mathcal{H}_1 = L^2(\mathbb{R})$,

$$\mathrm{Dom}(\chi_1(f)) := H^2\left(-\tfrac{\pi}{3}, 0\right)$$
$$(\chi_1(f))\xi(\theta) := \sqrt{2\pi|R|} f^+\left(\theta + \tfrac{\pi i}{3}\right)\xi\left(\theta - \tfrac{\pi i}{3}\right),$$

where $H^2(\alpha,\beta)$ is the space of analytic functions in $\mathbb{R} + i(\alpha,\beta)$ such that $\xi(\cdot + \gamma i)$ is uniformly bounded in L^2-norm, $\gamma \in (\alpha,\beta)$, and f^+ is analytic.

$$\chi_n(f) = nP_n\left(\chi_1(f) \otimes I \otimes \cdots \otimes I\right)P_n,$$
$$\chi(f) := \bigoplus \chi_n(f),$$
$$(\chi_1'(g)\xi)(\theta) := (J_1\chi(g_j)J_1)(\theta) = \sqrt{2\pi|R|}g^+\left(\theta - \frac{\pi i}{3}\right)\xi\left(\theta + \frac{\pi i}{3}\right),$$
$$\chi_n'(g) := J_n\chi_n(g_j)J_n, \;\; \chi'(g) := J\chi(g_j)J.$$

Wedge-local fields and weak commutativity

New field:

$$\tilde{\phi}(f) := \phi(f) + \chi(f) = z^\dagger(f^+) + \chi(f) + z(J_1 f^-),$$
$$\tilde{\phi}'(g) := J\tilde{\phi}(g_j)J, \;\; \chi'(g) = J\chi(g_j)J.$$

Theorem (Cadamuro-T. arXiv:1502.01313, CMP '15)

For real $f, g, \mathrm{supp} f \subset W_\mathrm{L}, \mathrm{supp} g \subset W_\mathrm{R}$, *then*

$$\langle \tilde{\phi}(f)\Phi, \tilde{\phi}'(g)\Psi\rangle = \langle \tilde{\phi}'(g)\Phi, \tilde{\phi}(f)\Psi\rangle, \;\; \Phi, \Psi \in \mathrm{Dom}(\tilde{\phi}(f)) \cap \mathrm{Dom}(\tilde{\phi}'(g)).$$

Proof)

$$\langle \chi(f)\Phi_1, \chi'(g)\Psi_1\rangle = 2\pi i R \int d\theta\, f^+\left(\theta + \frac{\pi i}{3}\right) g^+\left(\theta - \frac{2\pi i}{3}\right) \overline{\Phi(\theta)}\Psi_1(\theta)$$
$$= 2\pi i R \int d\theta\, f^+\left(\theta + \frac{\pi i}{3}\right) g^-\left(\theta + \frac{\pi i}{3}\right) \overline{\Phi(\theta)}\Psi_1(\theta)\ldots$$

Some features of the models

- No Reeh-Schlieder property for polynomials, but for the von Neumann algebra.

$$\widetilde{\phi}(f)\Omega = f^+ \text{ is not in the domain of } \widetilde{\phi}(f).$$

- No energy bound for $\widetilde{\phi}$ ($\Rightarrow$ no pointlike field?).

$$\widetilde{\phi}(f) = \phi(f) + \chi(f), \;\; \chi_1(f) = M_{f^+(\cdot + \frac{\pi i}{3})}\Delta_1^{\frac{1}{6}}.$$

- Non-temperate polarization-free generator (c.f. Borchers-Buchholz-Schroer '01).

$$(\chi_1(f)U_1(a)\Psi_1)(\theta) = \sqrt{2\pi|R|}f^+\left(\theta + \frac{\pi i}{3}\right)e^{ia\cdot p\left(\theta - \frac{\pi i}{3}\right)}\Psi_1\left(\theta - \frac{\pi i}{3}\right),$$

which grows exponentially.

- Bound states?

Open problems

- $\chi_1(f)$ is already unbounded and **not self-adjoint** on the domain $H^2(-\frac{\pi}{3}, 0)$. There is a nice self-adjoint extension of $\chi_1(f)$ (arXiv:1508.06402):
  the deficiency indices of $\chi_1(f) = \frac{1}{2}$ (the number of zeros of f^+).
- self-adjointness of $\chi(f) + \chi'(g) + cN$??
  $\Longrightarrow$ **strong commutativity** of $\widetilde{\phi}(f)$ and $\widetilde{\phi}'(g)$.
  $\Longrightarrow$ interacting Haag-Kastler net.
- extension to more complicated models (sine-Gordon model, Toda field theories,...): work in progress
- relation to conformal field theory