KMS states on the chiral components of two-dimensional CFT

Yoh Tanimoto

(with Roberto Longo)
University of Rome "Tor Vergata", JSPS fellow for research abroad

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Introduction

2d CFT

Important observables decompose into chiral components:
$\phi(x, t) = \phi_+(x + t) + \phi_-(x - t)$. $\phi_\pm$ are defined on $\mathbb{R}$, and by conformal invariance of the vacuum, extend to S^1. Furthermore, the algebra of these chiral observables are extremely restricted.

- Examples: stress-energy tensor (Virasoro algebra), currents (Heisenberg algebra, Kac-Moody algebra).

What thermal states are possible for these observables?

- The natural Hamiltonian as $1 + 1$-dim QFT: time translations (c.f. Camassa-Longo-Tanimoto-Weiner '12,'13): complete classification for rational CFTs, Virasoro algebra with $c = 1$, and the $\mathrm{U}(1)$-current (Heisenberg algebra), some thermal states for other examples.
- Conformal Hamiltonian L_0: the generator of rotations of S^1. Motivations from 3d quantum gravity.

Consider the classical black hole solutions of the Einstein equation which are asymptotically AdS$_3$ (the BTZ black holes). These solutions can be parametrized by $\mathrm{Diff}(S^1)/S^1 \times \mathrm{Diff}(S^1)/S^1$.

In the (hypothetical) quantum theory, an asymptotically AdS$_3$ black hole should be in a thermal state with the Bekenstein-Hawking temperature. The Virasoro algebra, the Lie algebra of $\mathrm{Diff}(S^1)$, should acts on the physical Hilbert space as the symmetry of the system.

Hence, **the Virasoro algebra should be in a thermal state**.

The time-translations correspond to the action of S^1. Hence we need to study thermal states on the Virasoro algebra with respect to S^1.

Thermal states

Gibbs states on a finite system

- $M_n(\mathbb{C})$: $(n \times n)$-matrix algebra, H: self-adjoint, $\sigma_t = \mathrm{Ad}\, e^{itH}$.
- A state (positive normalized linear functional) on $M_n(\mathbb{C})$ given by

$$\varphi(x) = \frac{\mathrm{Tr}(e^{-\beta H} x)}{\mathrm{Tr}(e^{-\beta H})}$$

has the **KMS condition**: $f(t) = \varphi(\sigma_t(x) y), f(t + i\beta) = \varphi(y \sigma_t(x))$.

KMS states on a C^*-dynamical system

- The KMS condition can be considered also for infinite systems, although H might not be trace class: $\mathcal{A}$: C^*-algebra, σ_t: one-parameter automorphisms group.
- We interpret that the states with the KMS condition represent states in thermal equilibrium.

Chiral components of 2d CFT

Conformal field theory in $d = 2$ Minkowski space is covariant with respect to the group $\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$.

Important observables live on the lightrays:

- The stress-energy tensor
  $T^{00} + T^{01} = 2T_-(t - x), T^{00} - T^{01} = 2T_-(t + x),$

- Conserved current $\partial_\mu j^\mu = 0$,
  $j^0 - j^1 = 2j_+(t + x), j^0 + j^1 = 2j_-(t - x),$

and mutually commute: $[X_+, Y_-] = 0$.

One can consider a QFT on a lightray $X(x_+)$, $x_+ := x + t$ and its local algebras $\mathcal{A}(I) = \overline{\{e^{iX_+(f)} : \operatorname{supp} f \subset I\}}^{\mathrm{vN}}$, $I \subset S^1$. The collection of local algebras $\{\mathcal{A}(I)\}$ is called a **conformal net**.

The universal C^*-algebra

$\mathcal{A}$: the conformal net on S^1.

What should the C^*-algebra of the system be?
$\overline{\bigcup_{I \subset S^1} \mathcal{A}(I)}^{\mathrm{vN}} = \mathcal{B}(\mathcal{H})$ is not interesting, and contains only the information on the vacuum sector.

One can consider an abstract C^*-algebra which contains the information of all charged sectors: for $x \in A(I)$, $\pi(x) := \bigoplus_\mu \pi_\mu(x)$, where the direct sum runs all cyclic (locally normal) sectors (generated by the GNS representations on the free $*$-algebra generated by $\mathcal{A}(I)$'s).
We define the **universal C^*-algebra** of $\mathcal{A}$ by

$$C^*(\mathcal{A}) := \overline{\{\pi(x) : x \in \mathcal{A}(I) \text{ for some } I \subset S^1\}}^{\|\cdot\|}.$$

There is a natural action of rotations σ_t.
Any representation (sector) of the net $\mathcal{A}$ lifts to $C^*(\mathcal{A})$.

Example: the Virasoro algebra

The stress-energy tensor satisfies the commutation relations in terms of the Fourier components $-2\pi T(z) = \sum L_n z^{-n-2}$ (in the S^1-picture),

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0}.$$

- A unitary irreducible representation is characterized by c and h, where $L_0\Omega = h\Omega$, $L_n\Omega = 0$ for $n \geq 1$.
- Possible values are $c = 1 - \frac{6}{m(m+1)}$, $m = 2, 3, \cdots$, $h = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)}$, $r = 1, \cdots, m - 1$, $s = 1, \cdots, r$ and $c \geq 1, h \geq 0$.
- **For a fixed c, $h = 0$,** $\mathcal{A}_c(I) = \overline{\{e^{iT_{c,0}(f)} : \operatorname{supp} f \subset I\}}^{\mathrm{vN}}$ is a conformal net.
- **For a fixed c,** any value h corresponds to a sector of $\mathcal{A}_c$.
- We need to study **non-irreducible representations** of $C^*(\mathcal{A}_c)$.

GNS representation and the type

For a state φ on a C^*-algebra $\mathfrak{A}$, there is the so-called **GNS representation** π_φ of $\mathfrak{A}$ where φ is represented by a vector Φ (and the whole Hilbert space is generated from Φ by $\pi_\varphi(\mathfrak{A})$). In the GNS representation, it is natural to consider the von Neumann closure $\overline{\pi_\varphi(\mathfrak{A})}^{\mathrm{vN}}$.

A general von Neumann algebra $\mathcal{M}$ (a weak-operator-topology-closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$) is a direct integral of **factors** (von Neumann algebras with trivial center). Factors can be classified into

- Type I: isomorphic to $\mathcal{B}(\mathcal{H})$.
- Type II: admits a tracial state/weight.
- Type III: all projections are equivalent, typical for **local** algebras in QFT.

Correspondingly, one can decompose the state φ. The case where $\pi_\varphi(\mathfrak{A})$ is type I is tractable.

Type I KMS states

Examples: take Virasoro net $\mathcal{A}_c$, a representation $\pi_{c,h}$ with $h > 0$.
$e^{-\beta L_0^h}$ is trace class $\implies$ one can define the Gibbs state

$$\varphi_{h,\beta}(x) = \frac{\mathrm{Tr}(\pi_{c,h}(x)e^{-\beta L_0^h})}{\mathrm{Tr}(e^{-\beta L_0^h})}, \ x \in C^*(\mathcal{A}_c).$$

The GNS representation with respect to $\varphi_{h,\beta}$ is an infinite direct sum of $\pi_{c,h}$ (the multiplicity corresponds to the eigenvectors of L_0^h). Especially, it is of type I.

In general, for a conformal net $\mathcal{A}$ and a KMS state φ, if the GNS representation is factorial of type I, then it is given as a Gibbs state in an irreducible representation.

The structure of the universal C^*-algebra: the rational case

Let $\mathcal{A}$ be a **completely rational** net (there are finitely many sectors and the conjugate sector exists, Kawahigashi-Longo-Müger).

Then any representation of $C^*(\mathcal{A})$ is a **direct sum** (not a direct integral) of irreducible representations and (Carpi-Conti-Hillier-Weiner)

$$C^*(\mathcal{A}) = \bigoplus_{\mu=1}^{n} \mathcal{B}(\mathcal{H}_\mu).$$

Especially, any KMS states on $C^*(\mathcal{A})$ is the **convex combination of Gibbs states** corresponding to irreducible representations.
(c.f. Stefano Iovieno, diploma thesis)

Non-rational examples

There are important non-rational nets.

- The Virasoro nets with $c \geq 1$.
- The $U(1)$-current (the Heisenberg algebra): $[J_m, J_n] = n\delta_{m+n,0}$.
- Finite tensor products of them.

Their irreducible representations have been classified.

- The Virasoro nets with c: lowest weight representations with $h \geq 0$.
- The $U(1)$-current: lowest weight representations $q \geq 0$.
- Finite tensor products of them.

And there are corresponding Gibbs states.

Problem: Are there non-type I representations for these nets?

Examples: classification of sectors and KMS states

Let $\mathcal{A}$ be a conformal net with **split property** and assume that **the irreducible sectors μ are classified by the lowest eigenvalue of L_0^μ**.

Examples:

- h for the Virasoro nets with $c \geq 1$.
- $\frac{q}{2}$ for the $U(1)$-current.

Let π any factorial representation of $\mathcal{A}$. $e^{itL_0^\pi}$ is contained in $\pi(C^*(\mathcal{A}))$ (D'Antoni-Fredenhagen-Köster), hence $e^{i2\pi L_0^\pi}$ is a scalar, hence the spectrum of L_0 is $l_0 + \mathbb{N}$.

Now, by split property, π can be further disintegrated into irreducible representations (in general this is not unique) and each of the component has the lowest eigenvalue $l_0 + n$, hence **countably many** irreducible representations.

On the other hand, **if π were non-type I, there must appear uncountably many irreducible representations** (Kawahigashi-Longo-Müger), which contradicts the above countability.

Open problems

- Is there any conformal net with type II_∞ or III representations?
  - cyclic orbifold: $(\mathcal{A} \otimes \cdots \otimes \mathcal{A})^{\mathbb{Z}_n}$, a classification of sectors not available in general (if $\mathcal{A}$ is not rational).
  - infinite tensor product $\mathcal{A} \otimes \mathcal{A} \cdots$, not conformal, $e^{-\beta L_0}$ not trace class...
- Should the trace class property of $e^{-\beta L_0^\rho}$ follow for a factorial representation ρ from any general assumption on the vacuum representation?
  $\implies \rho$ is type I.
- Which KMS states and which additional observables should appear in the BTZ black hole? Related mathematical problems?