

Ground state representations of some non-rational conformal nets

Yoh Tanimoto*

Dipartimento di Matematica, Università di Roma Tor Vergata

Via della Ricerca Scientifica, 1, I-00133 Roma, Italy

email: `hoyt@mat.uniroma2.it`

Abstract

We construct families of ground state representations of the $U(1)$ -current net and of the Virasoro nets Vir_c with central charge $c \geq 1$. We show that these representations are not covariant with respect to the original dilations, and those on the $U(1)$ -current net are not solitonic. Furthermore, by going to the dual net of with respect to the ground state representations of Vir_c , one obtains possibly new family of Möbius covariant nets on S^1 .

1 Introduction

A model of quantum field theory may be in various states: in quantum theory, a system is defined by the algebra of observables and a physical state is realized as a normalized positive linear functional on it [20]. Among them, the most important state on a quantum field theory is the vacuum state, and indeed, notable axiomatizations of quantum field theory are formulated in terms of the vacuum state or the vacuum correlation functions. Yet, some other states are of particular interest. For example, DHR representations correspond to charged states. KMS states represent thermally equilibrium states. Limiting cases of KMS states, where the temperature is zero, are called ground states and they enjoy particular properties. In this paper, we construct ground states on chiral components of certain two-dimensional conformal field theories.

Ground states can be characterized in various ways, see e.g. [4]. In one of them, ground states are invariant under the dynamics and the generator of the dynamics has positive generator. In the operator-algebraic setting, this is particularly interesting, because if one has a ground state on a one-dimensional Haag-Kastler net, one can take the dual net in the GNS representation, which can possibly give a new Haag-Kastler net, hence a different model.

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In this work, we focus on the simplest class of quantum field theory: chiral components of two-dimensional conformal field theory. Previously we studied KMS states on them [7, 8]. Differently from KMS representations, it appears to be difficult to classify ground states for a given net: the techniques to classify KMS states do not directly apply to ground states, because we do not have a direct connection between ground states and modular automorphisms, which was crucial in [7, Section 4.2.1, Appendix B]¹, nor extension results of KMS states to larger net (cf. [1][8, Appendix A]) are available for ground states. On the other hand, we classified ground states on loop algebras [29], yet it is still unclear whether the results passes to ground states on the corresponding Haag-Kastler nets. Therefore, rather than classifying them, we try to construct various examples.

Contrary to the case of loop algebras where ground state is unique [29], we find that the $U(1)$ -current net and the Virasoro nets with $c \geq 1$ admit continuously many ground states. It turns out that the $U(1)$ -current net admits a one-parameter family of automorphisms commuting with translations, and by composing them with the vacuum state, we obtain a family of ground states. As the Virasoro nets with $c \geq 1$ restricted to \mathbb{R} can be embedded in the $U(1)$ -current net [6], they also admit a family of ground states. In the GNS representations of these ground states, one can take the dual net and they must be different from the original net for the case of the Virasoro nets with $c > 1$, since the original nets are not strongly additive,

We also obtain explicit expressions of the current and the stress-energy tensor in the GNS representation. These expressions in turn serve to show that the GNS representations of the $U(1)$ -current net are not normal on half-lines. This shows that the implication “positivity of energy \implies solitonic representation”, conjectured in [21, Conjecture 34] for loop groups, does not hold for the $U(1)$ -current net.

This paper is organized as follows. In Section 2 we recall Möbius covariant nets, operator-algebraic setting of chiral components of conformal field theory and collect general facts on ground states. In Section 3 we introduce examples of Möbius covariant nets, the $U(1)$ -current net and the Virasoro nets. In Section 4 we construct ground states on these nets and study their property. We conclude with some open problems in Section 5.

2 Preliminaries

2.1 Möbius covariant net on circle

The following is a mathematical setting for chiral components of two-dimensional conformal field theory. These chiral components are essentially quantum field theories on the real line \mathbb{R} , and by conformal covariance they extend to S^1 . In the operator-algebraic setting, they are realized as nets (precosheaves) of von Neumann algebras on S^1 . More precisely, let \mathcal{I} be the set of open, non-dense, non empty intervals in S^1 . A **Möbius covariant net** (\mathcal{A}, U, Ω) is a triple of the map $\mathcal{A} : \mathcal{I} \ni I \mapsto \mathcal{A}(I) \subset \mathcal{B}(\mathcal{H})$, where $\mathcal{A}(I)$ is a von Neumann algebra on a Hilbert space \mathcal{H} , U a strongly continuous representation of the group $\mathrm{PSL}(2, \mathbb{R})$ on \mathcal{H} and a unit vector $\Omega \in \mathcal{H}$ satisfying the following conditions:

(MN1) **Isotony**: If $I_1 \subset I_2$, then $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)$.

¹ The arguments for the case of maximal nets [7, Theorem 4.7] might still work for ground states.

(MN2) **Locality**: If $I_1 \cap I_2 = \emptyset$, then $\mathcal{A}(I_1)$ and $\mathcal{A}(I_2)$ commute.

(MN3) **Möbius covariance**: for $\gamma \in \text{PSL}(2, \mathbb{R})$, $\text{Ad } U(\gamma)(\mathcal{A}(I)) = \mathcal{A}(\gamma I)$.

(MN4) **Positive energy**: the generator L_0 of rotations $U(\rho(t)) = e^{itL_0}$ is positive.

(MN5) **Vacuum**: Ω is the unique (up to a phase) invariant vector for $U(\gamma)$, $\gamma \in \text{PSL}(2, \mathbb{R})$ and $\overline{\mathcal{A}(I)\Omega} = \mathcal{H}$ (the **Reeh-Schlieder property**).

From (MN1)–(MN5), one can prove the following [17, 16]:

(MN6) **Haag duality**: $\mathcal{A}(I') = \mathcal{A}(I)'$, where I' is the interior of the complement of I .

(MN7) **Additivity**: If $I = \bigcup I_j$, then $\mathcal{A}(I) = \bigvee_j \mathcal{A}(I_j)$.

We consider also the following additional properties:

- **Strong additivity**: If I_1 and I_2 are the intervals obtained from I by removing one point, then $\mathcal{A}(I_1) \vee \mathcal{A}(I_2) = \mathcal{A}(I)$.
- **Conformal covariance**: U extends to a projective unitary representation of the group $\text{Diff}_+(S^1)$ of orientation preserving diffeomorphisms and $\text{Ad } U(\gamma)\mathcal{A}(I) = \mathcal{A}(\gamma I)$, and $\text{Ad } U(\gamma)(x) = x$ if $I \cap \text{supp } \gamma = \emptyset$.

Strong additivity does not follow from (MN1)–(MN5), and indeed the Virasoro nets with $c > 1$ fail to have strong additivity [6, Section 4]. If conformal covariance holds, (\mathcal{A}, U, Ω) is called a **conformal net**.

Concrete examples relevant to this work will be presented in Section 3.

2.2 Ground state representations

Let us identify \mathbb{R} with a dense interval in S^1 by the stereographic projection. By convention, the point of infinity ∞ corresponds to $-1 \in S^1 \subset \mathbb{C}$. We denote by $\mathcal{A}|_{\mathbb{R}}$ the net of algebras $\{\mathcal{A}(I) : I \in \mathbb{R}\}$ restricted to finite open intervals in the real line \mathbb{R} . Translations $\tau(t)$ and dilations $\delta(s)$ acts on $\mathcal{A}|_{\mathbb{R}}$, and they are elements of $\text{PSL}(2, \mathbb{R})$.

$\text{Ad } U(\tau(t))$ acts on $\overline{\bigcup_{I \in \mathbb{R}} \mathcal{A}(I)}^{\|\cdot\|}$ as an automorphism. A **state** φ on $\mathcal{A}|_{\mathbb{R}}$ is by definition a state on the C^* -algebra $\overline{\bigcup_{I \in \mathbb{R}} \mathcal{A}(I)}^{\|\cdot\|}$. Let φ be a state on $\mathcal{A}|_{\mathbb{R}}$ and let π_φ be the GNS representation with respect to φ . If π_φ extends to $\mathcal{A}(\mathbb{R}_\pm)$ in the σ -weak topology, then we say that π_φ is **solitonic**. If there is a representation U^φ of the translation group \mathbb{R} such that $\text{Ad } U^\varphi(\tau(t))(\pi_\varphi(x)) = \pi_\varphi(\text{Ad } U(\tau(t))(x))$, then we say that π_φ is **translation-covariant**. Furthermore, if the generator of U^φ can be taken positive, then we say π_φ is a **positive-energy representation**.

If a state φ on $\mathcal{A}|_{\mathbb{R}}$ is invariant under $\text{Ad } U(\tau(t))$, then π_φ is automatically translation-covariant, because one can define $U^\varphi(\tau(t))\pi_\varphi(x)\Omega_\varphi = \pi_\varphi(\text{Ad } U(\tau(t))(x))\Omega_\varphi$, where Ω_φ is the GNS vector for φ . If furthermore U^φ has positive energy, then we say that it is a **ground state representation**, and the state φ is called the **ground state**. By the Reeh-Schlieder argument (see e.g. [2, Theorem 1.3.2], [24, Theorem 3.2.1]), the GNS vector Ω_φ is cyclic for $\pi_\varphi(\mathcal{A}(I))$ for any $I \in \mathbb{R}$.

If π_φ is a ground state representation with the ground state vector Ω_φ , then one can consider the translation-covariant net $\{\pi_\varphi(\mathcal{A}(I))\}$ on intervals in \mathbb{R} on the GNS representation space \mathcal{H}_φ . Its dual net is defined by $\hat{\mathcal{A}}_\varphi(\mathbb{R}_+ + a) := \left(\bigvee_{I \in \mathbb{R}_+ + a} \pi_\varphi(\mathcal{A}(I))\right)''$ and $\hat{\mathcal{A}}_\varphi((a, b)) = \hat{\mathcal{A}}_\varphi(\mathbb{R}_+ + b) \cap \hat{\mathcal{A}}_\varphi(\mathbb{R}_+ + a)'$. By [19, Corollary 1.9], $\hat{\mathcal{A}}$ extends uniquely to a strongly additive Möbius covariant net on S^1 :

Theorem 2.1 (Guido-Longo-Wiesbrock). *There is a one-to-one correspondence between*

- *Isomorphism classes of strongly additive Möbius covariant nets*
- *Isomorphism classes of Borchers triples (\mathcal{M}, U, Ω) , (\mathcal{M} is a von Neumann algebra, U is a representation of \mathbb{R} with positive generator, Ω is cyclic and separating for \mathcal{M} and $U(t)\Omega = \Omega$ such that $\text{Ad } U(t)(\mathcal{M}) \subset \mathcal{M}$ for $t \geq 0$) with the property that Ω is cyclic for $\text{Ad } U(t)(\mathcal{M})' \cap \mathcal{M}$*

Even if the given net \mathcal{A} is conformally covariant, it is not known whether the net $\hat{\mathcal{A}}_\varphi$ is conformally covariant. Although $\hat{\mathcal{A}}_\varphi$ is $\text{Diff}_c(\mathbb{R})$ -covariant in the natural sense, where $\text{Diff}_c(\mathbb{R})$ is the group of diffeomorphisms of \mathbb{R} with compact support, we do not have the uniqueness results of $\text{Diff}_c(\mathbb{R})$ -action (cf. [10, Theorem 5.5] for a uniqueness theorem for $\text{Diff}_+(S^1)$ -action). See [31, Chapter 4] for some attempts to construct $\text{Diff}_+(S^1)$ -covariance), and [25, Section 4.2] for some examples in $(1+1)$ -dimensions where Möbius covariance gets lost by passing to the dual net.

The purpose of this paper is to construct ground state representations of certain nets. One of the ideas to construct ground state representations is the following: an **automorphism** α of the net $\mathcal{A}|_{\mathbb{R}}$ is a family $\{\alpha_I\}_{I \in \mathbb{R}}$ of automorphisms of $\{\mathcal{A}(I)\}$ such that $\alpha_{\tilde{I}}|_{\mathcal{A}(I)} = \alpha_I$ for $I \subset \tilde{I}$. α extends naturally to $\overline{\bigcup_{I \in \mathbb{R}} \mathcal{A}(I)}^{\|\cdot\|}$.

Lemma 2.2. *If α commutes with $\text{Ad } U(\tau(t))$, then $\omega \circ \alpha$ is a ground state.*

Proof. The GNS representation of $\omega \circ \alpha$ is given by $\pi_\omega \circ \alpha = \alpha$, where π_ω is the vacuum representation, the identity map. As α commutes with $\text{Ad } U(\tau(t))$, the same representation $U(\tau(t))$ with positive energy implements translations. \square

Furthermore, combining [7, Lemma 4.4, Lemma 4.5], we obtain the following dichotomy: if $\mathcal{A}|_{\mathbb{R}}$ admits one nontrivial ground state, then there must be continuously many ground states.

Proposition 2.3. *If there is an automorphism α of $\mathcal{A}|_{\mathbb{R}}$ commuting with translations and if $\omega \circ \alpha \neq \alpha$, then $\{\omega \circ \alpha \circ \text{Ad } U(\delta(s))\}_{s \in \mathbb{R}}$ are mutually different and their GNS representations $\{\text{Ad } U(\delta(-s)) \circ \alpha \circ \text{Ad } U(\delta(s))\}_{s \in \mathbb{R}}$ are mutually unitarily inequivalent.*

Note that α as in Proposition 2.3 cannot be implemented by a unitary operator, because Ω is unique up to a phase. Hence, the second statement implies that the dilation automorphisms $\{\text{Ad } U(\delta(s))\}$ are not unitarily implemented in the GNS representations (namely, there are no $U^\alpha(s)$ such that $\text{Ad } U^\alpha(s) \circ \alpha = \alpha \circ \text{Ad } U(\delta(s))$): if they were implemented, then it would imply that α and $\text{Ad } U(\delta(-s)) \circ \alpha \circ \text{Ad } U(\delta(s))$ are unitarily equivalent, which is a contradiction.

3 Main examples

Here we introduce examples of conformal nets. Although our emphasis is on their restriction to \mathbb{R} , they are most conveniently defined on S^1 . Our treatment of these models follows that of [8, Section 4.1].

3.1 The $U(1)$ -current net

Buchholz, Mack and Todorov obtained some fundamental results in the operator-algebraic treatment of the $U(1)$ -current algebra [5]. The current $J(f)$ can be defined by the Fourier modes $\{J_n\}_{n \in \mathbb{Z}}$, where J_n satisfy the following commutation relations $[J_m, J_n] = m\delta_{m+n,0}$ and hence the current J as a quantum field on S^1 satisfies the following relation: $[J(f), J(g)] = i \int_{S^1} f(\theta)g'(\theta)d\theta$, for $f, g \in C^\infty(S^1, \mathbb{R})$. It admits a vacuum state $\langle \Omega, \cdot \Omega \rangle$ such that $J_n \Omega = 0$ for $n \geq 0$, and the representation is uniquely determined by this property, unitarity $J_n^* = J_{-n}$ and the commutation relation. The smeared current $J(f) = \sum_n \hat{f}_n J_n$, where $f \in C^\infty(S^1, \mathbb{R})$ and $\hat{f}_n = \frac{1}{2\pi} \int f(\theta)e^{-in\theta}$ is the Fourier-coefficient, is essentially self-adjoint on the subspace \mathcal{H}^{fin} spanned by vectors $J_{-n_1} \cdots J_{-n_k} \Omega$, $n_j \in \mathbb{N}$.

The vacuum state is invariant under $\text{PSL}(2, \mathbb{R})$ -transformations $J(f) \mapsto J(f \circ \gamma^{-1})$, $\gamma \in \text{PSL}(2, \mathbb{R})$, hence this can be implemented by a unitary operator $U(\gamma)$, and in this way we obtain a unitary representation U of $\text{PSL}(2, \mathbb{R})$ with positive energy.

The exponential $W(f) = e^{iJ(f)}$ is called the Weyl operator. They satisfy $W(f)W(g) = e^{-\frac{i}{2}\sigma(f,g)}W(f+g)$, where $\sigma(f,g) = \int_{S^1} f(\theta)g'(\theta)d\theta$. If we define von Neumann algebras by $\mathcal{A}_{U(1)}(I) = \{W(f) : \text{supp } f \subset I\}''$, then $(\mathcal{A}_{U(1)}, U, \Omega)$ is a Möbius covariant net. This is called the **$U(1)$ -current net**. It is known that it satisfies strong additivity [6, (4.21) and below] and conformal covariance [27, Section 5.3].

We can also define J in the real line picture. To do this, note that the above commutation relation is invariant under diffeomorphisms. Therefore, if we introduce $J(f)$ for $f \in C_c^\infty(\mathbb{R}, \mathbb{R})$ by $J(f \circ C^{-1})$ where $C(t) = -\frac{t-i}{t+i}$ is the Cayley transform mapping $\mathbb{R} \rightarrow S^1$ (with a slight abuse of notation: the definition of $J(f)$ depends on whether $f \in C_c^\infty(\mathbb{R})$ or $f \in C^\infty(S^1)$), we obtain the same commutation relation

$$[J(f), J(g)] = i \int_{\mathbb{R}} f(t)g'(t)dt, \quad f, g \in C^\infty(\mathbb{R}, \mathbb{R}). \quad (1)$$

We call it J in the real line picture. Again by the diffeomorphism covariance, translations and dilations acts on J in the natural way: if γ is such a transformation, then $\text{Ad } U(\gamma)(J(f)) = J(f \circ \gamma^{-1})$.

Lemma 3.1. *For a function f on S^1 such that $\sum_{k \geq 0} k|\hat{f}_k|^2 < \infty$, $J(f)$ and $W(f)$ can be defined by continuity.*

Proof. Note first that if $J(f_n)\Omega \rightarrow J(f)\Omega$, then $W(f_n) \rightarrow W(f)$ in the strong operator topology. Indeed, recall that $\langle \Omega, W(g)\Omega \rangle = e^{-\frac{1}{2}\|J(g)\Omega\|^2}$ [24, Section 6.5] and $\sigma(f,g) = 2\text{Im} \langle J(f)\Omega, J(g)\Omega \rangle$. Therefore, it is straightforward to see that, if $J(f_n)\Omega \rightarrow J(f)\Omega$, then $W(f_n)\Omega \rightarrow W(f)\Omega$ and $\sigma(f_n, g) \rightarrow \sigma(f, g)$. It then follows that for a fixed g

$$\|(W(f_n) - W(f))W(g)\Omega\| \rightarrow 0,$$

and since $W(g)\Omega$ is total and $\{W(f_n)\}$ are unitary, this implies the claimed strong convergence.

For a smooth function f on S^1 , $\|J(f)\Omega\|^2 = \sum_{k \geq 0} k |\hat{f}_k|^2$, and hence J can be extended by continuity to any L^2 -function f on S^1 such that $\sum_{k \geq 0} k |\hat{f}_k|^2 < \infty$. \square

If f is piecewise smooth and continuous, then $\sum_{k \geq 0} k |\hat{f}_k|^2 < \infty$. Indeed, as an operator on $L^2(S^1)$, $i \frac{d}{d\theta}$ is self-adjoint on the following domain [28, X.1 Example 1]

$$\{f \in C(S^1) : f \text{ is absolutely continuous, } f' \text{ is in } L^2(S^1)\}$$

and $\sum_k |\hat{f}'_k|^2 = \sum_k k^2 |\hat{f}_k|^2 < \infty$, therefore, $\sum_{k \geq 0} k |\hat{f}_k|^2 < \infty$. These facts can be seen as an easier version of [10, Proposition 4.5][32, Lemma 2.2].

3.2 The Virasoro nets

On the Hilbert space of the $U(1)$ -current net, one can construct another quantum field. Let $L_n = \frac{1}{2} \sum_m : J_{-m} J_{n+m} :$ where the normal ordering means J_k with negative k comes to the left [22, (2.9)]. Then they satisfy the Virasoro algebra $[L_m, L_n] = (m - n)L_{m+n} + c \frac{m(m^2-1)}{12}$ with the central charge $c = 1$. The stress-energy tensor $T(f) = \sum \hat{f}_n L_n$ is essentially self-adjoint on \mathcal{H}^{fin} and satisfies the commutation relations $[T(f), J(g)] = iJ(fg')$. Especially, if f and g have disjoint support, they commute. By the linear energy bound [10, Proposition 4.5][3, Lemma 3.2] and the Driessler-Fröhlich theorem [13, Theorem 3.1](or its adaptation to the case of operators [3, Theorem 3.4]), they actually strongly commute. For $\gamma \in \text{Diff}(S^1)$, it holds that $\text{Ad } U(\gamma)(T(f)) = T(\gamma_* f) + \beta(\gamma, f)$ with a certain $\beta(\gamma, f) \in \mathbb{R}$, where $\gamma_* f$ is the pullback of f as a vector field by γ [15, Proposition 3.1]. Moreover, $T(f)$ integrates to the representation U in the sense that $U(\text{Exp}(tf)) = e^{itT(f)}$ up to a phase [18, Section 4], see also [15, Proposition 5.1].

Let us denote $\mathcal{B}_1(I) = \{e^{iT(f)} : \text{supp } f \subset I\}''$ and call it the **Virasoro subnet**. By Haag duality, it holds that $\mathcal{B}_1(I) \subset \mathcal{A}_{U(1)}(I)$. Furthermore, by the Reeh-Schlieder argument, $\mathcal{H}_{\text{Vir}_1} := \overline{\mathcal{B}_1(I)\Omega}$ does not depend on I . The restriction of $\{\mathcal{B}_1(I)\}$ and U to $\mathcal{H}_{\text{Vir}_1}$ together with $\Omega \in \mathcal{H}_{\text{Vir}_1}$ is called the **Virasoro net with $c = 1$** and we denote it by $(\text{Vir}_1, U_1, \Omega_1)$.

We can also consider the real line picture for T . For a vector field f on \mathbb{R} , or more precisely $f(t) \frac{d}{dt}$, we define $T(f) := T(C_*^{-1} f)$. Conversely, a vector field f on S^1 corresponds to $C_* f(t) = \frac{t^2+1}{2} f(C^{-1}(t))$.

In [8, Proof of Theorem 4.7], we also computed the exponentiation of the relation between $J(g)$ and $T(f)$:

$$\text{Ad } W(g)(T(f)) = T(f) - J(fg') + \frac{1}{2} \sigma(fg', g),$$

where this equality holds on \mathcal{H}^{fin} .

In order to obtain the Virasoro nets with $c > 1$, we need to perturb T as in [6, (4.6)]. We have computed its real-line picture [8, Section 5.3]:

$$T^\kappa(f) = T(f) + \kappa J(f'),$$

then T^κ satisfies the commutation relation

$$[T^\kappa(f), T^\kappa(g)] = iT^\kappa(fg' - f'g) + i\frac{1+\kappa^2}{12} \int_{\mathbb{R}} f'''(t)g(t)dt,$$

hence the central charge is $c = 1 + \kappa^2$. Define $\mathcal{B}_c(I) := \{e^{iT^\kappa(f)} : \text{supp } f \subset I\}''$. We have seen that $\mathcal{B}_c(I) \subset \mathcal{A}_{U(1)}(I)$, but $\mathcal{B}_c(I)$ is covariant only with respect to U restricted to translations and dilations. Yet, $\mathcal{H}_{\text{Vir}_c} := \overline{\mathcal{B}_c(I)}\Omega$ does not depend on I again by the Reeh-Schlieder argument. It was shown in [6, Section 4, (4.8)] that by exploiting the Möbius invariance of n -points functions of T^κ , the restriction of \mathcal{B}_c to $\mathcal{H}_{\text{Vir}_c}$ can be extended to a conformal net, the Virasoro net Vir_c with central charge $c = 1 + \kappa^2$. $\text{Vir}_c, c > 1$ does not satisfy strong additivity [6, Section 4, (4.13)]. It is an open problem whether the dual net of Vir_c is conformal ($\text{Diff}(S^1)$ -covariant).

4 Ground states

4.1 On the $U(1)$ -current net

Following Lemma 2.2, we construct automorphisms of $\mathcal{A}_{U(1)}|_{\mathbb{R}}$ commuting with translations. This has been done in [8, Proposition 4.1]. Let us recall it in our present notations.

Lemma 4.1. *For $q \in \mathbb{R}$, there is an automorphism α_q of $\mathcal{A}_{U(1)}|_{\mathbb{R}}$ commuting with translations such that $\alpha_q(W(f)) = e^{iq \int_{\mathbb{R}} f(t)dt} W(f)$.*

Proof. Let $I \Subset \mathbb{R}$. There is a smooth function s_I with compact support such that $s_I(t) = t$. By commutation relation (1), for f with $\text{supp } f \subset I$ we have

$$\text{Ad } W(qs_I)(W(f)) = e^{iq \int_{\mathbb{R}} s'_I(t)f(t)dt} W(f) = e^{iq \int_{\mathbb{R}} f(t)dt} W(f).$$

We define $\alpha_{q,I} = \text{Ad } W(qs_I)$. This does not depend on the choice of s_I within the restriction above, because $\mathcal{A}_{U(1)}(I)$ is generated by $W(f)$ with $\text{supp } f \subset I$. α_q commutes with translations because $\int_{\mathbb{R}} f(t)dt$ is invariant under translations. \square

From here, it is immediate to construct ground states by Lemma 2.2. We collect some properties of the resulting GNS representations.

Theorem 4.2. *We have the following.*

- The states $\{\omega \circ \alpha_q\}$ are mutually different ground states of $\mathcal{A}_{U(1)}|_{\mathbb{R}}$, and they are connected with each other by dilations. Dilations are not implemented unitarily.
- The GNS vector Ω_q of $\omega \circ \alpha_q$ is in the domain of $J_q(f)$, where $\alpha_q(W(f)) = e^{iJ_q(f)}$ and $J_q(f) = J(f) + q \int_{\mathbb{R}} f(t)dt$. Especially, it holds that $\omega \circ \alpha_q(J(f)) = q \int_{\mathbb{R}} f(t)dt$
- The GNS representations $\{\alpha_q\}$ do not extend to $\mathcal{A}(\mathbb{R}_{\pm})$ in the weak operator topology, hence are not solitonic.
- The dual nets $\{\hat{\alpha}_q(\mathcal{A}_{U(1)}(I))\}$ are equal to the original net $\mathcal{A}_{U(1)}$.

Proof. As $\{\alpha_q\}$ are automorphisms commuting with translations, the states $\{\omega \circ \alpha_q\}$ are ground states by Lemma 2.2, and their GNS representations are $\{\alpha_q\}$. They are different states because they give the scalar $e^{iq \int_{\mathbb{R}} f(t) dt}$ to $W(f)$. It follows from Proposition 2.3 that they are connected by dilations and dilations are not implemented unitarily.

Let us recall the action $\alpha_q(W(sf)) = e^{iqs \int_{\mathbb{R}} f(t) dt} W(sf)$. From this it is immediate that

$$\frac{d}{ds} \alpha_q(W(sf)) \Omega|_{s=0} = iJ(f)\Omega + iq \int_{\mathbb{R}} f(t) dt.$$

We also showed [8, Lemma 4.6] that $\mathcal{H}^\infty = \bigcap_n \text{Dom}(L_0^n)$ is invariant under Weyl operators $W(g)$, hence we have, the following equation of operators on \mathcal{H}^∞ :

$$\text{Ad } W(q\sigma_I)(J(f)) = J(f) + q \int_{\mathbb{R}} f(t) dt.$$

\mathcal{H}^∞ is a core of L_0 , and hence of $J(f)$ by the commutator theorem [28, Theorem X.37] and the estimate $\|J(f)\xi\| \leq c_f \|(L_0 + 1)^{\frac{1}{2}} \xi\|$, see the computations in [23, Proposition 1] (this estimate is known since [6, below (2.23)]). Since \mathcal{H}^∞ is invariant under $J(f)$, to apply the commutator theorem with slightly different assumptions [28, Theorem X.36], it is enough to have a linear bound $\|J(f)\xi\| \leq c_f \|(L_0 + 1)\xi\|$.

To show the non-normality on the half line \mathbb{R}_+ (\mathbb{R}_- is analogous), we take a sequence of smooth functions $g_n(-e^{i\theta})$ on S^1 :

$$g_n(-e^{i\theta}) = \begin{cases} 0 & \text{for } 0 \leq \theta < \pi \\ \theta - \pi & \text{for } \pi \leq \theta < \frac{3\pi}{2} \\ -\theta + 2\pi & \text{for } \frac{3\pi}{2} \leq \theta < 2\pi - \frac{1}{2n} \\ -2(\theta - 2\pi + \frac{1}{2n}) & \text{for } 2\pi - \frac{1}{n} \leq \theta < 2\pi - \frac{1}{2n} \\ 0 & \text{for } 2\pi - \frac{1}{2n} \leq \theta \leq 2\pi \end{cases}$$

It is clear that each g_n is piecewise smooth (linear) and continuous, and both g_n, g'_n converges

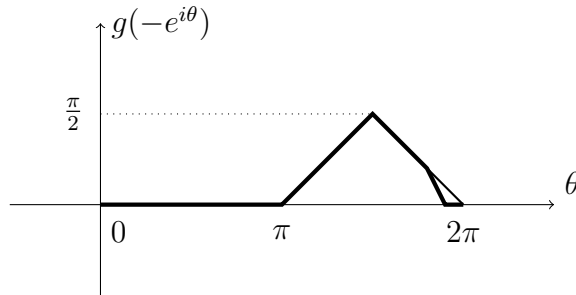


Figure 1: The functions $g_n(-e^{i\theta})$ (the thick line) and $g(-e^{i\theta})$ (the thin line above), restricted to $0 \leq \theta \leq 2\pi$.

in the L^2 -norm to

$$g(-e^{i\theta}) = \begin{cases} 0 & \text{for } 0 \leq \theta < \pi \\ \theta - \pi & \text{for } \pi \leq \theta < \frac{3\pi}{2} \\ -\theta + 2\pi & \text{for } \frac{3\pi}{2} \leq \theta < 2\pi \end{cases}.$$

Therefore, $J(g_n)\Omega \rightarrow J(g)\Omega$ and hence $W(g_n) \rightarrow W(g)$ in the strong operator topology, and in the weak operator topology, too. On the other hand, $\alpha_q(W(g_n)) = e^{iq_n}W(g_n)$, where

$$q_n = q \int_{\mathbb{R}} g_n(C(t))dt.$$

Note that, $C(t) = -\frac{t-i}{t+i}$, and $g(C(t))$ behaves as $\frac{1}{t}$ as $t \rightarrow \infty$, hence $q_n \rightarrow \infty$. Accordingly, $\alpha_q(W(g_n)) = e^{iq_n}W(g_n)$ is not convergent.

The last claim follows immediately because α_q are automorphisms, hence $\alpha_q(\mathcal{A}_{U(1)}(I)) = \mathcal{A}_{U(1)}(I)$ and by strong additivity of $\mathcal{A}_{U(1)}$. \square

In this case it is trivial that the dual net coincides with the original net. In contrast, we have a markedly different result for Vir_c .

The fact that $\omega \circ \alpha_q(J(f)) = q \int_{\mathbb{R}} f(t)dt$ suggests that q is an analogue of the lowest weight g of [5, Section 1C], cf. [29, Lemmas 5.3, 5.6] and the state Ω_q has uniform charge density q .

The $U(1)$ -current as a multiplier representation of the loop group SLS^1 . Let SLS^1 be the group of smooth maps from S^1 into S^1 with winding number 0 with pointwise multiplication [27, Section 5.1]. These elements can be identified with smooth functions on S^1 in \mathbb{R} by logarithm, and the group operation becomes addition. Recall the Weyl relation $W(f)W(g) = e^{-\frac{i}{2}\sigma(f,g)}W(f+g)$: this can be seen as a multiplier representation of SLS^1 with the cocycle $e^{-\frac{i}{2}\sigma(f,g)}$. From this perspective, one can also consider the subgroup $S\Omega S^1$ of functions f on S^1 such that $f^{(n)}(-1) = 0$ for all $n \in \mathbb{N}$ [21, Section 2.1.2].

Let us observe that $f \mapsto \alpha_q(W(f))$ is a multiplier representation of functions f whose support does not contain -1 , because for such f , $\alpha_q(W(f))$ is defined and respects product. We claim that this extends to $f \in S\Omega S^1$. Indeed, for such f , $f(C(t))$ is a rapidly decreasing smooth function, hence we can find compactly supported test functions $f_n \rightarrow f$ in the topology of Schwartz-class functions, and then $\alpha_q(W(f_n)) = e^{iq \int_{\mathbb{R}} f_n(t)dt}W(f_n) \rightarrow e^{iq \int_{\mathbb{R}} f(t)dt}W(f)$. On the other hand, we have seen in Theorem 4.2 that the representation α_q is not normal (continuous in the σ -weak topology, which coincides with the weak operator topology on norm-bounded sets) on any half-line.

Henriques conjectured that any positive-energy representation of ΩG should be normal on half-lines [21, Conjecture 34], where G is a simple and simply connected compact group. The above observation shows that an analogous conjecture for $S\Omega S^1$ does not hold. On the other hand, we have also shown that for such G , the only ground state of the algebra $\mathcal{S}\mathfrak{g}_{\mathbb{C}}$ of Schwartz-class maps in $\mathfrak{g}_{\mathbb{C}}$ is the vacuum state [29, Corollary 5.8]. This supports to some extent the conjecture for ΩG . Yet, there are many other positive-energy representations of ΩG without ground state, and the situation is not conclusive.

4.2 On Virasoro nets

We can simply restrict the states $\{\omega \circ \alpha_q\}$ to the Virasoro nets Vir_c . Let us denote $\mathcal{H}_q^\kappa = \overline{\alpha_q(\text{Vir}_c(I))\Omega}$, where $c = 1 + \kappa^2$ and this does not depend on I by the Reeh-Schlieder argument. Let us denote the GNS representation $P_{\mathcal{H}_q^\kappa}\alpha_q$ by $\rho_{\kappa,q}$, where $P_{\mathcal{H}_q^\kappa}$ is the projection onto \mathcal{H}_q^κ .

Theorem 4.3. *We have the following.*

- The states $\{\omega \circ \alpha_q\}$ are mutually different ground states of $\text{Vir}_c|_{\mathbb{R}}$ for different $\frac{q^2}{2}$, and they are connected with each other by dilations. Dilations are not implemented unitarily in $\rho_{\kappa,q}$.
- The GNS vector Ω_q of $\omega \circ \alpha_q$ is in the domain of $T_q^\kappa(f)$, where $\alpha_q(e^{isT^\kappa(f)}) = e^{iT_q^\kappa(f)}$ and $T_q^\kappa(f) = T(f) + qJ(f) + \kappa J(f') + \frac{q^2}{2} \int_{\mathbb{R}} f(t)dt$ on \mathcal{H}^∞ .
- If $c > 1$, the dual nets $\{\hat{\rho}_q(\text{Vir}_c(I))\}$ are not unitarily equivalent to any of $\text{Vir}_{c'}$, $c' > 1$.

Proof. Let us first prove the second statement. We have seen in [8, Lemma 4.6] that $W(g)$ preserves \mathcal{H}^∞ and for $I \supset \text{supp } f$ it holds on \mathcal{H}^∞ that

$$\text{Ad } W(qs_I)(T(f)) = T(f) + qJ(f) + \frac{q^2}{2} \int_{\mathbb{R}} f(t)dt.$$

Furthermore, we have $\text{Ad } W(qs_I)(J(f)) = J(f) + q \int_{\mathbb{R}} f(t)dt$, and $T^\kappa(f) = T(f) + \kappa J(f')$. Applying $\text{Ad } W(qs_I)$ term by term, we obtain on \mathcal{H}^∞

$$\text{Ad } W(qs_I)(T^\kappa(f)) = T(f) + qJ(f) + \kappa J(f') + \frac{q^2}{2} \int_{\mathbb{R}} f(t)dt,$$

since $\int_{\mathbb{R}} f'(t)dt = 0$, and this operator has Ω in its domain.

To show that they are different states for different $\frac{q^2}{2}$, it is enough to observe that

$$\frac{d}{ds} \omega \circ \alpha_q(e^{isT^\kappa(f)}) = \langle \Omega, \text{Ad } W(qs_I)(T^\kappa(f)) \Omega \rangle = \frac{q^2}{2} \int_{\mathbb{R}} f(t)dt,$$

giving different values.

Let us also check that these states depend only on $c = 1 + \kappa^2$ and not on κ . To see this, note that $J(f) \mapsto J(-f)$ is a vacuum-preserving automorphism of $\mathcal{A}|_{\mathbb{R}}$ commuting with translations. This automorphism α_- clearly intertwines α_q and α_{-q} . Under α_- , $T(f)$ is mapped to $T(f)$ itself because it is quadratic in J , therefore, $T^\kappa(f)$ is mapped to $T^{-\kappa}(f)$. Since $\omega \circ \alpha_q$ and $\omega \circ \alpha_q \circ \alpha_- = \omega \circ \alpha_- \circ \alpha_{-q} = \omega \circ \alpha_{-q}$ give the same state on Vir_c , they depend on only on $c = 1 + \kappa^2$.

They are connected by dilation $\delta(s)$ on the larger algebra $\mathcal{A}_{U(1)}|_{\mathbb{R}}$ and from the action² of dilations $\text{Ad } U(\delta(s))(J(f)) = J(f \circ \delta(-s))$ that

$$\begin{aligned} \text{Ad } [W(qs_I)U(\delta(s))](J(f)) &= J(f \circ \delta(-s)) + q \int_{\mathbb{R}} f \circ \delta(-s)(t)dt \\ &= J(f \circ \delta(-s)) + e^s q \int_{\mathbb{R}} f(t)dt \\ &= \text{Ad } U(\delta(s))(J_{e^s q}(f)) \end{aligned}$$

Hence $\omega \circ \alpha_q \circ \text{Ad } U(\delta(s)) = \omega \circ \alpha_{e^s q}$,

²Formally, J has scaling dimension 1: $\text{Ad } (U(\delta(s)))(J(t)) = e^s J(e^s t)$. By integrating this equation again f , we obtain $\text{Ad } U(\delta(s))(J(f)) = J(f \circ \delta(-s))$, where $f \circ \delta(-s)(t) = f(e^{-s}t)$. Similarly, the stress-energy tensor T has scaling dimension 2: $\text{Ad } (U(\delta(s)))(T(t)) = e^{2s} T(e^s t)$ and $\text{Ad } U(\delta(s))(T(f)) = e^s T(f \circ \delta(-s))$.

Since Ω_q is the unique translation-invariant vector, it follows that $\rho_{\kappa,q}$ are irreducible (see the arguments of [2, Corollary 1.2.3]). If dilations were implemented by unitary operators, they would bring a vector state to another vector state which are again translation-invariant, which contradicts the uniqueness of invariant vector.

As $\text{Vir}_c, c > 1$ are not strongly additive while dual nets are strong additive, they cannot be unitarily equivalent. \square

We expect that the GNS representations $\{\rho_{\kappa,q}\}$ are not normal on $\bigcup_{I \in \mathbb{R}_\pm} \text{Vir}_c(I)$. A candidate for a convergent sequence which is not convergent in $\{\rho_{\kappa,q}\}$ is $e^{T(f_n)}$, where f_n are smooth functions on \mathbb{R} , supported in \mathbb{R}_+ and converging to a function f which decays as $\frac{1}{t}$. We expect that $T(f) + \kappa J(f') + qJ(f)$ is self-adjoint, while $q^2 \int_{\mathbb{R}} f(t) dt$ is divergent. To do this, the commutator theorem would not suffice (because formally $[L_0, J(f)] = J(f')$ would not be defined for this f) but one would need an analysis similar to [10, Theorem 4.4]

Here the value $\frac{q^2}{2}$ can be viewed as the energy density of the state Ω_q , hence the representation as a whole has infinite energy with respect the original vacuum, cf. [14] where it was shown that, under certain conditions, translation-invariant states cannot be created by finite energy. This also supports that $\rho_{\kappa,q}$ are not normal even on half-lines.

Even for the case $c = 1$, the dual nets $\{\hat{\rho}_q(\text{Vir}_1(I))\}$ cannot be easily identified with any known conformal net. We remark that strong additivity of the original net Vir_1 does not appear to imply that $\{\rho_q(\text{Vir}_c(I))\}$ is already dual, cf. [7, Lemma 3.2].

5 Outlook

Classification. As we have seen, the ground states we constructed in this paper are characterized by a number q (for the $U(1)$ -current net) or $\frac{q^2}{2}$ (for the Virasoro nets). This is a structure very similar to the invariant ψ of [29, Lemma 5.3]. If one studies corresponding Lie algebras, it should be possible to classify ground state representations as in [29, Theorem 5.6].

More precisely, to the net $\mathcal{A}_{U(1)|\mathbb{R}}$ there corresponds the central extension of the Lie algebra of compactly supported smooth functions on \mathbb{R} with the relation $[f, g] = \sigma(f, g)$. To the Virasoro nets $\text{Vir}_c|_{\mathbb{R}}$ one considers the central extension of smooth vector fields on \mathbb{R} with $[f, g] = fg' - f'g + \frac{c}{12} \int f(t)g'''(t)dt$. By positivity of energy, one should be able to determine the representation in terms of q and $\frac{q^2}{2}$, respectively.

Actually, for the $U(1)$ -current net, the direct classification of ground states might be possible. Some techniques from [8, Section 4.2] used to classify KMS states should be useful, although the boundary condition ($t \rightarrow t + i\beta$ for β -KMS states) is missing for ground states. As for the Virasoro nets, it is still not clear whether such classification results pass to the Virasoro nets. For that implication, one needs that the ground state vector is infinitely differentiable. We are currently not able to do this, because the corresponding Lie group $\text{Diff}_c(\mathbb{R})$ of compactly supported diffeomorphisms of \mathbb{R} do not possess any compact subgroup (in contrast to $\text{Diff}(S^1)$ which contains finite covers of $\text{PSL}(2, \mathbb{R})$, which is the key to differentiate representations [9, Appendix], cf. [33]).

Dual nets. The most fundamental properties of the dual nets $\{\hat{\rho}_{\kappa,q}(\text{Vir}_c(I))\}$ remain open. Among them is conformal covariance. Conformal covariance implies the split property [26], and even the split property is unknown to hold in these dual nets. The split property may fail in the dual net in two-dimensional Haag-Kastler net [25, Section 4.2], therefore, it may be worthwhile to try to (dis)prove the split property in these nets.

More positive-energy representations. Ground states consist only a particular class of positive-energy representations. It was shown in [11, 12] that there is a huge class of locally normal positive-energy representations of the free massless fermion field in $(3+1)$ -dimensions. A similar construction should be possible in one dimension. Furthermore, important conformal nets, including some loop group nets, can be realized as subnets of (the tensor product of) the free fermion field nets (see [30, Examples 4.13-16]). Among them, there might be counterexamples to [21, Conjecture 34].

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