Self-adjointness of bound state operators (partly with D. Cadamuro)

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Introduction

Goal

Construct Haag-Kastler nets (local observables) for integrable models with bound states (factorizing S-matrices with **poles**).
Non-perturbative, non-trivial quantum field theories in $d = 2$.

- Sine-Gordon, Bulldough-Dodd, $Z(N)$-Ising...

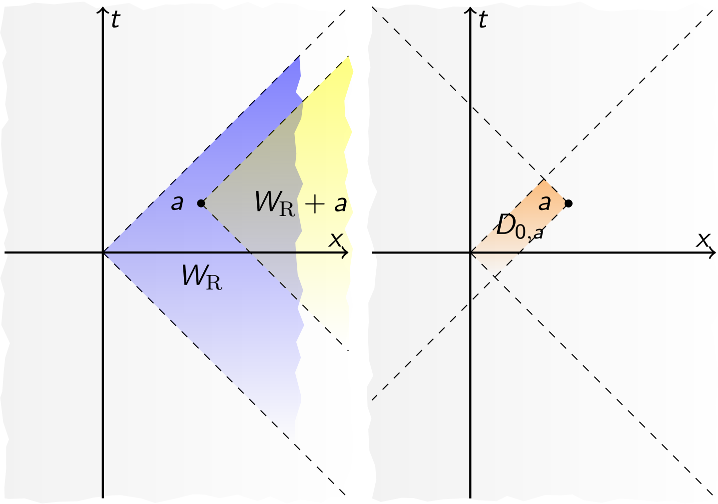
Methods and partial results (Cadamuro-T. '14)

Conjecture the S-matrix, construct first **observables localized in wedges**, then prove the existence of local observables indirectly (c.f. Lechner '08)

- Weakly commuting operators $\widetilde{\phi}(f) = z^\dagger(f) + \chi(f) + z(f), \widetilde{\phi}'(g)$.
- Wedge-algebras should be: $\mathcal{A}(W_\mathrm{L}) = \{e^{i\widetilde{\phi}(f)} : \mathrm{supp}\, f \subset W_\mathrm{L}\}$, $\mathcal{A}(W_\mathrm{R}) = \{e^{i\widetilde{\phi}'(g)} : \mathrm{supp}\, g \subset W_\mathrm{R}\}$.

Problem: $\chi(f)$, and $\widetilde{\phi}(f)$, are not self-adjoint on a naive domain.
Find a nice self-adjoint extension of $\chi(f)$.

Standard wedge and double cone



Self-adjointness

A linear operator A on a domain $\mathrm{Dom}(A) \subset \mathcal{H}$ is **symmetric** if

$$\langle \xi, A\eta \rangle = \langle A\xi, \eta \rangle, \xi, \eta \in \mathrm{Dom}(A).$$

Its **adjoint** A^* is defined on $\mathrm{Dom}(A^*) = \{\xi \mid \eta \mapsto \langle \xi, A\eta \rangle \text{is continuous}\}$, by

$$\langle \xi, A\eta \rangle = \langle A^*\xi, \eta \rangle, \eta \in \mathrm{Dom}(A),$$

If A is symmetric, then $A \subset A^*$. A is **self-adjoint** if and only if $A = A^*$.

Example: extensions ⇔ boundary conditions ⇔ different physics

$\mathrm{Dom}(A) = \{\xi \in L^2(0,1) \mid \xi \text{ is absolutely continuous}, \xi(0) = \xi(1) = 0\}$,

$$(A\xi)(t) = \xi'(t).$$

Then $A \neq A^*$, $\mathrm{Dom}(A^*) = \{\xi \in L^2(0,1) \mid \xi \text{ is absolutely continuous}\}$,

$$(A^*\xi)(t) = \xi'(t).$$

Many extensions: for $|\alpha| = 1, \mathrm{Dom}(A_\alpha) = \{\xi \in \mathrm{Dom}(A^*) \mid \xi(0) = \alpha\xi(1)\}$.

The bound state operator

S: S-matrix, $P_{S,n}$: S-symmetrization, $\mathcal{H} = \bigoplus P_{S,n}\mathcal{H}_1^{\otimes n}$, $\mathcal{H}_1 = L^2(\mathbb{R})$,

$$\mathrm{Dom}(\chi_1(f)) := H^2\left(-\tfrac{\pi}{3}, 0\right)$$
$$(\chi_1(f))\xi(\theta) := f^+\left(\theta + \tfrac{\pi i}{3}\right)\xi\left(\theta - \tfrac{\pi i}{3}\right),$$

where $H^2(\alpha,\beta)$ is the space of analytic functions in $\mathbb{R} + i(\alpha,\beta)$ such that $\xi(\cdot - \gamma i)$ is uniformly bounded in L^2-norm, $\gamma \in (\alpha,\beta)$, and f^+ is analytic.

$$\chi(f) := \bigoplus \chi_n(f),\ \chi_n(f) = nP_{S,n}\left(\chi_1(f) \otimes I \otimes \cdots \otimes I\right)P_{S,n},$$

Problem

- **Classify** self-adjoint extensions of $\chi_1(f)$.
- Choose a **physically correct** self-adjoint extension of $\chi_1(f)$, namely $\widetilde{\phi}(f) = z^\dagger(f) + \chi(f) + z(f)$ and $\widetilde{\phi}'(g)$ should **strongly commute**.

The one-particle bound state operator

- $\mathcal{H}_1 = L^2(\mathbb{R})$
- $h(\zeta)$: analytic in $\mathbb{R} + i(-\pi, 0)$, $\overline{h(\theta)} = h(\theta - \pi i)$.
- $\mathrm{Dom}(\chi_1(h)) = H^2(-\pi, 0)$: the space of analytic functions in $\mathbb{R} + i(-\pi, 0)$ such that $\xi(\cdot - \gamma i)$ is uniformly bounded in L^2-norm, $\gamma \in (-\pi, 0)$
- $(\chi_1(h))\xi(\theta) := h(\theta - \pi i)\,\xi(\theta - \pi i) \quad (= \overline{h(\theta)}\xi(\theta - \pi i))$

The main problem

What are self-adjoint extensions of $\chi_1(h)$?

- Write $\chi_1(h) = M_{\overline{h}}\Delta^{-\frac{1}{2}}$, $(\Delta^{-\frac{1}{2}}\xi)(\theta) = \xi(\theta - \pi i)$
- Classify extensions: **compute** $\ker(\chi_1(h)^* \pm i)$, $\chi_1(h)^* = \Delta^{-\frac{1}{2}} M_h$
- Self-ad. extensions of $\chi_1(h) \Leftrightarrow$ isometries between $\ker(\chi_1(h)^* \pm i)$.
- $n_\pm(\chi_1(h)) := \dim\ker(\chi_1(h)^* \pm i)$: **deficiency indices** of $\chi_1(h)$.

Case: Blaschke factors

Consider

$$h(\zeta) = \prod_{j=1}^{n} \frac{e^{\zeta} - \alpha_j}{e^{\zeta} + \alpha_j}, \quad h(\zeta - \pi i) = \prod_{j=1}^{n} \frac{e^{\zeta} + \alpha_j}{e^{\zeta} - \alpha_j}, \quad \alpha_j \text{ and } -\overline{\alpha_j} \text{ appear in pair.}$$

How many solutions ξ of $h(\zeta - \pi i)\xi(\zeta - \pi i) = \pm i\xi(\zeta)$, where $h\xi \in H^2(-\pi, 0)$?

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How many solutions ξ of $h(\zeta - \pi i)\xi(\zeta - \pi i) = \pm i\xi(\zeta)$, where $h\xi \in H^2(-\pi, 0)$?

$$\xi_k(\zeta) = e^{\left(\frac{1}{2}+k\right)\zeta} \prod_{j=1}^{n} \frac{1}{e^\zeta - \alpha_j}, \quad 0 \le k < n$$

$$\xi_k(\zeta - \pi i) = i(-1)^{n+k+1} e^{\left(\frac{1}{2}+k\right)\zeta} \prod_{j=1}^{n} \frac{1}{e^\zeta + \alpha_j}, \quad 0 \le k < n$$

The deficiency indices are: (m, m) for $n = 2m$, $(m+1, m)$ for $n = 2m+1$. The operator $\chi_1(h)$ has (infinitely many) self-adjoint extensions if and only if n is **even**. If n is **odd**, no self-adjoint extension.

Case: singular inner functions

Consider

$$h(\zeta) = \exp\left(-i\alpha_+ e^{\zeta} + i\alpha_- e^{-\zeta}\right), \quad h(\zeta - \pi i) = \exp\left(i\alpha_+ e^{\zeta} - i\alpha_- e^{-\zeta}\right),$$

$\alpha_\pm > 0$. How many solutions ξ of $h(\zeta - \pi i)\xi(\zeta - \pi i) = \pm i\xi(\zeta)$, where $h\xi \in H^2(-\pi, 0)$?

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Outside H^2, there is $g(\zeta) = h(\zeta - \pi i)g(\zeta - \pi i)$, $\frac{\xi(\zeta)e^{\pm\frac{1}{2}\zeta}}{g(\zeta)}$ is **periodic**: $g(\zeta) = \exp\left(\frac{1}{2}\left(i\alpha_+ e^{\zeta} - i\alpha_- e^{-\zeta}\right)\right)$.

$$\xi_{g_+,g_-}(\zeta) = \int dk_+ \cos\left(\frac{k_+}{2}e^{\zeta}\right) g_+(k_1) \cdot \int dk_- \cos\left(\frac{k_-}{2}e^{-\zeta}\right) g_-(k_-),$$

where $g_\pm$ are smooth, $g_\pm(k) = g_\pm(-k)$, $\mathrm{supp}\, g_\pm \subset (-\alpha_\pm, \alpha_\pm)$. Then the **product** $g(\zeta)\xi_{g_+,g_-}(\zeta)e^{\left(n+\frac{1}{2}\right)\zeta}$ is a solution.

The deficiency indices are: (∞, ∞).

The operator $\chi_1(h)$ has **infinitely many** self-adjoint extensions.

Case: outer functions

Consider

$$h(\zeta) = \exp\left(-i\int_{-\infty}^{\infty}\frac{ds}{1+s^2}\frac{1+e^{\zeta}s}{e^{\zeta}-s}|\log\phi(s)|\right),$$

$$|\log\phi(s)| < B|s|^{\alpha} + A, 0 < \alpha < 1.$$

How many solutions ξ of $h(\zeta - \pi i)\xi(\zeta - \pi i) = \pm i\xi(\zeta)$, where $h\xi \in H^2(-\pi, 0)$?

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$$g(\zeta) = \exp\left(-i\int_{-\infty}^{0}\frac{ds}{1+s^2}\frac{1+e^\zeta s}{e^\zeta - s}|\log\phi(s)|\right),$$

Divide an arbitrary solution ξ by g and $\frac{\xi}{g}$ is **periodic**, with an estimate of the form $\frac{\xi(\zeta)}{g(\zeta)} < Ae^{Be^{\alpha|\mathrm{Re}\,\zeta|}} \Rightarrow \xi(\zeta) = Cg(\zeta)$ (but ξ had to be H^2, $\lightning$).
The operator $\chi_1(h)$ is **essentially self-adjoint**.

Self-adjoint extensions, spectral calculus

Any H^∞-function h admits the decomposition $h = h_B h_{\mathrm{in}} h_{\mathrm{out}}$, where

- h_B is a Blaschke factor
- h_{in} is singular inner
- h_{out} is outer:

$$h(\zeta) = \exp\left(-i\int_{-\infty}^{\infty} \frac{ds}{1+s^2}\frac{1+e^\zeta s}{e^\zeta - s}|\log\phi(s)|\right) = h_+(\zeta)h_-(\zeta),$$

$$h_+(\zeta - \pi i) = \exp\left(-i\int_0^{\infty} \frac{ds}{1+s^2}\frac{1-e^\zeta s}{-e^\zeta - s}|\log\phi(s)|\right),$$

$$\theta\in\mathbb{R}, h_-(\theta) = \exp\left(-i\int_{-\infty}^{0} \frac{ds}{1+s^2}\frac{1+e^\theta s}{e^\theta - s}|\log\phi(s)|\right) = \overline{h_+(\theta-\pi i)}$$

If $h = f^2, f = f_B f_{\mathrm{in}} f_{\mathrm{out}}$, then f_B is a Blaschke factor, h_{in} is singular inner, $M_{f_B}, M_{f_{\mathrm{in}}}, M_{h_-}$ are unitary and

$$M_{\overline{h}}\Delta^{-\frac{1}{2}} \subset M_{\overline{f_B}} M_{\overline{f_{\mathrm{in}}}} M_{\overline{h_-}}\Delta^{-\frac{1}{2}} M_{f_B} M_{f_{\mathrm{in}}} M_{h_-}$$

Wedge-local fields (work in progress), Outlook

For a test function f on $\mathbb{R}^2$, $f^+(\theta) = \int da\, e^{ip(\theta)\cdot a} f(a)$. One takes $f * f$ so that $h = (f^+)^2$, and the previous decomposition applies.

- One can find a self-adjoint extension of $\chi_1(h)$.
- One can **explicitly** compute the square root of $\chi_1(h)$.
- $\chi_2(h) = P_{S,2}\,(\chi_1(h) \otimes I)\, P_{S,2}$ is self-adjoint, **without closure, without Friedrichs extension**.
- $\chi(h) + \chi'(g)$ is self-adjoint ??
- **Strong commutativity of** $\widetilde{\phi}(h), \widetilde{\phi}'(g)$ should follow by Driessler-Fröhlich argument.

Open problems

- self-adjointness of $\chi(h) + \chi'(g)$
- Bisognano-Wichmann property
- Modular nuclearity ($\Rightarrow$ Haag-Kastler net)