KMS states on conformal nets

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Introduction: KMS states

KMS state on C*-algebra

$\mathcal{A}$: C*-algebra, α_t: one-parameter automorphism group.
A β-KMS state φ on $\mathcal{A}$ with respect to α_t is a state with the following condition: for any $x, y \in \mathcal{A}$ there is an analytic function f such that

$$f(t) = \varphi(x\alpha_t(y)), f(t+i\beta) = \varphi(\alpha_t(y)x).$$

Example: matrix algebra

$\mathcal{A} = M_n(\mathbb{C}), \alpha_t = \mathrm{Ad}(e^{itH})$, H: positive. The state $\varphi(x) = \frac{\mathrm{Tr}(e^{-\beta H}x)}{\mathrm{Tr}(e^{-\beta H})}$ is a β-KMS state. The KMS condition characterizes this state.

Example: modular automorphism group

$\mathcal{A} = M$, a von Neumann algebra,φ: a faithful normal state, σ^φ: the modular automorphism. φ is a -1-KMS state.

Introduction: conformal nets

- Spacetime: the circle $S^1 = \mathbb{R} \cup \{\infty\}$.
- Möbius symmetry: translation, dilation, rotation.
- Diffeomorphism covariance.

Conformal net

A conformal net $\mathcal{A}$ is an assignment of von Neumann algebra $\mathcal{A}(I)$ to each interval $I \subset \mathcal{S}^1$ such that

- $I \subset J \Rightarrow \mathcal{A}(I) \subset \mathcal{A}(J)$
- $I \cap J = \emptyset \Rightarrow [A(I), \mathcal{A}(J)] = 0$.
- There is $U : \mathrm{PSL}(2, \mathbb{R}) \to \mathcal{U}(\mathcal{H})$ such that $U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI)$, with positivity of energy.
- There is Ω invariant under $\mathrm{PSL}(2, \mathbb{R})$.
- U extends to $\mathrm{Diff}(S^1)$.

Introduction: KMS states on conformal nets

Two copies of the theory on real line $\Longleftrightarrow$
Chiral conformal theory on two dimension.
The automorphism concerned: translation.

Observations

- Dilation covariance: the phase structure is uniform with respect to temperature.

$$\varphi \text{ is a } \beta\text{-KMS state } \Longleftrightarrow \varphi \circ \delta_s \text{ is a } \beta e^s\text{-KMS state.}$$

  We consider always $\beta = 1$.
- Diffeomorphism covariance: there is at least one geometric state.

Problem

Are there KMS states other than the geometric state?

Introduction: summary of results

Nets with several KMS states

- $U(1)$-current net: parametrized by $\mathbb{R}$.
- Virasoro net with $c \geq 1$: there are at least KMS states parametrized by $\mathbb{R}_+$.

Nets with unique KMS state

- completely rational nets.
  1. thermal completion gives extension.
  2. uniqueness for maximal nets.
  3. extension trick.

Geometric KMS state

Bisognano-Wichmann property

The vacuum state ω is a KMS state for $\mathcal{A}(\mathbb{R}_+)$ with respect to **dilation**.

Diffeomorphism covariance

The exponential map

$$t \longmapsto e^t$$

is a diffeomorphism between $\mathbb{R}$ and $\mathbb{R}_+$, and this intertwines translation and dilation.

This diffeomorphism Exp is implemented **locally** by a unitary U.

The geometric KMS state

The state $\omega \circ \mathrm{Exp}$ is well defined and a KMS state with respect to translation.

Case: $U(1)$-current net

$W(f)$: Weyl operator in Fock representation.

$$[W(f), W(g)] = \exp\left(-i \int f(\theta)g'(\theta)d\theta\right),$$
$$\mathcal{A}_{U(1)}(I) = \{W(f) : \mathrm{supp}(f) \subset I\}''.$$

There are automorphisms of the restricted net $\mathcal{A}_{\mathbb{R}}$ parametrized by $q \in \mathbb{R}$:

$$\alpha_q : W(f) \longmapsto \exp\left(iq \int_{\mathbb{R}} f(t)dt\right) W(f).$$

Theorem (Wang, 06)

For any $q \in \mathbb{R}$, $\varphi_{\mathrm{geo}} \circ \alpha_q$ is a KMS state. Any KMS state on the $U(1)$-current net is of this form.

Case: Virasoro nets with $c \geq 1$

Virasoro nets Vir_c are nets generated by the field T which satisfies the following commutation relation:

$$[T(f), T(g)] = iT(fg' - f'g) + \frac{ic}{12}\int_{\mathbb{R}} f'''(t)g(t)dt.$$

From the $U(1)$-current, we can construct a stress energy tensor:

$$T(z) = \frac{1}{2} : J(z)^2 :$$

or its deformations (Buchholz and Schulz-Mirbach, '90):

$$T_k(z) = T(z) + kJ'(z) + \frac{k^2}{2}$$

T_k satisfies the commutation relation with $c = 1 + k^2$.

Case: Virasoro nets with $c \geq 1$

$\mathrm{Vir}_c|_{\mathbb{R}}$ for $c \geq 1$ can be embedded in $\mathcal{A}_{U(1)}|_{\mathbb{R}}$ in a translation-covariant way. In particular, the automorphisms α_q can be applied to elements in Vir_c.

Theorem

The compositions $\varphi_{\mathrm{geo}} \circ \alpha_q|_{\mathrm{Vir}_c}$ give rise to different KMS states for different values of $\frac{q^2}{2}$.

We don't know if these states are all or not.

Complete rationality

Complete rationality

A conformal net $\mathcal{A}$ is said to be **completely rational** if it satisfies the following:

- The split property.
- Strong additivity.
- Finiteness of μ-index.

Examples of complete rational nets

- Loop group nets.
- Virasoro nets with $c < 1$.
- Finite index inclusions and extensions.

Complete rationality: finiteness of sectors and extensions

Fact

A completely rational net admits only finitely many inequivalent DHR representations.

In examples we saw:

- different KMS states $\Longleftrightarrow$ different "combination" of charges.

Translation invariance:

- a KMS state should contain an "infinite" amount of charge with some density.

Expectation: completely rational nets admit only the geometric state.

Thermal completion

Thermal completion

- ψ: a primary KMS state on a net $\mathcal{A}$.
- π: the GNS representation with respect to ψ.
- ξ: the corresponding GNS vector.

The inclusion $(\pi(\mathcal{A}(\mathbb{R}_+)) \subset \pi(\mathcal{A}(\mathbb{R})), \xi)$ is a half-sided modular inclusion. The corresponding Möbius covariant net is called the **thermal completion** of $\mathcal{A}$ with respect to ψ.

In completely rational case, the thermal completion is an irreducible extension of the original net with finite index.

Case: completely rational maximal nets

Fact

A completely rational net admits only finitely many extensions of net. Among extensions, there are **maximal** extensions.

Lemma

The thermal completion of the geometric KMS state φ_{geo} is the original net.

Lemma

Any KMS state ψ on a completely rational maximal net $\mathcal{A}$ is a composition of the geometric state φ_{geo} and an automorphism $\gamma = \pi_\psi \circ \pi_{\varphi_{\mathrm{geo}}}^{-1}$ of $\mathcal{A}|_{\mathbb{R}_+}$.

Automorphisms of nets on S^1 VS on $\mathbb{R}$

Lemma

There is a map from the set of automorphisms on $\mathbb{R}$ commuting with translation to the set of automorphisms on S^1 commuting with rotation. Two automorphisms on $\mathbb{R}$ are unitarily equivalent if and only if the images are unitarily equivalent.

Proof: by diffeomorphism covariance.

Example: on $U(1)$-current net,

$$\alpha_q(W(f)) = \exp\left(iq\int_{S^1} f(\theta)d\theta\right)W(f)$$

$$\gamma_q(W(f)) = \exp\left(iq\int_{\mathbb{R}} f(t)dt\right)W(f)$$

Dilation on automorphisms on $\mathbb{R}$

We want infinitely many automorphism to have a contradiction with the finiteness of sectors.

Lemma

γ: an automorphism on $\mathbb{R}$.
If $\varphi_{\mathrm{geo}} \circ \gamma \neq \varphi_{\mathrm{geo}}$, then $\omega \circ \gamma \neq \omega$.

Lemma

If $\varphi \circ \delta_s = \varphi$ for some $s \in \mathbb{R}$, then $\varphi = \omega$.

Lemma

If $\omega \circ \gamma \neq \omega$, then $\omega \circ \gamma \circ \delta_s$ are all different states. $\delta_{-s} \circ \gamma \circ \delta_s$ are all different automorphisms.

Uniqueness result for maxmal nets

For a completely rational maximal net $\mathcal{A}$:
Any KMS state ψ gives an automorphism γ on $\mathcal{A}_{\mathbb{R}_+}$: $\psi = \varphi_{\mathrm{geo}} \circ \gamma$.
$\Longrightarrow$ γ gives an automorphism α on $\mathcal{A}$.
$\Longrightarrow$ $\{\delta_s \circ \gamma \circ \delta_{-s}\}$ could give infinitely many sectors (Impossible).
$\Longrightarrow$ γ commutes with dilation.
$\Longrightarrow$ γ preserves the vacuum state.
$\Longrightarrow$ γ preserves the geometric state.

Theorem

Any completely rational maximal net admits only the geometric KMS state φ_{geo}.

Extension trick

There is a one-to-one correspondence:
KMS state w.r.t. translation on $\mathcal{A}|_\mathbb{R} \Longleftrightarrow$
KMS state w.r.t. dilation on $\mathcal{A}|_{\mathbb{R}_+}$.

ψ: KMS state, π: GNS representation, $\hat{\mathcal{A}}$: thermal completion.

Lemma

The inclusion $\pi(\mathcal{A}(a,b)) \subset \hat{\mathcal{A}}(a,b) := \pi(\mathcal{A}(a,\infty)) \cap \pi(\mathcal{A}(b,\infty))'$ is irreducible and of finite index, hence there is a conditional expectation $E : \hat{\mathcal{A}} \to \mathcal{A}$.

Lemma

The state $\omega \circ \mathrm{Exp} \circ \pi^{-1} \circ E$ is a KMS state on $\hat{\mathcal{A}}|_{\mathbb{R}_+}$.

Uniqueness result for general case

Extension trick

The initial net	vacuum state		KMS state
	$\Downarrow$	thermal completion	$\Downarrow$
The extended net	KMS state		vacuum state

- Repeating this construction, we arrive at a maximal net.
- For a maximal net, a KMS state is the geometric state.
- The KMS state of the starting point must be geometric.

Theorem

Any completely rational net admits only the geometric KMS state.

Summary and outlook

- Examples with continuously many KMS states
- Uniqueness for completely rational nets

- Similar result for ground states?
- General extension argument for finite index inclusions?