KMS states on conformal nets
(joint work with P. Camassa, R. Longo and M. Weiner)

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Introduction: KMS states

**KMS state on C*-algebra**

\( \mathcal{A} \): C*-algebra, \( \alpha_t \): one-parameter automorphism group. A \( \beta \)-KMS state \( \varphi \) on \( \mathcal{A} \) with respect to \( \alpha_t \) is a state with the following condition: for any \( x, y \in \mathcal{A} \) there is an analytic function \( f \) such that

\[
    f(t) = \varphi(x\alpha_t(y)), f(t + i\beta) = \varphi(\alpha_t(y)x).
\]

**Example: matrix algebra**

\( \mathcal{A} = M_n(\mathbb{C}) \), \( \alpha_t = \text{Ad}(e^{itH}) \), \( H \): positive. The state \( \varphi(x) = \frac{\text{Tr}(e^{-\beta H}x)}{\text{Tr}(e^{-\beta H})} \) is a \( \beta \)-KMS state. The KMS condition characterizes this state.

**Example: modular automorphism group**

\( \mathcal{A} = \mathcal{M} \), a von Neumann algebra, \( \varphi \): a faithful normal state, \( \sigma^\varphi \): the modular automorphism. \( \varphi \) is a \( -1 \)-KMS state.
Introduction: conformal nets

- Spacetime: the circle $S^1 = \mathbb{R} \cup \{\infty\}$.
- Möbius symmetry: translation, dilation, rotation.
- Diffeomorphism covariance.

Conformal net

A conformal net $\mathcal{A}$ is an assignment of von Neumann algebra $\mathcal{A}(I)$ to each interval $I \subset S^1$ such that

- $I \subset J \Rightarrow \mathcal{A}(I) \subset \mathcal{A}(J)$
- $I \cap J = \emptyset \Rightarrow [\mathcal{A}(I), \mathcal{A}(J)] = 0$.

There is $U : \text{PSL}(2, \mathbb{R}) \rightarrow \mathcal{U}(\mathcal{H})$ such that $U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI)$, with positivity of energy.

There is $\Omega$ invariant under $\text{PSL}(2, \mathbb{R})$.

$U$ extends to $\text{Diff}(S^1)$. 
Two copies of the theory on real line \(\iff\) Chiral conformal theory on two dimension. The automorphism concerned: translation.

**Observations**

- Dilation covariance: the phase structure is uniform with respect to temperature.

\[
\varphi \text{ is a } \beta\text{-KMS state } \iff \varphi \circ \delta_s \text{ is a } \beta e^s\text{-KMS state.}
\]

We consider always \(\beta = 1\).

- Diffeomorphism covariance: there is at least one geometric state.

**Problem**

Are there KMS states other than the geometric state?
Introduction: summary of results

Nets with several KMS states

- $U(1)$-current net: parametrized by $\mathbb{R}$.
- Virasoro net with $c \geq 1$: there are at least KMS states parametrized by $\mathbb{R}_+$.

Nets with unique KMS state

- completely rational nets.
  1. thermal completion gives extension.
  2. uniqueness for maximal nets.
  3. extension trick.
Geometric KMS state

Bisognano-Wichmann property
The vacuum state $\omega$ is a KMS state for $\mathcal{A}(\mathbb{R}_+)$ with respect to dilation.

Diffeomorphism covariance
The exponential map $t \mapsto e^t$ is a diffeomorphism between $\mathbb{R}$ and $\mathbb{R}_+$, and this intertwines translation and dilation.
This diffeomorphism $\text{Exp}$ is implemented locally by a unitary $U$.

The geometric KMS state
The state $\omega \circ \text{Exp}$ is well defined and a KMS state with respect to translation.
Case: $U(1)$-current net

$W(f)$: Weyl operator in Fock representation.

$$[W(f), W(g)] = \exp \left( -i \int f(\theta) g'(\theta) d\theta \right),$$

$$\mathcal{A}_{U(1)}(I) = \{ W(f) : \text{supp}(f) \subset I \}''.$$  

There are automorphisms of the restricted net $\mathcal{A}_\mathbb{R}$ parametrized by $q \in \mathbb{R}$:

$$\alpha_q : W(f) \mapsto \exp \left( iq \int_{\mathbb{R}} f(t) dt \right) W(f).$$

**Theorem (Wang, 06)**

*For any $q \in \mathbb{R}$, $\varphi_{\text{geo}} \circ \alpha_q$ is a KMS state. Any KMS state on the $U(1)$-current net is of this form.*
Virasoro nets $\text{Vir}_c$ are nets generated by the field $T$ which satisfies the following commutation relation:

$$[T(f), T(g)] = iT(fg' - f'g) + \frac{ic}{12} \int_{\mathbb{R}} f'''(t)g(t)dt.$$ 

From the $U(1)$-current, we can construct a stress energy tensor:

$$T(z) = \frac{1}{2} : J(z)^2 :$$

or its deformations (Buchholz and Schulz-Mirbach, ’90):

$$T_k(z) = T(z) + kJ'(z) + \frac{k^2}{2}$$

$T_k$ satisfies the commutation relation with $c = 1 + k^2$. 

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Case: Virasoro nets with $c \geq 1$

Vir$_c|_\mathbb{R}$ for $c \geq 1$ can be embedded in $A_{U(1)}|_\mathbb{R}$ in a translation-covariant way. In particular, the automorphisms $\alpha_q$ can be applied to elements in Vir$_c$.

**Theorem**

The compositions $\varphi_{\text{geo}} \circ \alpha_q|_{\text{Vir}_c}$ give rise to different KMS states for different values of $\frac{q^2}{2}$.

We don’t know if these states are all or not.
A conformal net $\mathcal{A}$ is said to be **completely rational** if it satisfies the following:

- The split property.
- Strong additivity.
- Finiteness of $\mu$-index.

Examples of complete rational nets:

- Loop group nets.
- Virasoro nets with $c < 1$.
- Finite index inclusions and extensions.
Complete rationality: finiteness of sectors and extensions

Fact
A completely rational net admits only finitely many inequivalent DHR representations.

In examples we saw:
- different KMS states $\iff$ different “combination” of charges.

Translation invariance:
- a KMS state should contain an “infinite” amount of charge with some density.

Expectation: completely rational nets admit only the geometric state.
Thermal completion

- $\psi$: a primary KMS state on a net $\mathcal{A}$.
- $\pi$: the GNS representation with respect to $\psi$.
- $\xi$: the corresponding GNS vector.

The inclusion $(\pi(\mathcal{A}(\mathbb{R}_+)) \subset \pi(\mathcal{A}(\mathbb{R})), \xi)$ is a half-sided modular inclusion. The corresponding Möbius covariant net is called the thermal completion of $\mathcal{A}$ with respect to $\psi$.

In completely rational case, the thermal completion is an irreducible extension of the original net with finite index.
Case: completely rational maximal nets

Fact
A completely rational net admits only finitely many extensions of net. Among extensions, there are **maximal** extensions.

Lemma
The thermal completion of the geometric KMS state $\varphi_{\text{geo}}$ is the original net.

Lemma
Any KMS state $\psi$ on a completely rational maximal net $\mathcal{A}$ is a composition of the geometric state $\varphi_{\text{geo}}$ and an automorphism $\gamma = \pi_\psi \circ \pi_{\varphi_{\text{geo}}}^{-1}$ of $\mathcal{A}|_{\mathbb{R}^+}$. 
Automorphisms of nets on $S^1$ VS on $\mathbb{R}$

**Lemma**

There is a map from the set of automorphisms on $\mathbb{R}$ commuting with translation to the set of automorphisms on $S^1$ commuting with rotation. Two automorphisms on $\mathbb{R}$ are unitarily equivalent if and only if the images are unitarily equivalent.

Proof: by diffeomorphism covariance.

Example: on $U(1)$-current net,

$$\alpha_q(W(f)) = \exp \left( iq \int_{S^1} f(\theta) d\theta \right) W(f)$$

$$\gamma_q(W(f)) = \exp \left( iq \int_{\mathbb{R}} f(t) dt \right) W(f)$$
Dilation on automorphisms on $\mathbb{R}$

We want infinitely many automorphism to have a contradiction with the finiteness of sectors.

**Lemma**

$\gamma$: an automorphism on $\mathbb{R}$.

If $\varphi_{\text{geo}} \circ \gamma \neq \varphi_{\text{geo}}$, then $\omega \circ \gamma \neq \omega$.

**Lemma**

If $\varphi \circ \delta_s = \varphi$ for some $s \in \mathbb{R}$, then $\varphi = \omega$.

**Lemma**

If $\omega \circ \gamma \neq \omega$, then $\omega \circ \gamma \circ \delta_s$ are all different states. $\delta_{-s} \circ \gamma \circ \delta_s$ are all different automorphisms.
For a completely rational maximal net $\mathcal{A}$:
Any KMS state $\psi$ gives an automorphism $\gamma$ on $\mathcal{A}_{\mathbb{R}^+}$: $\psi = \varphi_{\text{geo}} \circ \gamma$.
$\implies$ $\gamma$ gives an automorphism $\alpha$ on $\mathcal{A}$.
$\implies \{\delta_s \circ \gamma \circ \delta_{-s}\}$ could give infinitely many sectors (Impossible).
$\implies$ $\gamma$ commutes with dilation.
$\implies$ $\gamma$ preserves the vacuum state.
$\implies$ $\gamma$ preserves the geometric state.

**Theorem**

*Any completely rational maximal net admits only the geometric KMS state $\varphi_{\text{geo}}$.***
Extension trick

There is a one-to-one correspondence:
KMS state w.r.t. translation on $\mathcal{A}|_{\mathbb{R}} \iff$
KMS state w.r.t. dilation on $\mathcal{A}|_{\mathbb{R}^+}$.

$\psi$: KMS state, $\pi$: GNS representation, $\hat{\mathcal{A}}$: thermal completion.

**Lemma**

The inclusion $\pi(\mathcal{A}(a, b)) \subset \hat{\mathcal{A}}(a, b) := \pi(\mathcal{A}(a, \infty)) \cap \pi(\mathcal{A}(b, \infty))'$ is irreducible and of finite index, hence there is a conditional expectation $E: \hat{\mathcal{A}} \to \mathcal{A}$.

**Lemma**

The state $\omega \circ \text{Exp} \circ \pi^{-1} \circ E$ is a KMS state on $\hat{\mathcal{A}}|_{\mathbb{R}^+}$. 
### Extension trick

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- Repeating this construction, we arrive at a maximal net.
- For a maximal net, a KMS state is the geometric state.
- The KMS state of the starting point must be geometric.

### Theorem

*Any completely rational net admits only the geometric KMS state.*
Examples with continuously many KMS states
Uniqueness for completely rational nets
Similar result for ground states?
General extension argument for finite index inclusions?