Covariant representations of $\mathrm{Diff}(\mathbb{R})$

Yoh Tanimoto

Department of Mathematics, University of Rome "Tor Vergata"

15/01/10, Workshop "Foundations and Constructive Aspects of QFT"

Introduction

Conformal symmetry

- Poincaré group + dilation and special conformal group (preserving the angle).
- conformal symmetry appears in string theory, statistical mechanics, massless particles, scaling limit...

A traditional set of mathematical objects is

- Wightman field (operator valued distribution on Minkowski space)
- Unitary representation of conformal symmetry group with spectrum condition
- the vacuum vector

Introduction

Diff(S^1) symmetry

Why do we consider Diff(S^1) symmetry?

1. consider a conformal field on 2 dimensional space parametrized (t, x).
2. some important observables (e.g. stress energy tensor) decompose into components depending only on $t + x$ or $t - x$.
3. under the assumption of dilation symmetry and spectrum condition, each component of stress energy tensor can be extended to S^1 and has a certain commutation relations (Lüscher-Mack theorem).
4. the commutation relations are same as the Lie algebra of Diff(S^1).
5. the component of stress energy tensor is Diff(S^1) covariant.

Introduction

So minimal mathematical objects in conformal field theory are

- Projective unitary irreducible representations of $\mathrm{Diff}(S^1)$ with translations having the positive spectrum.

fact

Projective unitary, positive energy, irreducible representations of $\mathrm{Diff}(S^1)$ are completely classified by c and h. There exists such a representation if and only if there exist natural numbers m, r, s such that

$$\begin{aligned} c &= 1 - \frac{6}{(m+2)(m+3)}, 1 \leq m \\ h &= \frac{\{(m+3)r - (m+2)s\}^2 - 1}{4(m+2)(m+3)}, 1 \leq s \leq r \leq m+1, \end{aligned}$$

or $c \geq 1$ and $h \geq 0$. If $h = 0$, it is said to have the vacuum.

Introduction

What happens if we consider a similar problem for $\mathrm{Diff}(\mathbb{R})$?

Definition

A projective unitary representation ρ of $\mathrm{Diff}(\mathbb{R})$ is said to be covariant if there is a representation U of $\mathbb{R}$ with a positive generator such that

$$U(a)\rho(\gamma)U(a)^* = \rho(\tau_a \circ \gamma \circ \tau_{-a})$$

A vector Ω in the representation space is called a ground state vector if $U(a)\Omega = \Omega$.

And our main result is

Theorem

There are covariant irreducible representations with a ground state vector of $\mathrm{Diff}(\mathbb{R})$ *which do not extend to* $\mathrm{Diff}(S^1)$.

Physical interpretation of ground state representations

We identify the circle S^1 and the real line $\mathbb{R}$ by the Kayley transformation.

$$x = i\frac{1+z}{1-z} \iff z = \frac{x-i}{x+i}, x \in \mathbb{R}, z \in S^1 \subset \mathbb{C}$$

Virasoro net

With a vacuum representation ρ_c of $\mathrm{Diff}(\mathrm{S}^1)$, for any interval I of S^1 we define

$$\mathrm{Vir}_c(I) = \{\rho_c(\gamma) : \mathrm{supp}\gamma \subset I\}''$$

Physical interpretation of ground state representations

KMS states

A β-KMS state ϕ for the Virasoro net is a state on $\overline{\bigcup_{-1\notin I} \mathrm{Vir}_c(I)}^{\|\cdot\|}$ such that for any two element a, b there is analytic function F on $0 < \mathrm{Im} z < \beta$ with

$$F(t) = \phi(a\tau_t(b)), F(t + i\beta) = \phi(\tau_t(b)a),$$

where τ is the automorphism of translations.

Taking the limit $\beta \longrightarrow \infty$, we have the following definition.

ground states

A ground state ϕ for the Virasoro net is a state on $\overline{\bigcup_{-1\notin I} \mathrm{Vir}_c(I)}^{\|\cdot\|}$ invariant for translations and in whose GNS representation the generator of translations has a positive spectrum.

Physical interpretation of ground state representations

ρ_c: the representation of $\mathrm{Diff}(S^1)$
ϕ: a ground state on Vir_c
π: corresponding GNS representation
$\Longrightarrow \pi \circ \rho_c|_{\mathrm{Diff}(\mathbb{R})}$ is a ground state representation of $\mathrm{Diff}(\mathbb{R})$.

Classification program

A (locally normal) ground state on Vir net
$\Longrightarrow$ A (locally normal) ground state representation of $\mathrm{Diff}(\mathbb{R})$
Classification of ground state representations
$\Longrightarrow$ Classification of ground states on Vir net

Physical interpretation of ground state representations

Theorem (T. in preparation)

For Virasoro net with $c \geq 1$, there are ground states parametrized by a positive number k. The case $k = 0$ corresponds to the vacuum representation.

cf.

Conjecture (Longo and Weiner)

For Virasoro net with $c < 1$, there is unique β-KMS state for each β.

Representations of Diff(S^1)

Let U be a representation of Diff(S^1).
Diff(S^1) includes important one-parameter subgroups:

Möbius group

$$\begin{aligned} \text{rotation } \rho_s(z) &= e^{is}z, \text{ for } z \in S^1 \subset \mathbb{C} \\ \text{translation } \tau_s(x) &= x + s, \text{ for } x \in \mathbb{R} \\ \text{dilation } \delta_s(x) &= e^s x, \text{ for } x \in \mathbb{R} \end{aligned}$$

Spectrum condition
$\Longleftrightarrow$ the spectrum of translation is positive
$\Longleftrightarrow$ the spectrum of rotation is positive
The rotation group is compact $\Longrightarrow$ the rotation group has the lowest eigenvector with eigenvalue h.

Representations of Diff(S^1)

U is a projective representation.
$\iff U(g)U(h) = c(g,h)U(gh)$ where $c(g,h) \in S^1 \subset \mathbb{C}$
The function $c(g,h)$ must satisfy the cocycle identity.

fact

the second cohomology of Diff(S^1) is isomorphic to $\mathbb{R}$.

$c(g,h)$ is determined by a real number c.

Lie algebra of Diff(S^1)

group/algebra	elements	operation
Diff(S^1)	C^∞ diffeomorphisms of S^1	$f \circ g$ (composition)
Vect(S^1)	C^∞ vector fields on S^1	$[f, g] := fg' - f'g$
Witt	$L_n(\theta) = ie^{in\theta}$	$[L_m, L_n] = (m-n)L_{m+n}$

fact

Witt is simple. In particular, the first cohomology of Witt on $\mathbb{C}$ is trivial (Witt has no nontrivial one dimensional representation).
The second cohomology of Witt on $\mathbb{C}$ is one dimensional (Witt has the unique central extension Vir).

The Virasoro algebra has the following commutation relations.

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{Cn(n^2-1)}{12}\delta_{m,-n}$$

Concrete constructions of representations of $\mathrm{Diff}(\mathbb{R})$

Take the U(1)-current algebra.

$$[J_m, J_n] = im\delta_{m,-n}$$

Define an operator valued distribution J on S^1 as follows.

$$J : C^\infty(S^1) \ni f(z) = \sum f_n z^n \mapsto \sum f_n J_n$$

fact

J satisfies locality, namely, if $\mathrm{supp} f$ and $\mathrm{supp} g$ are disjoint, then $[J(f), J(g)] = 0$.

Concrete constructions of representations of $\mathrm{Diff}(\mathbb{R})$

Define

$$L_n = \frac{1}{2}\sum_m J_m J_{n-m}, L_0 = \sum_{m>0} J_{-m} J_m.$$

fact

Operators $\{L_n\}$ consist a representation of the Virasoro algebra with $C = I$. $\{L_n\}$ generate a representation ρ of $\mathrm{Diff}(\mathrm{S}^1)$ with $c = 1$.

fact

The current J is covariant under the action of ρ, namely,

$$\rho(\gamma)J(f)\rho(\gamma)^{-1} = J(f \circ \gamma)$$

Define nets of von Neumann algebras:

$$\begin{aligned} \mathrm{Vir}_1(I) &= \{\rho(\gamma) : \mathrm{supp}\gamma \subset I\}'' \\ \mathcal{A}_{U(1)}(I) &= \{e^{iJ(f)} : \mathrm{supp} f \subset I\}'' \end{aligned}$$

fact

For an interval I of S^1, it holds that $A_{U(1)}(I)' = A_{U(1)}(I')$ where I' is the complement of I (the Haag duality).

$$\mathrm{Vir}_1(I) \subset A_{U(1)}(I).$$

Concrete constructions of representations of Diff($\mathbb{R}$)

Definition

An endomorphism of the net $A_{U(1)}$ is a family of endomorphisms $\{\sigma_I\}$ of $\{A_{U(1)}(I)\}$ such that if $I \subset J$ then $\sigma_J|_I = \sigma_I$.

For a real function α on $\mathbb{R} = S^1 \setminus 1$, we define a new field

$$J_\alpha(f) = J(f) + \int \alpha(z) f(z) \frac{dz}{2\pi i z}$$

fact (Buchholz-Mack-Todorov automorphism)

The Fourier components of J_α satisfy the same commutation relations of J. The map $\sigma_\alpha : e^{iJ(f)} \mapsto e^{iJ(f_\alpha)}$ extends to an automorphism of $A_{U(1)}$.

Concrete constructions of representations of $\mathrm{Diff}(\mathbb{R})$

If $\{\sigma_I\}$ is an endomorphism of $\mathcal{A}_{U(1)}$, then $\sigma \circ \rho_1$ is a (possibly new) representation of $\mathrm{Diff}(\mathbb{R})$.

Theorem (T. in preparation)

Let α be a function on S^1, invariant under translations as a vector field. Then $\sigma_\alpha \circ \rho_1$ is a ground state representation of $\mathrm{Diff}(\mathbb{R})$ with the cocycle $c = 1$. $\sigma_\alpha \circ \rho_1$ is covariant with respect to the original translation with the original vacuum vector.
The set of these functions α are parametrized by a real number q. These representations are equivalent if and only if $q^2 = q'^2$.

There are constructions also for $c > 1$.

Verma modules for $\mathrm{Vect}(\mathbb{R})$

Let V be a covariant reprsentation T of $\mathrm{Vect}(\mathbb{R})$ with a ground state Ω with a cocycle corresponding to $c \in \mathbb{R}$. We set

$$k = \frac{\langle T(f)\Omega, \Omega\rangle}{\int f(x)dx} \text{ “energy density” } .$$

Observation

- If the Fourier transform of f has the support in $\mathbb{R}_+$, then $T(f)\Omega = 0$.
- If the Fourier transform of f has the support only around 0, then T(f) is “almost a scalar multiple of the identity”.
- We can decompose $T(f)$ into three parts, namely positive, zero, negative frequency parts.

Verma modules for Vect($\mathbb{R}$)

Observation (continued)

- The representation space is spanned by the vectors $T(f_1)T(f_2)\cdots T(f_n)\Omega$.
- The inner product is determined by the n-point function $\langle T(f_1)T(f_2)\cdots T(f_n)\Omega, \Omega\rangle$.
- The n-point function is reduced to (n-1)-point functions by the commutation relations and the support property.
- The inner product is determined by k (and c).

Verma modules for $\mathrm{Vect}(\mathbb{R})$

The dilation acts on $\mathrm{Vect}(\mathbb{R})$ as automorphisms and the composition of a dilation with a representation changes "the energy density" k. In particular, if a representation $V_{c,k}$ has a positive definite scalar product for some $c, k > 0$, then $V_{c,k'}$ has a positive definite scalar product for any $k' > 0$.

The case $k = 0$ extends to a representation of Vir.

Theorem (T. in preparation)

The irreducible representation V of $\mathcal{S}(\mathbb{R})$ with a ground state and with is classified c and k.
Conversely, for any $c, h \in \mathbb{R}$ we can construct a module of $\mathcal{S}(\mathbb{R})$ with an invariant sesquilinear form. The form is positive semidefinite if $c \geq 1$ and $k \geq 0$. The form is not positive semidefinite if $k > 0$ and $c \neq 1 - \frac{6}{(m+2)(m+3)}$ or $c = \frac{1}{2}$.

Verma modules for $\mathrm{Vect}(\mathbb{R})$

Conjecture

The representation V has an invariant inner product if and only if $c \geq 1$ and $k \geq 0$ or $k = 0$ and c is in the discrete series.

Conjecture

- The ground state on Virasoro net with $c < 1$ is unique.
- The ground state on a loop group net is unique.
- The ground state on a completely rational net is unique.

Further covariant representations

We consider B_0, the group of stabilizers of 1 of $\mathrm{Diff}(S^1)$.

group/algebra	elements	operation
B_0	diffeomorphisms stabilizing $\theta = 0$	$f \circ g$ (composition)
$\mathrm{Vect}(S^1)_0$	vector fields f with $f(0) = 0$	$[f, g] := fg' - f'g$
$\mathcal{K}_0$	$K_n(\theta) = i(1 - e^{in\theta})$	Restriction of Vir

Theorem (T. '09)

The ideal structure of $\mathcal{K}_0$ is determined as an infinite sequence of ideals

$$\mathcal{K}_0 \supset \mathcal{K}_1 \supset \mathcal{K}_2 \supset \cdots$$

and an exceptional ideal $\mathcal{K}_{1,3} \supset \mathcal{K}_3$ and it holds that $[\mathcal{K}_n, \mathcal{K}_n] = \mathcal{K}_{2n+1}$. In particular, $\mathcal{K}_1 = [\mathcal{K}_0, \mathcal{K}_0]$ has codimension one in $\mathcal{K}_0$ and $\mathcal{K}_0$ has one dimensional representation.
The second cohomology of $\mathcal{K}_0$ on $\mathbb{C}$ is one dimensional ($\mathcal{K}_0$ has the unique central extension $\mathcal{K}$).

Further covariant representations

Theorem (T. '09, T. in preparation)

For any $c \in \mathbb{R}$, $h, \lambda \in \mathbb{C}$, *there is a representation of* $\mathcal{K}$ *with a contravariant sesquilinear form* $\langle \cdot, \cdot \rangle$ *and a lowest weight vector* v *such that*

$$Cv = cv, K_n v = (h + n\lambda)v \text{ for } n > 0.$$

(Note that $K_n = L_0 - L_n$ *in Vir. If* $h \in \mathbb{R}, \lambda = 0$, *this module reduces to a restriction of Vir module.)*
The sesquilinear form is positive semidefinite for some values of $c \geq 1$, $h \in \mathbb{C} \setminus \mathbb{R}$, *and* $\lambda \in \mathbb{C}$. *These representations integrate to representations of the group* B_0 *with positive spectrum of translations, without a ground state.*

These representations are expected not to extend to $\mathrm{Diff}(S^1)$.

Summary and outlook

- several covariant representations (with and without ground states) are constructed.
- partial classification of Verma modules of $\mathrm{Diff}(\mathbb{R})$ with conditions.

- differentiability of ground state representations?
- positive definiteness of general Verma modules?
- relations with KMS states?
- ground states for loop group models?