Wedge-local fields in integrable models with bound states
(partly with D. Cadamuro, arXiv:1502.01313)

Yoh Tanimoto

JSPS SPD fellow, Tokyo / Göttingen

February 11th 2015, Frascati

Towards more 2d QFTs

Goal

Construct Haag-Kastler nets (local observables) for integrable models with bound states (factorizing S-matrices with **poles**).
Non-perturbative, non-trivial quantum field theories in $d = 2$.

- Sine-Gordon, Bullough-Dodd, $Z(N)$-Ising...

Methods and partial results

Conjecture the S-matrix with **poles**, construct first **observables localized in wedges**, then prove the existence of local observables indirectly.

- **Weakly commuting** fields: $\widetilde{\phi}(f) = z^\dagger(f^+) + \chi(f) + z(J_1 f^-), \widetilde{\phi}'(g)$ (c.f. Lechner '08, $\phi(f) = z^\dagger(f) + z(J_1 f^-)$ for S-matrix without poles).
- Wedge-algebras: $\mathcal{A}(W_\mathrm{L}) = \{e^{i\widetilde{\phi}(f)} : \mathrm{supp}\, f \subset W_\mathrm{L}\}$, $\mathcal{A}(W_\mathrm{R}) = \{e^{i\widetilde{\phi}'(g)} : \mathrm{supp}\, g \subset W_\mathrm{R}\}$. $\widetilde{\phi}(f)$ and $\widetilde{\phi}'(g)$ strongly commute? Arguments for **modular nuclearity**.

Overview of the strategy

- Haag-Kastler net $(\{\mathcal{A}(O)\}, U, \Omega)$: local observables $\mathcal{A}(O)$, spacetime symmetry U and the vacuum Ω.
- Wedge-algebras first: construct $\mathcal{A}(W_{\mathrm{R}}), U, \Omega$, then take the intersection

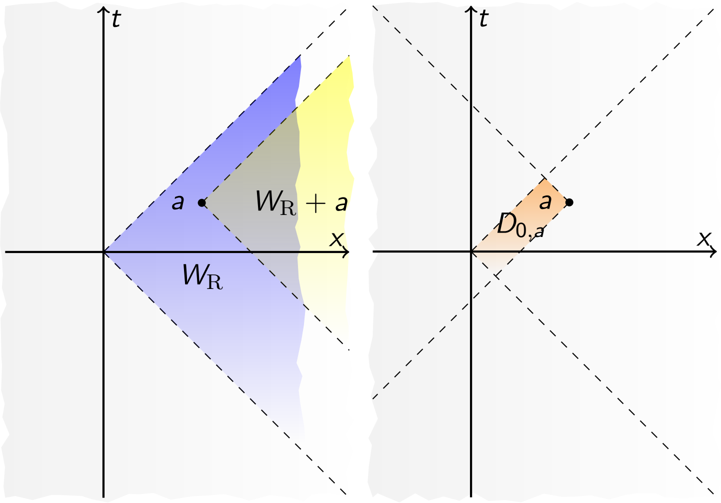
$$\mathcal{A}(D_{a,b}) = U(a)\mathcal{A}(W_{\mathrm{R}})U(a)^* \cap U(b)\mathcal{A}(W_{\mathrm{R}})'U(b)^*$$

  The intersection is large enough if **modular nuclearity** or wedge-splitting holds.
- **Wedge-local fields**: a pair of operator-valued distributions ϕ, ϕ' such that $[e^{i\phi(f)}, e^{i\phi'(g)}] = 0$ if $\mathrm{supp}\, f \subset W_{\mathrm{L}}, \mathrm{supp}\, g \subset W_{\mathrm{R}}$.

Examples: scalar analytic factorizing S-matrix (Lechner '08), twisting by inner symmetry (T., '14), diagonal S-matrix (Alazzawi-Lechner '15)...

More example? **S-matrices with poles**.

Standard wedge and double cone



Factorizing S-matrix models (Lechner, Schroer)

- **Input**: analytic function $S : \mathbb{R} + i(0,\pi) \to \mathbb{C}$,

$$\overline{S(\theta)} = S(\theta)^{-1} = S(-\theta) = S(\theta + \pi i), \quad \theta \in \mathbb{R}.$$

- S-symmetric Fock space: $\mathcal{H}_1 = L^2(\mathbb{R}, d\theta)$, $\mathcal{H}_n = P_n \mathcal{H}_1^{\otimes n}$, where P_n is the projection onto S-**symmetric** functions:

$$\Psi_n(\theta_1, \cdots, \theta_n) = S(\theta_{k+1} - \theta_k)\Psi_n(\theta_1, \cdots, \theta_{k+1}, \theta_k, \cdots, \theta_n).$$

- Zamolodchikov-Faddeev algebra: S-symmetrized creation and annihilation operators $z^\dagger(\xi) = Pa^\dagger(\xi)P, z(\xi) = Pa(\xi)P, P = \bigoplus_n P_n$.
- **Wedge-local field**: $\phi(f) = z^\dagger(f^+) + z(J_1 f^-)$,

$$f^\pm(\theta) = \int dx e^{ix\cdot p(\theta)} f(x), \quad p(\theta) = (m\cosh\theta, m\cosh\theta),$$

J_1 is the one-particle CPT operator, $\phi'(g) = J\phi(g_j)J$, $g_j(x) = \overline{g(-x)}$. If $\mathrm{supp}\, f \subset W_{\mathrm{L}}, \mathrm{supp}\, g \subset W_{\mathrm{R}}$, then $[e^{i\phi(f)}, e^{i\phi'(g)}] = 0$.

S-matrix with poles

If S has a pole:

$$[\phi(f), \phi'(g)]\Psi_1(\theta_1) = \\ -\int d\theta \left(f^+(\theta)g^-(\theta)S(\theta_1-\theta) - f^+(\theta+\pi i)g^-(\theta+\pi i)S(\theta_1-\theta+\pi i)\right) \\ \times \Psi_1(\theta_1)$$

obtains the **residue** of S and does not vanish.

- Example (Bullough-Dodd models): poles at $\theta = \frac{\pi i}{3}, \frac{2\pi i}{3}$, residues $-R, R$

$$S_B(\theta) = \frac{\tanh\frac{1}{2}\left(\theta+\frac{2\pi i}{3}\right)}{\tanh\frac{1}{2}\left(\theta-\frac{2\pi i}{3}\right)} \cdot \frac{\tanh\frac{1}{2}\left(\theta+\frac{(B-2)\pi i}{3}\right)}{\tanh\frac{1}{2}\left(\theta-\frac{(B-2)\pi i}{3}\right)} \frac{\tanh\frac{1}{2}\left(\theta-\frac{B\pi i}{3}\right)}{\tanh\frac{1}{2}\left(\theta+\frac{B\pi i}{3}\right)},$$

where $0 < B < 2, B \neq 1$. $S(\theta) = S\left(\theta+\frac{\pi i}{3}\right)S\left(\theta-\frac{\pi i}{3}\right)$.

New wedge-local field?

The bound state operator

S: two-particle S-matrix, P_n: S-symmetrization,
$\mathcal{H} = \bigoplus P_n \mathcal{H}_1^{\otimes n}$, $\mathcal{H}_1 = L^2(\mathbb{R})$,

$$\mathrm{Dom}(\chi_1(f)) := H^2\left(-\tfrac{\pi}{3}, 0\right)$$
$$(\chi_1(f))\xi(\theta) := \sqrt{2\pi|R|} f^+\left(\theta + \tfrac{\pi i}{3}\right)\xi\left(\theta - \tfrac{\pi i}{3}\right),$$

where $H^2(\alpha, \beta)$ is the space of analytic functions in $\mathbb{R} + i(\alpha, \beta)$ such that $\xi(\cdot - \gamma i)$ is uniformly bounded in L^2-norm, $\gamma \in (\alpha, \beta)$, and f^+ is analytic.

$$\chi_n(f) = nP_n\left(\chi_1(f) \otimes I \otimes \cdots \otimes I\right)P_n,$$
$$\chi(f) := \bigoplus \chi_n(f),$$
$$(\chi_1'(g)\xi)(\theta) := (J_1\chi(g_j)J_1)(\theta) = \sqrt{2\pi|R|} g^+\left(\theta - \frac{\pi i}{3}\right)\xi\left(\theta + \frac{\pi i}{3}\right),$$
$$\chi_n'(g) := J_n\chi_n(g_j)J_n, \quad \chi'(g) := J\chi(g_j)J.$$

Wedge-local fields and weak commutativity

New field:

$$\widetilde{\phi}(f) := \phi(f) + \chi(f) = z^\dagger(f^+) + \chi(f) + z(J_1 f^-),$$
$$\widetilde{\phi}'(g) := J\widetilde{\phi}(g_j)J, \ \ \chi'(g) = J\chi(g_j)J.$$

Theorem (Cadamuro-T. arXiv:1502.01313)

For real $f, g, \mathrm{supp}\, f \subset W_{\mathrm{L}}, \mathrm{supp}\, g \subset W_{\mathrm{R}}$, *then*

$$\langle \widetilde{\phi}(f)\Phi, \widetilde{\phi}'(g)\Psi\rangle = \langle \widetilde{\phi}'(g)\Phi, \widetilde{\phi}(f)\Psi\rangle, \ \ \Phi, \Psi \in \mathrm{Dom}(\widetilde{\phi}(f)) \cap \mathrm{Dom}(\widetilde{\phi}'(g)).$$

Proof)

$$\begin{aligned}\langle \chi(f)\Phi_1, \chi'(g)\Psi_1\rangle &= 2\pi i R \int d\theta\, f^+\left(\theta + \frac{\pi i}{3}\right) g^+\left(\theta - \frac{2\pi i}{3}\right) \overline{\Phi(\theta)}\Psi_1(\theta) \\ &= 2\pi i R \int d\theta\, f^+\left(\theta + \frac{\pi i}{3}\right) g^-\left(\theta + \frac{\pi i}{3}\right) \overline{\Phi(\theta)}\Psi_1(\theta)\ldots\end{aligned}$$

Towards modular nuclearity

By assuming the strong commutativity ($\Longrightarrow \Delta^{it} = $ boosts)...

Modular nuclearity: $\mathcal{A}(W_{\mathrm{R}}) \ni A \mapsto \Delta^{\frac{1}{4}} U(a) A\Omega \in \mathcal{H}$, $a = (0, a_1)$.

$$(\Delta^{\frac{1}{4}} U(a)A\Omega)_n(\theta_1, \cdots, \theta_n) =$$
$$e^{-ia_1 \sum_k \sinh(\theta_k - \frac{\pi i}{2})}(A\Omega)_n \left(\theta_1 - \frac{\pi i}{2}, \cdots, \theta_n - \frac{\pi i}{2}\right)$$

$A \in \mathcal{A}(W_{\mathrm{R}}) \Longrightarrow A\Omega \in \mathrm{Dom}(\widetilde{\phi}(f)) \Longrightarrow (A\Omega)_n \in \mathrm{Dom}(\chi_n(f))$, where $\chi_1(f) = u_h^* \Delta^{\frac{1}{6}} u_h$, see later.

$$\langle \chi_n(f)(A\Omega)_n, (A\Omega)_n \rangle = n \|(\mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes u_{h_{\frac{\pi i}{6}}} \Delta^{\frac{1}{12}} \cdot (A\Omega)_n\|^2$$
$$= \langle (\widetilde{\phi}(f) - \phi(f))(A\Omega)_n, (A\Omega)_n \rangle$$
$$= \langle (Af^+ - \phi(f)A\Omega)_n, (A\Omega)_n \rangle \leq 3\sqrt{n+1} \|f^+\| \cdot \|A\Omega\|^2$$

$\Longrightarrow$ Estimate of $(AU(\frac{a}{2})\Omega)_n$ around $\left(\theta_1, \cdots, \theta_{n-1}, \theta_n - \frac{\pi i}{6}\right)$ by $\|A\|$

$\Longrightarrow$ Estimate of $\Delta^{\frac{1}{4}} U(\frac{a}{2}) A\Omega$ (S-symmetry + $\Delta^{\frac{1}{2}} A\Omega = JA^*\Omega$ + Flat tube)

$\Longrightarrow$ nuclearity for minimal distance (Alazzawi-Lechner '15).

Some features of the models

- No Reeh-Schlieder property for polynomials, but for the von Neumann algebra.

  $$\widetilde{\phi}(f)\Omega = f^+ \text{ is not in the domain of } \widetilde{\phi}(f).$$

- No energy bound for $\widetilde{\phi}$ ($\Rightarrow$ no pointlike field?).

  $$\widetilde{\phi}(f) = \phi(f) + \chi(f), \ \ \chi_1(f) = x_f \Delta_1^{\frac{1}{6}}.$$

- Non-temperate polarization-free generator (c.f. Borchers-Buchholz-Schroer '01).

  $$(\chi_1(f)U_1(a)\Psi_1)(\theta) = \sqrt{2\pi|R|}f^+\left(\theta + \frac{\pi i}{3}\right)e^{ia\cdot p\left(\theta - \frac{\pi i}{3}\right)}\Psi_1\left(\theta - \frac{\pi i}{3}\right),$$

  which grows exponentially.

- Bound states?

Summary

- input: two-particle factorizing S-matrix with **poles**
- **new field** $\widetilde{\phi}(f) = \phi(f) + \chi(f)$
- weak commutativity
- modular nuclearity (by assuming strong commutation)
- features of $\widetilde{\phi}(f)$: no polynomial Reeh-Schlieder property, no energy bound, non-temperateness

Open problems

- **strong commutativity**
- non-scalar models (Sine-Gordon, Z(N)-Ising...): weakly commuting fields (with D. Cadamuro), strong commutativity and modular nuclearity more difficult

Towards proof of strong commutativity

If $\chi(f) + \chi'(g)$ is self-adjoint...

- $\chi(f) + \chi'(g) + cN$ is self-adjoint.
- $T(f, g) := \widetilde{\phi}(f) + \widetilde{\phi}'(g) + cN$ is self-adjoint by Kato-Rellich. $(= \chi(f) + \chi'(g) + cN + \phi(f) + \phi'(g))$
- $[T(f, g), \widetilde{\phi}(f)] = [cN, \widetilde{\phi}(f)] = [cN, \phi(f)]$ is small, $\|\widetilde{\phi}(f)\Psi\| \leq \|T(f, g)\Psi\|$.
- use Driessler-Fröhlich theorem (including commutator theorem of Nelson) with $T(f, g)$ **as the reference operator**.

Why is self-adjointness of $\chi(f) + \chi'(g)$ difficult?

- $\chi(f)$ should have different domain of self-adjointness, depending on f.

$$(\chi_1(f))\xi(\theta) := \sqrt{2\pi|R|}f^+\left(\theta + \frac{\pi i}{3}\right)\xi\left(\theta - \frac{\pi i}{3}\right).$$

  ξ might have poles at zeros of f^+.
- From two particles on, the operator is of the form PAP...

The one-particle bound state operator

- $\mathcal{H}_1 = L^2(\mathbb{R})$
- $f(\zeta)$: analytic in $\mathbb{R} + i(-\frac{\pi}{3}, 0)$, $\overline{f(\theta)} = f(\theta - \frac{\pi i}{3})$.
- $\mathrm{Dom}(\chi_1(f)) = H^2(-\frac{\pi}{3}, 0)$: analytic functions in $\mathbb{R} + i(-\frac{\pi}{3}, 0)$
- $(\chi_1(f))\xi(\theta) := f\left(\theta - \frac{\pi i}{3}\right)\xi\left(\theta - \frac{\pi i}{3}\right) \quad \left(= \overline{f(\theta)}\xi\left(\theta - \frac{\pi i}{3}\right)\right)$

Problem

What are self-adjoint extensions of $\chi_1(f)$?

- Write $\chi_1(f) = M_{\overline{f}}\Delta^{\frac{1}{6}}$, $(\Delta^{\frac{1}{6}}\xi)(\theta) = \xi(\theta - \frac{\pi i}{3})$
- Classify extensions: **compute** $\ker(\chi_1(f)^* \pm i)$, $\chi_1(f)^* = \Delta^{\frac{1}{6}} M_f$
- $\mathrm{Dom}(\Delta^{\frac{1}{6}} M_f) =$ functions ξ such that $f(\zeta)\xi(\zeta)$ is analytic in $\mathbb{R} + i(-\frac{\pi}{3}, 0)$
- zeros of $f \implies$ nontrivial function in $\ker(\chi_1(f)^* \pm i)$.

Open problem: self-adjointness of n-particle bound state operators

Why difficult?

Different domain of self-adjointness for different f

$\Longrightarrow$ standard tools (the commutator theorem, the analytic vector theorem, perturbation...) do not apply.

- Take particular f

If $f = g * \bar{g}$, then one can write $f^+(\theta) = \overline{h(\theta)}h(\theta - \frac{\pi i}{3})$, $|h(\theta)| = 1$,

$$\chi_1(h) \subset u_h^* \Delta^{\frac{1}{6}} u_h$$

and the right-hand side is self-adjoint.

- $\widetilde{\phi}(f) = \phi(f) + \chi(f)$ is no longer a linear map.

Open problem: self-adjointness of n-particle bound state operators

Two-particle case

- $P_2(u_h^*\Delta^{\frac{1}{6}}u_h \otimes \mathbb{1})P_2 = u_h^* \otimes u_h^* \cdot P_2(\Delta^{\frac{1}{12}} \otimes \mathbb{1}) \cdot (\Delta^{\frac{1}{12}} \otimes \mathbb{1})P_2 \cdot u_h \otimes u_h$
- $\overline{P_2(\Delta^{\frac{1}{12}} \otimes \mathbb{1})} \cdot (\Delta^{\frac{1}{12}} \otimes \mathbb{1})P_2$ is self-adjoint.
- it is enough to show that

$$\overline{P_2(\Delta^{\frac{1}{12}} \otimes \mathbb{1})} \cdot (\Delta^{\frac{1}{12}} \otimes \mathbb{1})P_2 = P_2(\Delta^{\frac{1}{12}} \otimes \mathbb{1}) \cdot (\Delta^{\frac{1}{12}} \otimes \mathbb{1})P_2$$

Take $\xi_n \to \xi \in \mathrm{Dom}(\overline{P_2(\Delta^{\frac{1}{12}} \otimes \mathbb{1})}), P_2(\Delta^{\frac{1}{12}} \otimes \mathbb{1})\xi_n \to \eta$.

- $(\Delta^{\frac{1}{12}} \otimes \mathbb{1})\xi + (\mathbb{1} \otimes \Delta^{\frac{1}{12}})R_{\frac{\pi i}{3}}\xi$ is square-integrable.
- Need to prove that $(\Delta^{\frac{1}{12}} \otimes \mathbb{1})\xi$ is square-integrable.
- The crossing term $\langle(\Delta^{\frac{1}{12}} \otimes \mathbb{1})\xi, (\mathbb{1} \otimes \Delta^{\frac{1}{12}})R_{\frac{\pi i}{3}}\xi\rangle$ is at worst **negative infinite**. By **Fubini**, positive. $\Longrightarrow (\Delta^{\frac{1}{12}} \otimes \mathbb{1})\xi$ is square-integrable.