Scale and Möbius covariance in two-dimensional Haag-Kastler net

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Scale invariance $\Longrightarrow$ conformal invariance?

- Scale invariant theory: universality tells that there are properties of statistical systems that do not depend on the detail of the system. Universal properties can be studied in the IR scaling limit.
- High-energy properties may reveal simpler structure, such as asymptotic freedom, in the UV scaling limit.

Question

Dilation covariance $\Longrightarrow$ conformal covariance?

- "proofs" by Zamolodchikov-Polchinski in 2d
- ✓ **A proof in the Haag-Kastler framework in 2d of Möbius covariance** (under a variation of modular covariance or strong additivity)
- ✓ **Counterexample** (the dual net of the Brunetti-Guido-Longo construction)

Dilation and Möbius covariance in 2d

Conformal transformations preserve the metric up to a scalar.

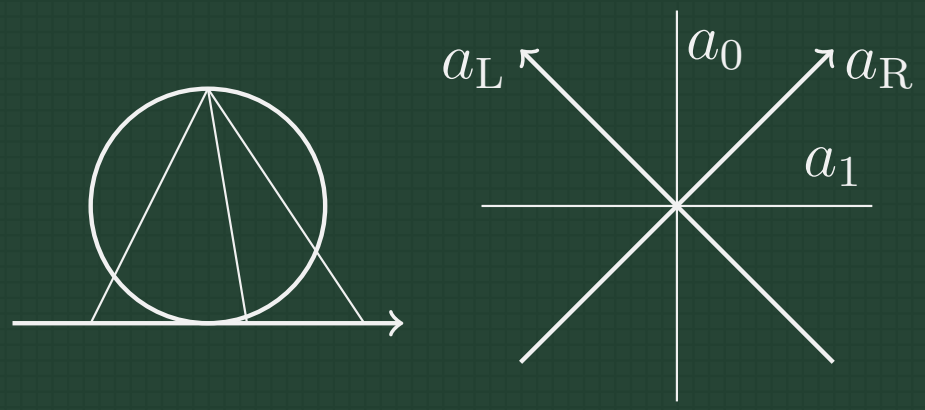
- Poincaré group
- Dilation: $a \mapsto \lambda a$, where $a = (a_0, a_1)$
- Special conformal transformation (acting locally)

The **2d** Lorentz metric splits: $\langle a, a \rangle = (a_0 + a_1)(a_0 - a_1)$.

- The conformal group is the diffeomorphism group of lightrays.
- **Möbius group** corresponds to the higher-dimensional conformal groups generated by Poincaré transformations, dilations and special conformal transformations.
- Möbius group of the lightray is generated by translations, dilations and rotations.
- There are Haag-Kastler nets that are dilation covariant but not Möbius covariant. **When is Möbius covariance guaranteed**?

1d and 2d Möbius covariant nets

Möbius group of S^1 (linear fractional transformations) contains translations, dilations and rotations.



- $\mathcal{A}(I)$: local algebras, $I \subset S^1 = \mathbb{R} \cup \{\infty\}$.
- U: unitary rep of the Möbius group.
- Ω: the "vacuum".

$(\mathcal{A}, U, \Omega)$ is a **Möbius covariant net** on S^1 if it satisfies isotony, locality, Möbius covariance, positivity of energy, vacuum.
In 2d, the Minkowski space admits a local action of $\widetilde{\mathrm{M\ddot{o}b}} \times \widetilde{\mathrm{M\ddot{o}b}}$.
A **2d Möbius covariant net** $(\mathcal{A}, U, \Omega)$ is a $\widetilde{\mathrm{M\ddot{o}b}} \times \widetilde{\mathrm{M\ddot{o}b}}$-covariant net of observables associated with open regions in $\mathbb{R}^2$.

Zamolodchikov-Polchinski "theorem"

(1) unitarity

(2) Poincaré covariance

(3) (unbroken) dilation covariance

(4) discrete spectrum in scaling dimension: each of the pointlike observables $O_k(x)$ has a definite "scaling dimension" $\mathrm{Ad}\, U(\delta(t) \times \delta(t))(O_k(x)) = t^{\Delta_k} O_k(tx)$.

(5) existence of scale current: there are a stress-energy tensor $T_{\mu\nu}(x)$ and a current $J_\mu(x)$ such that $\int d^{d-1}x\, [x^\rho T_{\rho 0}(x) - J_0(x)]$ is the generator of dilations.

"Proof": (1) one can replace $T_{\mu\nu}$ by one satisfying the canonical scaling dimension: $\mathrm{Ad}\, U(\delta(t)) T_{\mu\nu}(x) = t^2 T_{\mu\nu}(tx)$. (2) By "$c$-theorem", T is traceless and satisfies the algebra of vector fields. **((3) Smear T by $x^2 + 1$ to obtain rotations)**

Non-uniqueness of stress-energy tensor

$\mathrm{U}(1)$-current algebra (derivative of the massless free field) on S^1:

$$[J(f), J(g)] = i \int f' g.$$

The Fourier components $J_n = \int e^{-in\theta} J(e^{i\theta}) d\theta$ satisfy $[J_m, J_n] = m\delta_{m,-n}$.
Vacuum representation: $J_n \Omega = 0$ for $n \geq 0$.
Stress-energy tensor: $L_n = \sum_k : J_n J_{k-n} :$, $T(f) = \sum_n \hat{f}_n L_n$, where $\hat{f}_n = \int e^{-in\theta} J(e^{i\theta}) d\theta$.
New stress-energy tensor $T_\kappa(f) = T(f) + \kappa J'(f)$. T_κ generates the same translations and dilations, but does not extend to S^1 on the same Hilbert space (Buchholz and Schulz-Mirbach '90).

- Stress-energy tensor T (local, generating translations and dilations) is not unique.
- T must be smeared by $x^2 + 1$.

Möbius covariant net on S^1 and half-sided modular inclusion

- $\mathcal{N} \subset \mathcal{M}$: von Neumann algebras,
- Ω: cyclic and separating for $\mathcal{M}, \mathcal{N}$
- $\Delta_{\mathcal{M}}^{it}$: the modular group of $\mathcal{M}$ with respect to Ω:
  $S_{\mathcal{M}} : x\Omega \longmapsto x^*\Omega, S_{\mathcal{M}} = J_{\mathcal{M}}\Delta_{\mathcal{M}}^{\frac{1}{2}}$.

Theorem (Wiesbrock '93, Araki-Zsido '05)

If $\mathcal{N} \subset \mathcal{M}$ is a **half-sided modular inclusion** (HSMI):
$\mathrm{Ad}\,\Delta_{\mathcal{M}}^{it}(\mathcal{N}) \subset \mathcal{N}$ for $t \leq 0$, then $\Delta_{\mathcal{M}}^{it_1}, \Delta_{\mathcal{N}}^{it_2}$ generate $ax + b$ group.

Theorem (Guido-Longo-Wiesbrock '98)

There is a **one-to-one correspondence** between

- standard (Ω is cyclic for $\mathcal{M} \cap \mathcal{N}'$) HSMIs
- Möbius covariant nets on S^1 which are "strongly additive"

$\Delta_{\mathcal{M}}^{it_1}, \Delta_{\mathcal{M}\cap\mathcal{N}'}^{it_2}, \Delta_{\mathcal{N}}^{it_3}$ generate a representation of $\widetilde{\mathrm{Möb}}$.

Enhancement of symmetry to $\widetilde{\mathrm{M\ddot{o}b}} \times \widetilde{\mathrm{M\ddot{o}b}}$

Let $(\mathcal{A}, U, \Omega)$ be a 2d Haag-Kastler net satisfying isotony, locality, Poincaré-dilation covariance, positivity of the energy, vacuum, the **Bisognano-Wichmann property** for wedges and light cones ($\Delta^{it}_{\mathcal{A}(W_{\mathrm{L}})}$ are Lorents boosts, $\Delta^{it}_{\mathcal{A}(V_+)}$ are dilations) and one of the following:

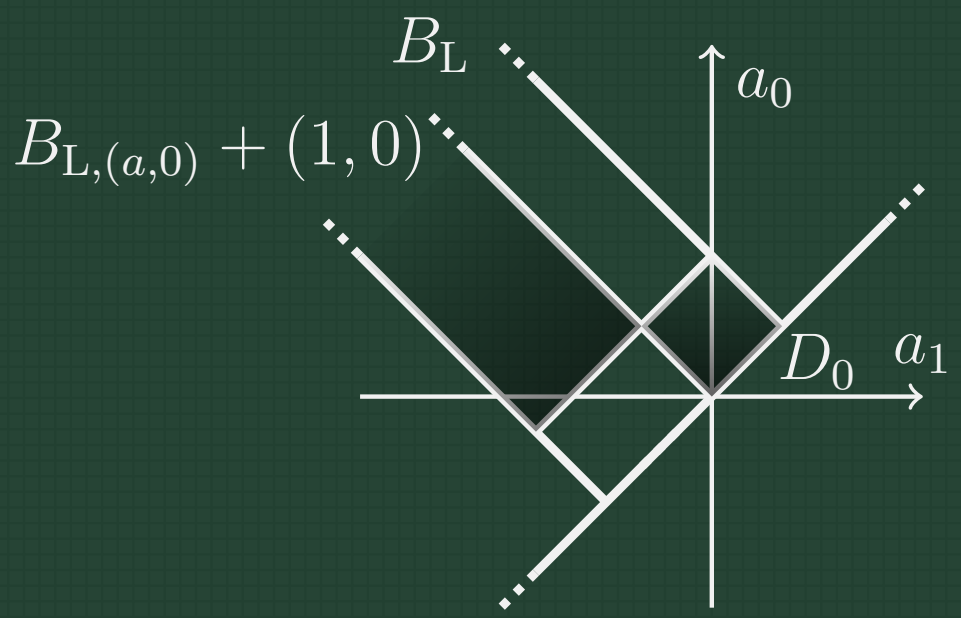
(a) **Modular covariance for** B_{L}:
$\mathrm{Ad}\,\Delta^{it}_{B_{\mathrm{L}}}(\mathcal{A}(D_0)) = \mathrm{Ad}\,U(\delta(-2\pi t) \times \iota)(\mathcal{A}(D_0))$. In particular, $\mathcal{A}(D_0) \subset \mathcal{A}(B_{\mathrm{L}})$ is a HSMI.

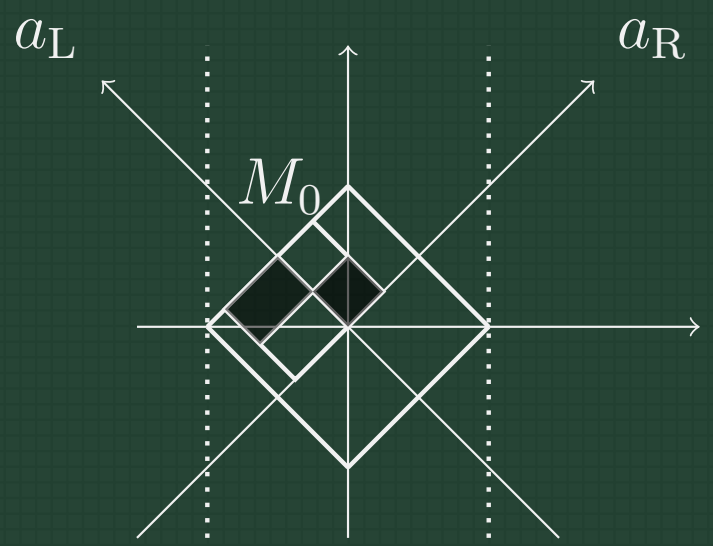
(b) **strong additivity in cylinder:** Let $a < b < c$ and $d > 0$ in $\mathbb{R}$ and $B_{L,(a,c)}$ and $B_{L,(a,b)} + (d, 0)$ two half-band with a common edge then

$$\mathcal{A}(D_{(0,d)(b,c)}) = \mathcal{A}(B_{L,(a,c)}) \cap \mathcal{A}(B_{L,(a,b)} + (d, 0))'.$$

Then $(\mathcal{A}, U, \Omega)$ is $\widetilde{\mathrm{M\ddot{o}b}} \times \widetilde{\mathrm{M\ddot{o}b}}$-covariant.

Half-bands and double cone

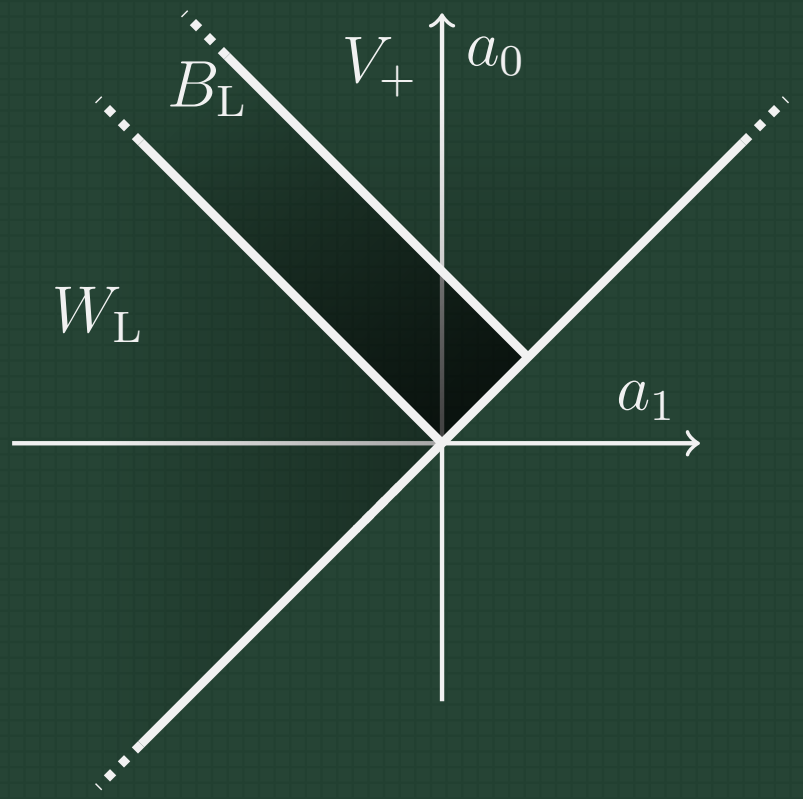


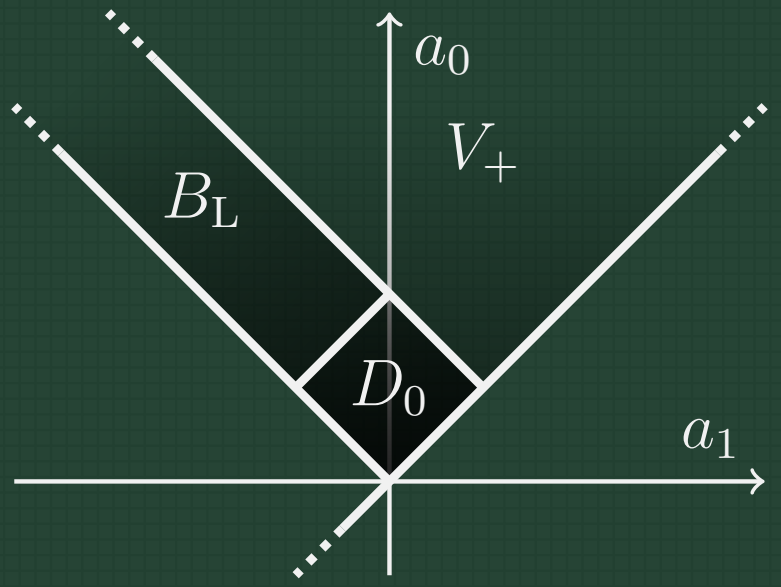


Proof of enhancement

- (b) $\Longrightarrow$ (a). We only need (a) (and (a) is a necessary condition).
- $\mathcal{A}(B_{\mathrm{L}}) \subset \mathcal{A}(W_{\mathrm{L}} + (1,1))$ and $\mathcal{A}(B_{\mathrm{L}}) \subset \mathcal{A}(V_+)$ are HSMI. By Bisognano-Wichmann property for W_{L} and W_+, their modular groups satisfy the right commutation relations $\Longrightarrow$ a representation of $\widetilde{\mathrm{M\ddot{o}b}}$. With "dilation $\delta_{(0,1)}$ associated with $(0,1)$", $U_{\mathrm{R}}(\iota \times \delta_{(0,1)}(2\pi t)) := \Delta_{B_{\mathrm{L}}}^{it} U(\delta(2\pi t) \times \iota)$.
- Do a similar construction U_{L} with $W_{\mathrm{R}}, B_{\mathrm{R}}$.
- In $\mathcal{A}(D_0) \subset \mathcal{A}(B_{\mathrm{L}}) \subset \mathcal{A}(V_+)$ there are three HSMIs by (a). $[\Delta_{D_0}^{is} U_{\mathrm{R}}(\iota \times \delta_{(0,1)}(2\pi s)), U(\iota \times \delta(2\pi t))] = 0$.
- $\Delta_{D_0}^{it} = U_{\mathrm{R}}(\iota \times \delta_{(0,1)}(-2\pi t)) U_{\mathrm{L}}(\delta_{(0,1)}(-2\pi t) \times \iota)$ by dilating the relations above.
- U_{R} and U_{L} commute $\Longrightarrow \widetilde{\mathrm{M\ddot{o}b}} \times \widetilde{\mathrm{M\ddot{o}b}}$.

Wedge, half-band and double cone





Counterexample

- Brunetti-Guido-Longo construction: given a positive-energy representation of $\widetilde{\mathrm{M\ddot{o}b}} \times \widetilde{\mathrm{M\ddot{o}b}} \rtimes \mathbb{Z}_2$, there is a second quantized net.
- Take the tensor product of the representations of $\widetilde{\mathrm{M\ddot{o}b}}$ with lowest weight 1. The resulting net $\mathcal{A}$ is $\widetilde{\mathrm{M\ddot{o}b}} \times \widetilde{\mathrm{M\ddot{o}b}} \rtimes \mathbb{Z}_2$-covariant **without chiral components**.
- Take its dual net: $\hat{\mathcal{A}}(D) := \mathcal{A}(W_{\mathrm{L}} + a) \cap \mathcal{A}(W_{\mathrm{R}} + b)$. This coincides with the dual net of a generalized free field with continuous measure, hence $\hat{\mathcal{A}}(V_+) = \mathcal{B}(\mathcal{H})$.
  - $\hat{A}$ remains dilation covariant.
  - No $\widetilde{\mathrm{M\ddot{o}b}} \times \widetilde{\mathrm{M\ddot{o}b}}$-covariance.
- $\widetilde{\mathrm{M\ddot{o}b}} \times \widetilde{\mathrm{M\ddot{o}b}}$-**covariance, the split property and various nuclearity properties are lost by going to the dual net**.

Summary, outlook and conclusion

- ✓ In 2d, dilation covariance can be enhanced to Möbius covariance under the Bisognano-Wichmann property for wedges and light cones, modular covariance for bands or strong additivity on cylinder.

- Dilation $\Longrightarrow$ conformal in higher dimensions?
- The free Maxwell theory in d $<>$ 4 is not conformal (El-Showk-Nakayama-Rychkov)

- Can strong additivity in cylinder be removed?
- When does conformal (diffeomorphism) covariance hold?
- Is the dual net of $\mathrm{Vir}_c, c > 1$ conformal?