Scale and Möbius covariance in two-dimensional Haag-Kastler net

Yoh Tanimoto (University of Rome “Tor Vergata”)  

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Scale invariance $\Longrightarrow$ conformal invariance?

- Scale invariant theory: universality tells that there are properties of statistical systems that do not depend on the detail of the system. Universal properties can be studied in the IR scaling limit.
- High-energy properties may reveal simpler structure, such as asymptotic freedom, in the UV scaling limit.

Question

Dilation covariance $\Longrightarrow$ conformal covariance?

- “proofs” by Zamolodchikov-Polchinski in 2d
- A proof in the Haag-Kastler framework in 2d of Möbius covariance (under a variation of modular covariance or strong additivity)
- Counterexample (the dual net of the Brunetti-Guido-Longo construction)
Dilation and Möbius covariance in 2d

**Conformal** transformations preserve the metric up to a scalar.
- Poincaré group
- Dilation: $a \mapsto \lambda a$, where $a = (a_0, a_1)$
- Special conformal transformation (acting locally)

The 2d Lorentz metric splits: $\langle a, a \rangle = (a_0 + a_1)(a_0 - a_1)$.
- The conformal group is the diffeomorphism group of light rays.
- **Möbius group** corresponds to the higher-dimensional conformal groups generated by Poincaré transformations, dilations and special conformal transformations.
- Möbius group of the light ray is generated by translations, dilations and rotations.
- There are Haag-Kastler nets that are dilation covariant but not Möbius covariant. When is Möbius covariance guaranteed?
1d and 2d Möbius covariant nets

Möbius group of $S^1$ (linear fractional transformations) contains translations, dilations and rotations.

- $\mathcal{A}(I)$: local algebras, $I \subset S^1 = \mathbb{R} \cup \{\infty\}$.
- $U$: unitary rep of the Möbius group.
- $\Omega$: the “vacuum”.

$(\mathcal{A}, U, \Omega)$ is a Möbius covariant net on $S^1$ if it satisfies isotony, locality, Möbius covariance, positivity of energy, vacuum.

In 2d, the Minkowski space admits a local action of $\tilde{\text{Möb}} \times \tilde{\text{Möb}}$.

A 2d Möbius covariant net $(\mathcal{A}, U, \Omega)$ is a $\tilde{\text{Möb}} \times \tilde{\text{Möb}}$-covariant net of observables associated with open regions in $\mathbb{R}^2$. 
Zamolodchikov-Polchinski “theorem”

(1) unitarity
(2) Poincaré covariance
(3) (unbroken) dilation covariance
(4) discrete spectrum in scaling dimension: each of the pointlike observables $O_k(x)$ has a definite “scaling dimension”
$$\text{Ad} \ U(\delta(t) \times \delta(t))(O_k(x)) = t^{\Delta_k}O_k(tx).$$

(5) existence of scale current: there are a stress-energy tensor $T_{\mu\nu}(x)$ and a current $J_\mu(x)$ such that
$$\int d^{d-1}x \left[ x^\rho T_{\rho 0}(x) - J_0(x) \right]$$
is the generator of dilations.

“Proof”: (1) one can replace $T_{\mu\nu}$ by one satisfying the canonical scaling dimension: $\text{Ad} \ U(\delta(t))T_{\mu\nu}(x) = t^2T_{\mu\nu}(tx)$. (2) By "c-theorem", $T$ is traceless and satisfies the algebra of vector fields. (3) Smear $T$ by $x^2 + 1$ to obtain rotations)
Non-uniqueness of stress-energy tensor

$\text{U}(1)$-current algebra (derivative of the massless free field) on $S^1$:

$$[J(f), J(g)] = i \int f' g.$$

The Fourier components $J_n = \int e^{-in\theta} J(e^{i\theta}) d\theta$ satisfy

$$[J_m, J_n] = m \delta_{m,-n}.$$

Vacuum representation: $J_n \Omega = 0$ for $n \geq 0$.

Stress-energy tensor: $L_n = \sum_k : J_n J_{k-n} :$, $T(f) = \sum_n \hat{f}_n L_n$, where

$$\hat{f}_n = \int e^{-in\theta} J(e^{i\theta}) d\theta.$$

New stress-energy tensor $T_\kappa(f) = T(f) + \kappa J'(f)$. $T_\kappa$ generates the same translations and dilations, but does not extend to $S^1$ on the same Hilbert space (Buchholz and Schulz-Mirbach ‘90).

- Stress-energy tensor $T$ (local, generating translations and dilations) is not unique.
- $T$ must be smeared by $x^2 + 1$. 
Möbius covariant net on $S^1$ and half-sided modular inclusion

- $\mathcal{N} \subset \mathcal{M}$: von Neumann algebras,
- $\Omega$: cyclic and separating for $\mathcal{M}, \mathcal{N}$
- $\Delta^i_{\mathcal{M}}$: the modular group of $\mathcal{M}$ with respect to $\Omega$:
  $$S_{\mathcal{M}} : x\Omega \mapsto x^*\Omega, S_{\mathcal{M}} = J_{\mathcal{M}} \Delta^\frac{1}{2}_{\mathcal{M}}.$$

**Theorem (Wiesbrock ‘93, Araki-Zsido ‘05)**
If $\mathcal{N} \subset \mathcal{M}$ is a **half-sided modular inclusion (HSMI)**: $\text{Ad} \Delta^i_{\mathcal{M}}(\mathcal{N}) \subset \mathcal{N}$ for $t \leq 0$, then $\Delta^i_{\mathcal{M}}, \Delta^i_{\mathcal{N}}$ generate $ax + b$ group.

**Theorem (Guido-Longo-Wiesbrock ‘98)**
There is a **one-to-one correspondence** between
- standard ($\Omega$ is cyclic for $\mathcal{M} \cap \mathcal{N}^\prime$) HSMIs
- Möbius covariant nets on $S^1$ which are “strongly additive”
  $\Delta^i_{\mathcal{M}}, \Delta^i_{\mathcal{M} \cap \mathcal{N}^\prime}, \Delta^i_{\mathcal{N}}$ generate a representation of $\widehat{\text{M"ob}}$. 
Enhancement of symmetry to \( \tilde{\mathbb{Möb}} \times \tilde{\mathbb{Möb}} \)

Let \((\mathcal{A}, U, \Omega)\) be a 2d Haag-Kastler net satisfying isotony, locality, Poincaré-dilation covariance, positivity of the energy, vacuum, the Bisognano-Wichmann property for wedges and light cones (\(\Delta_{it}^{it}(W_L)\) are Lorentz boosts, \(\Delta_{it}^{it}(V_+)\) are dilations) and one of the following:

(a) **Modular covariance for** \(B_L\):
\[
\text{Ad} \Delta_{it}^{it}(B_L)(\mathcal{A}(D_0)) = \text{Ad} U(\delta(-2\pi t) \times t)(\mathcal{A}(D_0)).
\]
In particular, \(\mathcal{A}(D_0) \subset \mathcal{A}(B_L)\) is a HSML.

(b) **strong additivity in cylinder:** Let \(a < b < c\) and \(d > 0\) in \(\mathbb{R}\) and \(B_{L,(a,c)}\) and \(B_{L,(a,b)} + (d,0)\) two half-band with a common edge then
\[
\mathcal{A}(D_{(0,d)(b,c)}) = \mathcal{A}(B_{L,(a,c)}) \cap \mathcal{A}(B_{L,(a,b)} + (d,0))'.
\]

Then \((\mathcal{A}, U, \Omega)\) is \(\tilde{\mathbb{Möb}} \times \tilde{\mathbb{Möb}}\)-covariant.
Half-bands and double cone

\[ B_{L,(a,0)} + (1, 0) \]
Proof of enhancement

- (b) \implies (a). We only need (a) (and (a) is a necessary condition).
- \( \mathcal{A}(B_L) \subset \mathcal{A}(W_L + (1, 1)) \) and \( \mathcal{A}(B_L) \subset \mathcal{A}(V_+) \) are HSMI. By Bisognano-Wichmann property for \( W_L \) and \( W_+ \), their modular groups satisfy the right commutation relations \implies \text{a representation of } \widetilde{\text{M"ob}}. \) With “dilation \( \delta_{(0,1)} \) associated with \( (0, 1) \)”, \( U_R(\iota \times \delta_{(0,1)}(2\pi t)) := \Delta^it_{B_L} U(\delta(2\pi t) \times \iota) \).
- Do a similar construction \( U_L \) with \( W_R, B_R \).
- In \( \mathcal{A}(D_0) \subset \mathcal{A}(B_L) \subset \mathcal{A}(V_+) \) there are three HSMIIs by (a). 
  \[ [\Delta^i_{D_0} U_L(\iota \times \delta_{(0,1)}(2\pi s)), U(\iota \times \delta(2\pi t))] = 0. \]
- \( \Delta^it_{D_0} = U_R(\iota \times \delta_{(0,1)}(-2\pi t)) U_L(\delta_{(0,1)}(-2\pi t) \times \iota) \) by dilating the relations above.
- \( U_R \) and \( U_L \) commute \implies \( \widetilde{\text{M"ob}} \times \widetilde{\text{M"ob}}. \)
Wedge, half-band and double cone
Counterexample

- Brunetti-Guido-Longo construction: given a positive-energy representation of $\tilde{\text{M"ob}} \times \tilde{\text{M"ob}} \rtimes \mathbb{Z}_2$, there is a second quantized net.

- Take the tensor product of the representations of $\tilde{\text{M"ob}}$ with lowest weight 1. The resulting net $\mathcal{A}$ is $\tilde{\text{M"ob}} \times \tilde{\text{M"ob}} \rtimes \mathbb{Z}_2$-covariant without chiral components.

- Take its dual net: $\hat{\mathcal{A}}(D) := \mathcal{A}(W_L + a) \cap \mathcal{A}(W_R + b)$. This coincides with the dual net of a generalized free field with continuous measure, hence $\hat{\mathcal{A}}(V_+) = \mathcal{B}(\mathcal{H})$.
  - $\hat{\mathcal{A}}$ remains dilation covariant.
  - No $\tilde{\text{M"ob}} \times \tilde{\text{M"ob}}$-covariance.

- $\tilde{\text{M"ob}} \times \tilde{\text{M"ob}}$-covariance, the split property and various nuclearity properties are lost by going to the dual net.
Summary, outlook and conclusion

✓ In 2d, dilation covariance can be enhanced to Möbius covariance under the Bisognano-Wichmann property for wedges and light cones, modular covariance for bands or strong additivity on cylinder.

○ Dilation $\mapsto$ conformal in higher dimensions?
○ The free Maxwell theory in $d < 4$ is not conformal (El-Showk-Nakayama-Rychkov)

○ Can strong additivity in cylinder be removed?
○ When does conformal (diffeomorphism) covariance hold?
○ Is the dual net of $Vir_c, c > 1$ conformal?