Operator-algebraic construction of integrable QFT and CFT

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Constructing QFT

What is quantum field theory? cf.

- classical field theory: PDE
- quantum mechanics: Hilbert space, unitary evolution (+ measurement theory)

What is a quantum field?

- definition?
- example?
- integrability?

Wightman axioms

Quantum field ϕ

- ϕ: operator valued distribution ($\phi(f)$ is an (unbouded) operator on a Hibert space)
- satisfies locality, Poincaré covariance, positivity of energy, existence of vacuum

n-point functions

- consider $F_n(x_1, \cdots x_n) := \langle \Omega, \phi(x_1) \cdots \phi(x_n) \Omega \rangle$
- $\{F_n\}$ satisfy locality, Poincaré invariance, analyticity
- Reconstruction theorem: one can recover ϕ from $\{F_n\}$.

Particle spectrum and S-matrix can be constructed from ϕ.
Examples: (d=2) $P(\phi)$, exponential interaction (sine/sinh-Gordon), Yukawa, Gross-Neveu, Thirring, local gauge theories, CFT...
(d=3) ϕ^4, abelian gauge theories...

Integrable QFT

(Pertubatively check that there is no particle production, e.g. sine-Gordon)

Factorizing S-matrix

- Conjecture particle contents from Lagrangian
- Determine the symmetry of the model and fusion relations
- Conjecture the S-matrix

Form factor programme (Babujian, Karowski, Smirnov...)

- Factorizing S-matrix is given.
- Form factors ${}^{\mathrm{out}}\langle q_1, \cdots, q_m | O(x) | p_1, \cdots, p_n \rangle^{\mathrm{in}}$ are conjectured.
- n-point functions $\langle \Omega, O(x)O(0)\Omega\rangle = \sum_n \int dp_1 \cdots dp_n \, \langle \Omega, O(x) | p_1, \cdots, p_n\rangle^{\mathrm{in}\ \mathrm{in}}\langle p_1, \cdots, p_n | O(0)\Omega\rangle$ should be computed.
- Convergence? Locality in e.g. the sine-Gordon model?

Alternative approach to integrable QFT

Haag-Kastler axioms

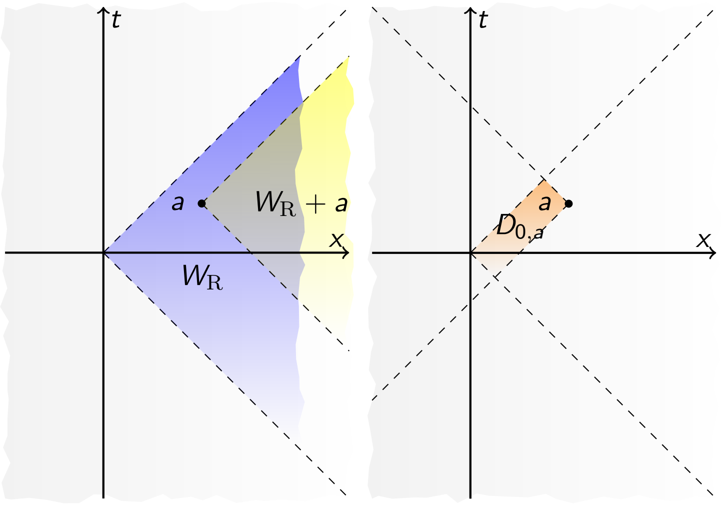
- Concerned with algebras of observables (**bounded operators**) $\mathcal{A}(O)$ in spacetime regions O.
- **Isotony**, locality, Poincaré covariance, positivity of energy, existence of vacuum
- ϕ: quantum (Wightman) field $\Longrightarrow \mathcal{A}(O) := \overline{\{e^{i\phi(f)} : \mathrm{supp}\, f \subset O\}}^{\mathrm{vN}}$

Isotony: $O_1 \subset O_2 \Longrightarrow \mathcal{A}(O_1) \subset \mathcal{A}(O_2)$ means that **larger** regions contain **more** observables, also **simpler** ones. **Wedge**: $W_{\mathrm{R}} := \{(t,x) : x > |t|\}$.

Why wedges?

- Form factors ${}^{\mathrm{out}}\langle q_1, \cdots, q_m | O(x)\Omega\rangle^{\mathrm{in}}$ of interacting pointlike fields $O(x)$ are complicated.
- For a large region W_{R}, there might be simpler observables.

Standard wedge and double cone



Constructing Haag-Kastler net from wedge observables

Prepare

- Hilbert space $\mathcal{H}$
- Representation U of the Poincaré group
- Vacuum vector Ω
- A (von Neumann) algebra $\mathcal{M}$ (of observables in W_R)

They should satisfy $U(a,\lambda)\mathcal{M}U(a,\lambda) \subset \mathcal{M}$ if $a \in W_\mathrm{R}$, and $\overline{\mathcal{M}\Omega} = \overline{\mathcal{M}'\Omega} = \mathcal{H}$ (the Reeh-Schlieder property), where $\mathcal{M}' = \{x \in \mathcal{B}(\mathcal{H}) : [x,y] = 0 \text{ for } y \in \mathcal{M}\}$.

Wedge to double cones

- For a double cone $D_{a,b} = (W_\mathrm{R} + a) \cap (W_\mathrm{L} + b)$,
  $\mathcal{A}(D_{a,b}) = U(a,0)\mathcal{M}U(a,0)^* \cap U(b,0)\mathcal{M}'U(b,0)^*$

If $\mathcal{A}(D_{a,b})$ is sufficiently large, $(\mathcal{A}, U, \Omega)$ is a Haag-Kastler net.

Example: massive free field

- $\mathcal{H}_1 = L^2(\mathbb{R}, d\theta), \mathcal{H} = \bigoplus_n P_n \mathcal{H}_1^{\otimes n}$ (symmetric Fock space), where P_n is the symmetrizer. $z^\dagger, z$: creation/annihilation operators
- $(U_1(a,\lambda)\Psi_1)(\theta) = e^{ia\cdot(m\cosh\theta, m\sinh\theta)}\Psi(\theta-\lambda)$, $U = \Gamma(U_1)$ (second quantization)
- Ω: the Fock vacuum
- free field ϕ: for a test function f, $\phi(f) = z^\dagger(f^+) + z(f^+)$, where $f^+(\theta) = \int d^2a\, e^{ia\cdot(m\cosh\theta, m\sinh\theta)} f(a)$ and

$$\mathcal{M} = \overline{\{e^{i\phi(f)} : \mathrm{supp}\, f \subset W_\mathrm{R}\}}^{\mathrm{vN}}$$

The free field net

We get the usual free field net:

$$\begin{aligned}\mathcal{A}(D_{a,b}) &:= U(a,0)\mathcal{M}U(a,0)^* \cap U(b,0)\mathcal{M}'U(b,0)^* \\ &= \overline{\{e^{i\phi(f)} : \mathrm{supp}\, f \subset D_{a,b}\}}^{\mathrm{vN}}\end{aligned}$$

Example: twisting the free field

Let ϕ be the massive **complex** free field, with the charge operator Q, and $\mathcal{M} = \overline{\{e^{i\phi(f)} : \mathrm{supp}\, f \subset W_{\mathrm{R}}\}}^{\mathrm{vN}}$ be the wedge-algebra, take the usual U, Ω on $\mathcal{H}_{\mathrm{c}}$. **Double** the Hilbert space: $\mathcal{H}_{\mathrm{c}} \otimes \mathcal{H}_{\mathrm{c}}$.

Theorem (T. arXiv:1301.6090)

- $\tilde{\mathcal{H}}_{\mathrm{c}} = \mathcal{H}_{\mathrm{c}} \otimes \mathcal{H}_{\mathrm{c}}$
- $\tilde{U} = U \otimes U$
- $\tilde{\Omega} = \Omega \otimes \Omega$
- $\tilde{\mathcal{M}}_t = (\mathcal{M} \otimes \mathbb{C}\mathbb{1}) \bigvee e^{itQ\otimes Q}(\mathbb{C}\mathbb{1} \otimes \mathcal{M})e^{-itQ\otimes Q}$

give an interacting Haag-Kastler net for $t \notin \mathbb{R}/2\pi\mathbb{Z}$. The S-matrix is $e^{itQ\otimes Q}$, very similar to that of the Federbush model.

Proof of wedge-localization: $x \mapsto e^{sQ}xe^{-sQ}$ is an **automorphism** of $\mathcal{M}$. Differently from $\mathcal{M} = \mathcal{A}(W_{\mathrm{R}})$, observables in $\mathcal{A}(D_{b,a})$ are **not explicitly known**.

Example: twisting the free field

A variation: Let ϕ be the massive **real** free field, φ an inner symmetric function (almost two-particle scattering function), $\mathcal{M}$ the wedge-algebra. One can construct an operator $\tilde{R}_\varphi$:

Theorem (T. arXiv:1301.6090, Alazzawi-Lechner arXiv:1608.02359)

- $\tilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{H}$
- $\tilde{U} = U \otimes U$
- $\tilde{\Omega} = \Omega \otimes \Omega$
- $\tilde{\mathcal{M}}_\varphi = (\mathcal{M} \otimes \mathbb{C}\mathbb{1}) \bigvee \tilde{R}_\varphi(\mathbb{C}\mathbb{1} \otimes \mathcal{M})\tilde{R}_\varphi^*$

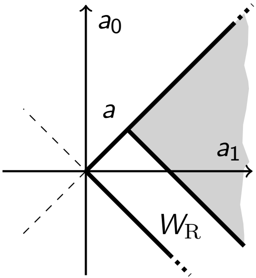
give an interacting Haag-Kastler net, and the S-matrix is $\tilde{R}_\varphi$.

Proof of wedge-localization: $x \mapsto \Gamma(\varphi(P_1))x\Gamma(\varphi(P_1))^*$ is an **endomorphism** of $\mathcal{M}$, P_1 the generator of lightlike translation. $\tilde{R}_\varphi$ is a **diagonal** factorizing S-matrix, interaction only between two different species of particles.

Relations to CFT?

ϕ: massive **free field**, $\mathcal{M}$: the wedge algebra.

- One obtaines a 1d CFT (the Heisenberg algebra) by restricting to the lightray. Net of observables on intervals I on the lightray $\mathbb{R}$.
- Negative lightlike translations implement **Longo-Witten endomorphisms**: $V(s)\mathcal{M}V(s)^* \subset \mathcal{M}$ and $V(s)$ commutes with the positive lightlike translations.
- e^{isQ} and $\Gamma(\varphi(P_1))$ implement Longo-Witten endomorphisms
- $\Rightarrow$ massive integrable model with $\tilde{R}_\varphi$ on $\mathcal{H} \otimes \mathcal{H}$.

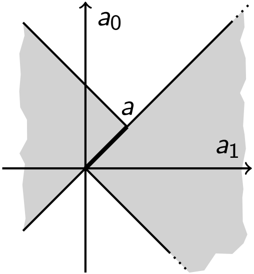


CFT + nice Longo-Witten endomorphisms (commuting with both lightlike translations) $\Rightarrow$ integrable CFT. **Integrable perturbation of CFT**?

Relations to CFT?

Take an **interacting** QFT $(\mathcal{A}, U, \Omega)$, the wedge algebra $\mathcal{M} = \mathcal{A}(W_{\mathrm{R}})$ and lightlike translation.
Question: How large is the lightlike intersection $\mathcal{M} \cap U(a)\mathcal{M}' U(a)^*$?
If it is nontrivial, one obtains a chiral component of a CFT (Guido-Longo-Wiesbrock '98).
$\Rightarrow$ **Scaling limit** of CFT? (c.f. Bostelmann -Lechner-Morsella '11)
(Non-)examples:

- For the twisting $(\tilde{M}_t, \tilde{U}, \tilde{\Omega})$ of the massive complex field, it is too small (T. '14).
- The intersection can be trivial in general (Longo-T.-Ueda '17).



Conjecture:

- $\mathrm{SU}(2)$-symmetric Thirring $\Leftrightarrow$ $\mathrm{SU}(2)$-current algebra (WZW model)

Example: analytic factorizing S-matrix

- **analytic** two-parrticle S-matrix (e.g. the **sinh-Gordon** model) $S : \mathbb{R} + i(0,\pi) \to \mathbb{C}$,
$$\overline{S(\theta)} = S(\theta)^{-1} = S(-\theta) = S(\theta + \pi i), \quad \theta \in \mathbb{R}.$$
- S-**symmetric Fock space**: $\mathcal{H}_1 = L^2(\mathbb{R}, d\theta)$, $\mathcal{H}_n = P_n \mathcal{H}_1^{\otimes n}$, where P_n is the projection onto S-symmetric functions: $\Psi_n(\theta_1, \cdots, \theta_n) = S(\theta_{k+1} - \theta_k)\Psi_n(\theta_1, \cdots, \theta_{k+1}, \theta_k, \cdots, \theta_n)$.
- Zamolodchikov-Faddeev algebra: S-symmetrized creation and annihilation operators $z^\dagger(\xi) = Pa^\dagger(\xi)P, z(\xi) = Pa(\xi)P, P = \bigoplus_n P_n$.
- **Wedge-local field**: $\phi(f) = z^\dagger(f^+) + z(f^+)$.

Wedge-localization (Lechner '03)

If $\mathcal{M} = \overline{\{e^{i\phi(f)} : \mathrm{supp}\, f \subset W_{\mathrm{R}}\}}^{\mathrm{vN}}$, then $\overline{\mathcal{M}\Omega} = \overline{\mathcal{M}'\Omega} = \mathcal{H}$.

Local observables

- $\mathcal{H}$: S-symmetric Fock space
- $U = \Gamma(U_1)$: second quantization
- Ω: Fock vacuum
- $\mathcal{M} = \overline{\{e^{i\phi(f)} : \mathrm{supp}\, f \subset W_{\mathrm{R}}\}}^{\mathrm{vN}}$

Question: are there sufficiently many observable in $\mathcal{A}(D_{a,b}) = \mathcal{A}(W_{\mathrm{R}} + a) \cap \mathcal{A}(W_{\mathrm{L}} + b)$?

Theorem (Lechner '08)

If S is analytic, satisfies a regularity condition (not too many CDD factors) and and $S(0) = -1$, there are local observables in $\mathcal{A}(D_{a,b})$ for $b - a$ sufficiently large. The **Haag-Kastler net** $(\mathcal{A}, U, \Omega)$ has S as the two-particle S-matrix.

This works also with diagonal S-matrices (Alazzawi-Lechner '17).

Example: scalar S-matrices with poles (bound states)

If S has a pole (e.g. the **Bullough-Dodd** model), $\phi(f) = z^\dagger(f^+) + z(f^+)$ is **no longer wedge-local**.

S: scalar, poles at $\theta = \frac{\pi i}{3}, \frac{2\pi i}{3}$, $S(\theta) = S\left(\theta + \frac{\pi i}{3}\right) S\left(\theta - \frac{\pi i}{3}\right)$

P_n: S-symmetrization, $\mathcal{H} = \bigoplus P_n \mathcal{H}_1^{\otimes n}$, $\mathcal{H}_1 = L^2(\mathbb{R})$,

$$(\chi_1(f))\xi(\theta) := \sqrt{2\pi|R|} f^+\left(\theta + \frac{\pi i}{3}\right) \xi\left(\theta - \frac{\pi i}{3}\right),$$

$$\chi_n(f) := nP_n\left(\chi_1(f) \otimes I \otimes \cdots \otimes I\right) P_n, \quad \chi(f) := \bigoplus \chi_n(f).$$

Theorem (Cadamuro-T. arXiv:1502.01313, Bostelmann-Cadamuro-T., in preparation)

Set $\tilde\phi(f) := \phi(f) + \chi(f)$, $\mathcal{M} = \overline{\{e^{i\tilde\phi(f)} : f = h^2, \operatorname{supp} h \subset W_\mathrm{R}\}}^{\mathrm{vN}}$, *then* $\mathcal{H}, U, \Omega, \mathcal{M}$ *generates an interacting Haag-Kastler net with two-particle S-matrix* S.

Conclusion

Summary:

- Some integrable QFT, including the **sinh-Gordon model**, the **Bullough-Dodd** model and others with diagonal S-matrices (with CDD factors), have been constructed in a mathematically satisfactory way (Haag-Kastler nets).
- Some of them can be realized on the same Hilbert space as the free field, by twisting the observables in wedges, thus by "perturbing" the Heisenberg algebra (CFT).

Outlook:

- Work in progress: A_n-affine Toda, sine-Gordon/Thirring, Gross-Neveu...
- Complete the proof of modular nuclearity for nondiagonal S-matrices ($\mathrm{O}(N)$-symmetric S-matrices).
- Study the lightlike intersection ($\Longrightarrow$ CFT from integrable models?).
- Quantum group symmetry? Yangian as observables?