Axiomatic and algebraic Quantum Field Theory

Yoh Tanimoto

University of Rome "Tor Vergata"

Theoretical high energy physics seminar/Group therapy, Buenos Aires
17 November 2022

Mathematical Quantum Field Theory

- Quantum mechanics: Hilbert space, Hamiltonian (a specific self-adjoint operator), $[Q, P] = i\hbar$, spectral analysis, observables...
- Quantum field theory (QFT): infinite degrees of freedom on continuum configulation space (infrared and ultraviolet difficulties)
- Axiomatic approaches: **Wightman**, Osterwalder-Schrader, **Araki-Haag-Kastler**.
- Examples: free fields, $\mathcal{P}(\phi)_2$ models (and more "(super)renormalizable" models), some gauge theories in $d = 1+1, 1+2$, ϕ_3^4 model, integrable models in $d = 1+1$, conformal field theories (CFT) in $d = 1+1$.
- **No known interacting example in** $d = 3+1$. Constructing the Yang-Mills theory is a Millenium problem.
- Research topics: Constructing examples, studying representations (states, fusion rules), calculating entanglement measures, curved spacetime...

What is a quantum field?

- A (scalar) classical field ϕ is a function on the Minkowski space. Together with its momentum Π, it satisfies a certain equation of motion.
- A quantum field should be an **operator-valued** object on the Minkowski space, acting on a certain Hilbert space, satisfying the same equation of motion.
- They should also satisfy the equal time canonical commutation relations (CCR)
$$[\phi(x), \Pi(y)] = i\hbar\delta(x - y)$$
so they must be operator-valued **distributions**. $\phi(x)^n$ do not make sense directly.
- There should be the vaccum vector Ω.
- We are interested in correlation functions
$$\langle \Omega, \phi(x_1) \cdots \phi(x_n)\Omega \rangle,$$
invariant under the Poincaré transformations.

Wightman axioms

A **Wightman field theory** on a Hilbert space $\mathcal{H}$ consists of a (family of) operator-valued distribution(s) ϕ on a dense common invariant domain $\mathscr{D}$, a unitary representation U of the Poincaré group and a vacuum $\Omega \in \mathcal{H}$ satisfying

- Locality: $[\phi(f), \phi(g)] = 0$ if f, g have spacelike separated supports.
- Covariance: $U(g)\phi(x)U(g)^* = \phi(g \cdot x)$.
- Positive energy: The spectrum of $U|_{\mathbb{R}^{d+1}}$ is contained in the future lightcone.
- Vacuum: Ω is unique s.t. $U(g)\Omega = \Omega$ and $\phi(f_1)\cdots\phi(f_n)\Omega$ span $\mathcal{H}$.

From the set of correlation functions $\langle \Omega, \phi(x_1)\cdots\phi(x_n)\Omega\rangle$ satisfying locality, invariance, spectrum condition and the clustering property, one can (re)construct Wightman fields.

Examples: free fields in all d, $\mathcal{P}(\phi)_2$ models, Yukawa model $d = 2, 3$, ϕ^4_3 model, some gauge fields $d = 2, 3$ (except the uniqueness of vacuum), many CFT in $d = 2$.

Araki-Haag-Kastler axioms

An **Araki-Haag-Kastler net** consists of a family of von Neumann algebras $\{\mathcal{A}(O)\}$, a unitary representation U of the Poincaré group and a vacuum $\Omega \in \mathcal{H}$ satisfying

- Isotony: If $O_1 \subset O_2$, then $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$.
- Locality: If O_1 and O_2 are spacelike separated, then $[\mathcal{A}(O_1), \mathcal{A}(O_2)] = \{0\}$.
- Covariance: $U(g)\mathcal{A}(O)U(g)^* = \mathcal{A}(g \cdot O)$.
- Positive energy: The spectrum of $U|_{\mathbb{R}^{d+1}}$ is contained in the future lightcone.
- Vacuum: Ω is unique s.t. $U(g)\Omega = \Omega$ and $\bigcup_O \mathcal{A}(O)\Omega$ span $\mathcal{H}$.

Assume that a Wightman field satisfies a technical condition (linear energy bounds). For spacetime regions O, define $\mathcal{A}(O) = \{e^{i\phi(f)} : \operatorname{supp} f \subset O\}''$. Here, for a set M of bounded operators, M' is called the **commutant** of M and it is the set of all bounded operators on $\mathcal{H}$ commuting with all elements of M. M'' is the double commutant, and is the smallest von Neumann algebra including M. Then $\mathcal{A}, U, \Omega$ satisfy the AHK axioms.

FAQ

Q1 We don't have a good definition of QFT. Why is AQFT good?

A1 The axioms of AQFT are very weak. If we don't have a QFT as Wightman/AQFT, we haven't understood it enough.

Q2 But the Lagrangian approach is better. Why should one care about AQFT?

A2 Some Lagrangian QFTs have been constructed as Wightman/AQFT, including some (just, non-super) renormalizable models (Federbush model, Gross-Neveu model, Thirring model in 2d). If we don't have a QFT as Wightman/AQFT, we haven't understood it enough. **4d** Yang-Mills, **4d** QED, the standard model in **4d**...

Q3 For Yang-Mills, we have a good lattice theory. Why AQFT?

A3 Numerically yes. But should we be satisified with $\mathbb{Q}^{d+1}$ instead of $\mathbb{R}^{d+1}$? Poincaré covariance?

Q4 So is AQFT useful?

A4 See below.

How useful is AQFT?

Von Neumann algebras $\mathcal{A}(O)$ are algebras of bounded operators. This is convenient because one can consider

- states as normalized positive functionals
- representations for states (charged, thermal), application of subfactor theory
- fusion of representations (composition of endomorphisms)
- the Tomita-Takesaki theory (modular operator, relative entropy)

As consequenes, one has

- classification of chiral components of 2d CFT
- construction of full 2d CFT
- (rigorous) computations of relative entropy and mutual information in some models
- construction of 2d massive integrable models.

Two-dimensional chiral conformal field theory

- In relativistic QFT in $d = 1 + 1$, one puts the Lorentzian metric $(x, y) = x_0 y_0 - x_1 y_1$ on $\mathbb{R}^2$.
- The conformal group (transformations of $\mathbb{R}^2$ which preserve the metric up to a function) is $\mathrm{Diff}(\mathbb{R}) \times \mathrm{Diff}(\mathbb{R})$, acting on the lightrays $x_0 \pm x_1 = 0$.
- In a quantum theory, $\mathrm{Diff}(\mathbb{R}) \times \mathrm{Diff}(\mathbb{R})$ gets a (projective) unitary representation.
- There are observables that are invariant by $\iota \times \mathrm{Diff}(\mathbb{R})$ (or $\mathrm{Diff}(\mathbb{R}) \times \iota$): chiral observables.
- Chiral observables are quantum fields living on the lightray $\mathbb{R}$. By conformal symmetry, they extend to the one-point compactification (the circle S^1 under the stereographic projection, and have $\mathrm{Diff}(S^1)$ as the symmetry group).
- The full CFT is a certain extension of a pair of chiral components.

Chiral conformal net

Definition

A **conformal net** on S^1 is $(\mathcal{A}, U, \Omega)$, where $\mathcal{A}$ is a map from the set of intervals in S^1 into the set of von Neumann algebras on $\mathcal{H}$ which satisfies

- Isotony: $I \subset J \Rightarrow \mathcal{A}(I) \subset \mathcal{A}(J)$.
- Locality: $I \cap J = \emptyset \Rightarrow [\mathcal{A}(I), \mathcal{A}(J)] = \{0\}$.
- Diffeomorphism covariance: U is a projective unitary representation of $\mathrm{Diff}(S^1)$ such that $U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI)$ and if $\mathrm{supp}\, g \cap I = \emptyset$, then $U(g)xU(g)^* = x$ for $x \in \mathcal{A}(I)$.
- Positive energy: the restriction of U to rotations has the positive generator L_0.
- Vacuum: there is a unique (up to a scalar) unit vector Ω such that $U(g)\Omega = \Omega$ for $g \in \mathrm{PSL}(2, \mathbb{R})$ and cyclic for $\mathcal{A}(I)$.

Many examples: $U(1)$-current (free massless boson), Free massless fermion, Virasoro nets (stress energy tensor), WZW models.

Chiral conformal net

Let $\mathcal{M}$ be a von Neumann algebra, Ω a cyclic and separating vector for M, i.e., $\overline{\mathcal{M}\Omega} = \mathcal{H}$ and $x\Omega \neq 0$ for $x \neq 0, x \in \mathcal{M}$.
Define S to be the closure of the map

$$x\Omega \longmapsto x^*\Omega$$

Polar decomposition $S = J\Delta^{\frac{1}{2}}$.
Tomita's theorem: $\Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M}, J\mathcal{M}J = \mathcal{M}'$.

Some consequences of the axioms of conformal net $(\mathcal{A}, U, \Omega)$:

- The Reeh-Schlieder property: Ω is cyclic for single $\mathcal{A}(I)$, that is, $\overline{\mathcal{A}(I)\Omega} = \mathcal{H}$.
- The Bisognano-Wichmann property. $\Delta^{it}_{\mathcal{A}(I)} = U(D_I(-2\pi t))$ (the dilation preserving I).
- Haag duality. $\mathcal{A}(I') = \mathcal{A}(I)'$, where I' is the internal points of $S^1 \setminus I$.
- The split property. If $\overline{I_1} \subset I_2$, then there is a type I factor $\mathcal{R}$ (a von Neumann algebra isomorphic to some $\mathcal{B}(\mathcal{K})$) such that $\mathcal{A}(I_1) \subset \mathcal{R} \subset \mathcal{A}(I_2)$.

Representations of chiral CFT

- A **representation** of a conformal net $(\mathcal{A}, U, \Omega)$ is a family of representations $\{\rho_I\}$ on $\mathcal{H}_\rho$ of single algebras $\{\mathcal{A}(I)\}$ satisfying the compatibility: if $I_1 \subset I_2$, then $\rho_{I_2}|_{\mathcal{A}(I_1)} = \rho_{I_1}$.
- For many $\mathcal{A}$, any such representation can be decomposed into irreducible ones.
- In a nice class of "rational" CFT, there are only finitely many inequivalent irreducibles.
- Any such representation is unitarily equivalent to a (Doplicher-Haag-Roberts) endomorphism localized in some I: $\mathcal{H}_\sigma = \mathcal{H}$, $\sigma_{I'} = \mathrm{id}$ by Haag duality.
- Two localized endomorphisms σ_1, σ_2 can be composed (fusion) and give a new endomorphism $\sigma_1 \circ \sigma_2$. Fused representations decompose into irreducibles, giving rise to the fusion rules.

Chiral CFT (Böckenhauer, Evans, Kawahigashi, Longo, Müger, Rehren, Wassermann, Xu...)

- Take a pair of disjoint intervals I_1, I_3. The complement is I_2, I_4. Consider the subfactor $\mathcal{A}(I_1 \cup I_3) \subset \mathcal{A}(I_2 \cup I_4)'$. If the index is finite, the net $\mathcal{A}$ is called **completely rational**.
- For a completely rational net $\mathcal{A}$, we want to classify extensions $\mathcal{A} \subset \mathcal{B}$. From a representation σ of $\mathcal{A}$ and this extension, one considers α-induction (an analogue of induced representation) and obtain extended representations $\alpha_\sigma^\pm$. $Z_{\lambda,\sigma} = \dim \mathrm{Hom}(\alpha_\lambda^+, \alpha_\sigma^-)$ gives a **modular invariant**.
- Classification follows from the classification of modular invariant.
- Carried out for the Virasoro net with $c < 1$, WZW model with $\mathrm{SU}(2)$ with different levels.
- Recent works by Carpi, Gui, Tener, Weiner on Vertex Operator Algebras and conformal nets.

Full 2d CFT (Kawahigashi, Longo, Rehren...)

- A full CFT contains a left and a right chiral components, $\mathcal{A}_{\mathrm{L/R}}$.
- A full theory is an extension $\mathcal{A}_{\mathrm{L}} \otimes \mathcal{A}_{\mathrm{R}} \subset \mathcal{A}$.
- Consider the case where $\mathcal{A}_{\mathrm{L}} = \mathcal{A}_{\mathrm{R}}$. Take a family Δ of irreducible representation that is closed under fusion and decomposition. Then one such $\mathcal{A}$ can be constructed in such a way that the extension is given on

$$\mathcal{H} = \bigoplus_{\lambda \in \Delta} \mathcal{H}_\lambda \otimes \mathcal{H}_{\bar{\lambda}},$$

  where $\bar{\lambda}$ is the congjugate representation.
- Some more complicated combination of representations can be obtained again from modular invariants.
- For a fixed combination of representations, is there a unique extension $\mathcal{A}$? This is a question of cohomology. Solved for Virasoro net with $c < 1$, $\mathrm{SU}(2)$-WZW models but not in general.

Entanglement measures

- Let $\mathcal{M}_1 \otimes \mathcal{M}_2$ be a bipartite system of quantum mechanics. A state ρ is given by a density matrix M_ρ and the trace: $\rho(x) = \mathrm{tr}(M_\rho x)$.
- For a density matrix M_ρ, the von Neumann entropy is given by $-\mathrm{tr}(M_\rho \log M_\rho)$.
- The entanglement entropy is the von Neumann entropy of the restriction $\rho|_{\mathcal{M}_1}$ (partial trace).
- These are interesting objects also in QFT. Unfortunately, there is no trace in QFT (local algebras are of type III).
  - Consider a quantity that generalize the entanglement entropy.
  - Consider other entanglement measures (relative entropy, mutual information...).

Entanglement measures (Hollands, Longo, Sanders, Tanimoto, Otani, Xu...)

- Let Φ be another vector. Let $\omega = \langle \Omega, \cdot\Omega\rangle, \varphi = \langle \Phi, \cdot\Phi\rangle$. The relative modular objects is given by $S_{\omega,\varphi} = J_{\Omega,\varphi}\Delta_{\Omega,\varphi}^{\frac{1}{2}}$, where
$$S_{\omega,\varphi} x\Psi = x^*\Omega.$$
- The relative entropy of $\mathcal{M}$ with respect to ω, φ is
$$S(\omega, \varphi') = \langle \Omega, \log \Delta_{\omega,\varphi}\Omega\rangle.$$
(mathematically well-defined, although it can be infinite).
- For a pure state on QM, $S(\varphi) = \sup\{\sum_j \lambda_j S(\varphi_j, \varphi) : \sum_j \lambda_j = \varphi\} = \inf\{S(\varphi, \sigma) : \sigma$ is a separable state $\}$. In this sense, relative entropy is more fundamental.
- In some models and some states, $S(\omega, \varphi)$ can be calculated and also $\inf\{S(\varphi, \sigma) : \sigma$ is a separable state $\}$ can be estimated for (a pair of) local algebra(s).

Integrable QFT

Some massive 2d QFT are believed to be integrable.

- sine/sinh-Gordon, Gross-Neveu, Thirring...

Form factor programme (Babujian, Karowski, Smirnov...)

- Conjecture the S-matrix for a model.
- Form factors ${}^{\mathrm{out}}\langle q_1, \cdots, q_m | O(x) | p_1, \cdots, p_n \rangle^{\mathrm{in}}$ of local operators $O(x)$ should satisfy certain relations. In some cases, they can be calculated.
- n-point functions $\langle \Omega, O(x)O(0)\Omega \rangle = \sum_n \int dp_1 \cdots dp_n \, \langle \Omega, O(x) | p_1, \cdots, p_n \rangle^{\mathrm{in}} \, {}^{\mathrm{in}}\langle p_1, \cdots, p_n | O(0)\Omega \rangle$ should be computed.
- Convergence? Locality in e.g. the sine-Gordon model?

Construction of integrable models in 2d (Alazzawi, Bostelmann, Buchholz, Cadamuro, Lechner, Schroer, Tanimoto...)

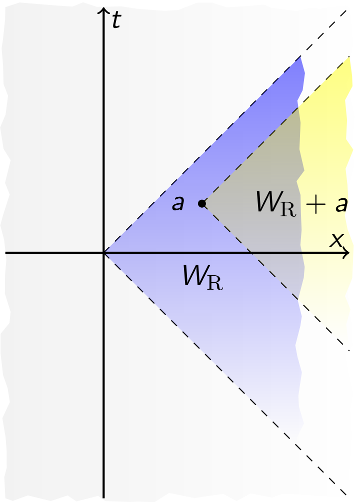
Isotony: $O_1 \subset O_2 \Longrightarrow \mathcal{A}(O_1) \subset \mathcal{A}(O_2)$ means that **larger** regions contain **more** observables, also **simpler** ones. **Wedge**: $W_\mathrm{R} := \{(t,x) : x > |t|\}$.

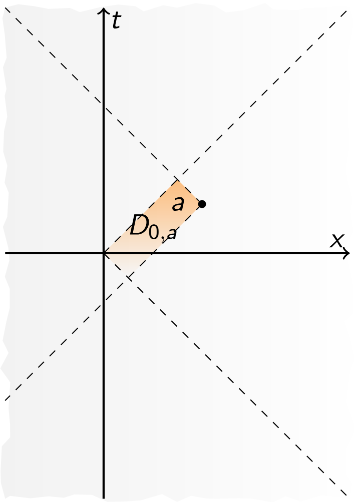
Why wedges?

- Form factors ${}^{\mathrm{out}}\langle q_1, \cdots, q_m | O(x)\Omega\rangle^{\mathrm{in}}$ of interacting pointlike fields $O(x)$ are complicated.
- For a large region W_R, there might be simpler observables.

Construct first the algebras $\mathcal{A}(W_\mathrm{R})$ of observables in wedges. Local observables are obtained by $\mathcal{A}(O) = \mathcal{A}(W_\mathrm{R} + a) \cap \mathcal{A}(W_\mathrm{L} + b)$.

Standard wedge and double cone





Construction of integrable models in 2d

- analytic two-parrticle S-matrix (e.g. the sinh-Gordon model) $S : \mathbb{R} + i(0,\pi) \to \mathbb{C}$,
$$\overline{S(\theta)} = S(\theta)^{-1} = S(-\theta) = S(\theta + \pi i), \ \ \theta \in \mathbb{R}.$$
- S-symmetric Fock space: $\mathcal{H}_1 = L^2(\mathbb{R}, d\theta)$, $\mathcal{H}_n = P_n \mathcal{H}_1^{\otimes n}$, where P_n is the projection onto S-symmetric functions: $\Psi_n(\theta_1, \cdots, \theta_n) = S(\theta_{k+1} - \theta_k)\Psi_n(\theta_1, \cdots, \theta_{k+1}, \theta_k, \cdots, \theta_n)$.
- Zamolodchikov-Faddeev algebra: S-symmetrized creation and annihilation operators $z^\dagger(\xi) = Pa^\dagger(\xi)P, z(\xi) = Pa(\xi)P, P = \bigoplus_n P_n$.
- **Wedge-local field**: $\phi(f) = z^\dagger(f^+) + z(f^+)$.
- $\mathcal{A}(W_\mathrm{R}) = \overline{\{e^{i\phi(f)} : \mathrm{supp}\, f \subset W_\mathrm{R}\}}^{\mathrm{vN}}$.
- Proving that $\mathcal{A}(O) = \mathcal{A}(W_\mathrm{R} + a) \cap \mathcal{A}(W_\mathrm{L} + b)$ can be reduced to proving that $\mathcal{A}(W_\mathrm{R} + a) \subset \mathcal{A}(W_\mathrm{R})$ is split. **Modular nuclearity** helps.
- Some models constructed in this way: Sinh-model with CDD factors, Bullough-Dodd model (work in progress), some diagonal S-matrices.

Curved spacetimes

- One can generalize the definition of Haag-Kastler net (of C^*-algebras) to curved spacetimes, omitting covariance and vacuum.
- One can study energy inequalities (ANEC, QNEC...) for free fields on various spacetimes.
- For the de Sitter space, there is a generalization of the notion of covariance and vacuum. 2d CFT and $\mathcal{P}(\phi)_2$ models can be constructed.

Outlook (personal, with Adamo, Giorgetti, Jäkel, Neeb...)

- Construct 2d CFT with Wightman fields.
  - Patch chiral primary fields together, following Longo-Rehren.
  - Explicit construction of inequivalent full CFT with the same chiral decomposition?
- Prove reflection positivity for unitary VOA
  - Vertex Operator Algebras are about formal series $Y(a, z)$. z appears to be the Wich-rotated two-dimensional variable.
  - Are they genuine Euclidean QFT? Do they satisfy reflection positivity?
- Construct massive field theory by "perturbing" a CFT on the de Sitter space.
  - Zamolodchikov claims that some integrable models can be obtained by perturbing 2d CFT by "relevant fields".
  - Primary fields with low scaling dimensions can be restricted to the time-zero circle. Can they be added to the Hamiltonian to deform the dynamics?
  - Do primary fields make sense in the Euclidean space? Can the Euclidean vacuum perturbed by them?
- 4d Yang-Mills? The methods of Bałaban-Dimock?