Thermal states in conformal QFT

(joint work with P. Camassa, R. Longo and M. Weiner)

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25/04/11, Bucharest

Introduction: KMS states

$\mathcal{A}$: C*-algebra. “algebra of observables”.
α_t: one-parameter automorphism group of $\mathcal{A}$. “time-translation”.
We want to study thermal equilibrium states.

KMS state on C*-algebra

A β-**KMS state** φ on $\mathcal{A}$ with respect to α_t is a state with the following condition: for any $x, y \in \mathcal{A}$ there is an analytic function f such that

$$f_{x,y}(t) = \varphi(x\alpha_t(y)), f_{x,y}(t + i\beta) = \varphi(\alpha_t(y)x).$$

$T = \frac{1}{\beta}$ is called the **temperature** of φ.

Introduction: examples of KMS states

Example (matrix algebra)

$\mathcal{A} = M_n(\mathbb{C}), \alpha_t = \mathrm{Ad}(e^{itH})$, H: positive. The state $\varphi(x) = \frac{\mathrm{Tr}(e^{-\beta H}x)}{\mathrm{Tr}(e^{-\beta H})}$ is a β-KMS state.

The **Gibbs state** in grand canonical ensemble.

Remark

For finite dimensional systems, the KMS condition **characterizes** the Gibbs state.

For infinite dimensional $\mathcal{A}$, the Hamiltonian H is typically not trace-class.

Example (modular automorphism group)

$\mathcal{A} = M$, a von Neumann algebra,φ: a faithful normal state, σ^φ: the modular automorphism. φ is a β-KMS state with $\beta = -1$.

Introduction: conformal nets

- Spacetime: the circle $S^1 = \mathbb{R} \cup \{\infty\}$.
- Möbius symmetry $\mathrm{PSL}(2,\mathbb{R})$: translation $\tau_s : t \mapsto t+s$, dilation $\delta_s : t \mapsto e^s t$, rotation $\rho_s : z \mapsto e^{is} z$.
- Diffeomorphism covariance.

Conformal net

A conformal net $\mathcal{A}$ is an assignment of von Neumann algebra $\mathcal{A}(I)$ to each interval $I \subset \mathcal{S}^1$ such that

- $I \subset J \Rightarrow \mathcal{A}(I) \subset \mathcal{A}(J)$
- $I \cap J = \emptyset \Rightarrow [A(I), \mathcal{A}(J)] = 0$.
- There is a projective representation $U : \mathrm{Diff}(S^1) \to \mathcal{U}(\mathcal{H})$ such that $U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI)$, with positive generator of rotation.
- There is a vector Ω invariant under $\mathrm{PSL}(2,\mathbb{R}) \subset \mathrm{Diff}(S^1)$.

KMS states on conformal nets

Translation on $S^1 = \mathbb{R} \cup \{\infty\}$ corresponds to the translation in lightlike direction in two-dimension.

Main objects:

$$\mathcal{A} = \overline{\bigcup_{I \Subset \mathbb{R}} \mathcal{A}(I)}^{\|\cdot\|}, \text{ the quasilocal algebra.}$$

$$\alpha_t = \mathrm{Ad}\, U(\tau_t), \text{ where } \tau_t \text{ is translation.}$$

Uniformity of the phase structure

Dilation covariance: correspondence between KMS states in different temperatures.

$$\varphi \text{ is a } \beta\text{-KMS state } \iff \varphi \circ \mathrm{Ad}\, U(\delta_s) \text{ is a } \beta e^s\text{-KMS state.}$$

We consider always $\beta = 1$.

Geometric KMS state

Fact: Bisognano-Wichmann property

The vacuum state ω is a KMS state for $\mathcal{A}(\mathbb{R}_+)$ with respect to **dilation**.

Fact: the exponential map

The exponential map

$$t \longmapsto e^t$$

is a diffeomorphism between $\mathbb{R}$ and $\mathbb{R}_+$, and this intertwines translation and dilation.
This diffeomorphism Exp is implemented **locally** by a unitary U.

Theorem (The geometric KMS state)

The state $\omega \circ \mathrm{Exp}$ is well defined and a KMS state with respect to translation.

Complete rationality

Complete rationality

A conformal net $\mathcal{A}$ is said to be **completely rational** if it satisfies the following:

- Split property: for $I \Subset J$,$\exists F$ type I factor s.t $\mathcal{A}(I) \subset F \subset \mathcal{A}(J)$.
- Strong additivity: $\mathcal{A}(I) = \mathcal{A}(I_1) \vee \mathcal{A}(I_2)$ if $I_1 \cup I_2 = I \setminus \{p\}$.
- The index of $\mathcal{A}(I_1 \cup I_3) \subset \mathcal{A}(I_2 \cup I_4)'$ is finite ($S^1 = \overline{I_1 \cup I_2 \cup I_3 \cup I_4}$).

Examples of complete rational nets

- Loop group nets (current algebras with compact group G).
- Virasoro nets with $c < 1$ (the algebras of stress-energy tensor).
- Finite index inclusions and extensions.

Theorem (Uniqueness of KMS state)

Any completely rational net admits only the geometric KMS state.

Thermal completion

Thermal completion

- φ: a primary KMS state on a net $\mathcal{A}$.
- π_φ: the GNS representation with respect to φ.
- Φ: the corresponding GNS vector.

The inclusion $(\pi_\varphi(\mathcal{A}(\mathbb{R}_+)) \subset \pi_\varphi(\mathcal{A}(\mathbb{R})), \Phi)$ is a **standard half-sided modular inclusion**.
We can construct a covariant called the **thermal completion** of $\mathcal{A}$ with respect to φ.

In completely rational case, the thermal completion is an irreducible conformal **extension** of the original net with finite index.

Complete rationality: finiteness of sectors and extensions

Fact

A completely rational net admits only finitely many inequivalent DHR representations.

- A completely rational net admits only "finite charges".
- A thermal state should contain "infinite charges" (**Contradiction?**).

A completely rational net admits only finitely many extensions of net. Among extensions, there are **maximal** extensions.

Lemma

The thermal completion of the geometric KMS state φ_{geo} is the original net.

Lemma

Any KMS state φ on a completely rational maximal net $\mathcal{A}$ is $\varphi = \varphi_{\mathrm{geo}} \circ \gamma$ where $\gamma = \pi_\varphi \circ \pi_{\varphi_{\mathrm{geo}}}^{-1}$ is an automorphism of $\mathcal{A}|_{\mathbb{R}_+}$.

Proof of uniqueness

Lemma

If γ is an automorphism which does not preserve $\varphi_{\rm geo}$, then $\{\delta_s \circ \gamma \circ \delta_s\}$ are **unitarily inequivalent***.*

Theorem

A maximal completely rational net admits only $\varphi_{\rm geo}$.

Proof:

$\varphi = \varphi_{\rm geo} \circ \gamma$ and there would be infinitely many sectors $\{\delta_s \circ \gamma \circ \delta_{-s}\}$ (**contradiction**).

Theorem

Any completely rational net admits only $\varphi_{\rm geo}$.

Further results

Theorem

For non-completely rational nets, we have classified KMS states on:

- *the* $U(1)$*-current net:* $\varphi^q, q \in \mathbb{R}$
- *the Virasoro net* Vir_c *with* $c = 1$*:* $\varphi_1^q, q \in \mathbb{R}_+$,

and constructed KMS states $\varphi_c^q, q \in \mathbb{R}_+$ *on* Vir_c, $c > 1$.

Theorem

Any two-dimensional completely rational conformal net admits only the geometric state.