Longo-Witten endomorphisms and interacting two-dimensional models

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Introduction

Historical problem

- Long standing open problem: interacting QFT models in 4-dim
- Construction of nets of von Neumann algebras

Recent progress:

- Wedge-local net in 2-dim, based on a single von Neumann algebra and the modular theory (Borchers '92)
- Factorizing S-matrix models (Lechner '08).

Present approach:

- Chiral conformal net on S^1: many examples
- Endomorphisms of the half-line algebra (Longo-Witten '11)

Main result

Interacting massless wedge-local nets in 2-dim

Fundamental notions

- $\mathcal{H}$: a Hilbert space
- $B(\mathcal{H})$: the *-algebra of all bounded operators on $\mathcal{H}$.

The weak operator topology on $B(\mathcal{H})$: a net of bounded operators x_n is convergent to x if and only if $\langle \xi, x_n \eta \rangle \to \langle \xi, x\eta \rangle$ for any $\xi, \eta \in \mathcal{H}$.

von Neumann algebras

A *-subalgebra $\mathcal{M}$ (closed under addition, multiplication and the *-operation) of $B(\mathcal{H})$ is a **von Neumann algebra** if it contains the identity operator I and is closed in the weak operator topology.

Theorem (von Neumann)

A *-subalgebra $\mathcal{M}$ of $B(\mathcal{H})$ is a von Neumann algebra if and only if $\mathcal{M} = (\mathcal{M}')'$, where $\mathcal{M}' = \{y \in B(\mathcal{H}) : [x, y] = 0 \text{ for } x \in \mathcal{M}\}$.

A vector ξ is **cyclic** for $\mathcal{M}$ if $\overline{\mathcal{M}\xi} = \mathcal{H}$ and **separating** if $x\xi \neq 0$ for $0 \neq x \in \mathcal{M}$.

Quantum fields

Conventional quantum field

- ϕ: operator valued distribution on $\mathbb{R}^d$
- U: implementation of the spacetime symmetry
- Ω the vacuum vector

- Locality: $x \perp y \Rightarrow [\phi(x), \phi(y)] = 0$.
- Poincaré covariance: $\exists U$: positive energy rep of $\mathcal{P}_+^\uparrow$ such that $\mathrm{Ad}\,U(g)\phi(x) = \phi(g \cdot x)$.
- Vacuum: $\exists \Omega$ such that $U(g)\Omega = \Omega$.

One can smear ϕ by a compactly supported function f on $\mathbb{R}^d$ to obtain an unbounded operator $\phi(f)$ " $=$ " $\int \phi(x)f(x)dx$, then a bounded operator by exponentiation $e^{i\phi(f)}$.

Net of von Neumann algebras

Definition

A **Poincaré covariant net** of von Neumann algebras is $\mathcal{A}(O)$: von Neumann algebras parametrized by open regions $O \Subset \mathbb{R}^d$ such that

- Isotony: $O_1 \subset O_2 \Rightarrow \mathcal{A}(O_1) \subset \mathcal{A}(O_2)$.
- Locality: $O_1 \perp O_2 \Rightarrow [\mathcal{A}(O_1), \mathcal{A}(O_2)] = 0$.
- Poincaré covariance: $\exists U$: positive energy rep of $\mathcal{P}_+^\uparrow$ such that $U(g)\mathcal{A}(O)U(g)^* = \mathcal{A}(gO)$.
- Vacuum: $\exists\Omega$ such that $U(g)\Omega = \Omega$ and cyclic for $\mathcal{A}(O)$.

If one has a quantum field ϕ, he can construct a net by $\mathcal{A}(O) = \{e^{i\phi(f)} : \mathrm{supp} f \subset O\}''$. Conversely, if a net $\mathcal{A}$ has a conformal symmetry (see below), one can reconstruct the field from $\mathcal{A}$.

Problem

To construct nets of von Neumann algebras.

Chiral conformal net

Definition

A **conformal net** on S^1 is a map $\mathcal{A}$ from the set of intervals in S^1 into the set of von Neumann algebras on $\mathcal{H}$ which satisfies

- Isotony: $I \subset J \Rightarrow \mathcal{A}(I) \subset \mathcal{A}(J)$.
- Locality: $I \cap J \Rightarrow [\mathcal{A}(I), \mathcal{A}(J)] = 0$.
- Möbius covariance: $\exists U$: positive energy rep of $\mathrm{PSL}(2, \mathbb{R})$ such that $\mathrm{Ad}U(g)\mathcal{A}(I) = \mathcal{A}(gI)$.
- Vacuum: $\exists\Omega$ such that $U(g)\Omega = \Omega$ and cyclic for $\mathcal{A}(I)$.

Many examples: $U(1)$-*current* (free massless boson), *Free massless fermion*, Virasoro nets (stress energy tensor), Loop group nets (noncommutative currents).
In the present work, important is the $U(1)$-current net which admits the **Fock space structure**.

Example: the $U(1)$-current net

The abelian current algebra

$$[J(f), J(g)] = i \int f(x) g'(x) dx$$

where $f, g \in C^\infty(\mathbb{R}, \mathbb{R})$ admits the vacuum representation π_0 on the Fock space $F(L^2(\mathbb{R}, dp))$ (there is a representation U of $\mathrm{M\ddot{o}b}$ which renders W covariant). One defines the $U(1)$-**current net** by

$$\mathcal{A}^{(0)}(I) = \{e^{i\pi_0(J(f))} : \mathrm{supp} f \subset I\}'',$$

where we identified $\mathbb{R}$ with $S^1 \setminus \{-1\}$.

- The derivative of the free massless bosonic field on 2 dimensions decomposes into the tensor product of two copies of the $U(1)$-current net.
- For a unitary V_1 on $\mathrm{L}^2(\mathbb{R}, dp)$, one defines the second quantization $\Gamma(V_1) = \mathbb{1} \oplus V_1 \oplus (V_1 \otimes V_1) \otimes \cdots$

Endomorphisms of $\mathcal{A}^{(0)}(\mathbb{R}_+)$

Definition

A Longo-Witten endomorphism of a net $\mathcal{A}$ on S^1 is an endomorphism of $\mathcal{A}(\mathbb{R}_+)$ implemented by a unitary V commuting with translation $T(t)$.

Simplest examples: $\mathrm{Ad}\, T(s)$ for $s \geq 0$, inner symmetry (automorphism which preserves each local algebra $\mathcal{A}(I)$ and the vacuum state)

An **inner symmetric** function φ is the boundary value of a bounded analytic function on the upper-half plane with $|\varphi(p)| = 1, \varphi(p) = \overline{\varphi(-p)}$ for $p \in \mathbb{R}$. Example: $\varphi(p) = e^{i\kappa p}$ with $\kappa \geq 0$, $\frac{p-i\kappa}{p+i\kappa}$ with $\kappa > 0$

Theorem (Longo-Witten '11)

$\mathcal{A}^{(0)}$: the $U(1)$-current net
$V_\varphi := \Gamma(\varphi(P_1))$ implements a Longo-Witten endomorphism of $\mathcal{A}^{(0)}$, where P_1 is the generator of the translation on the one-particle space.

Boundary quantum field net

Definition (Two-dimensional net with boundary)

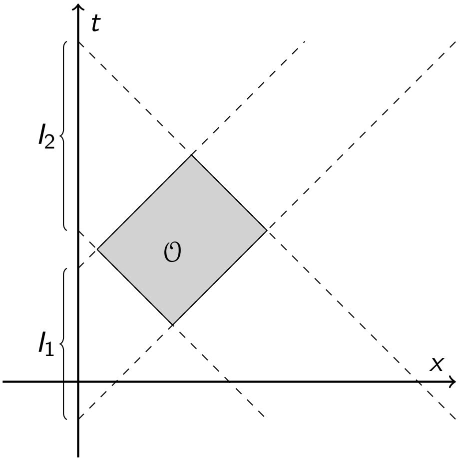
A net $\mathcal{B}$ of von Neumann algebras in two dimensions with boundary is an assignment of von Neumann algebra $\mathcal{B}(D)$ to a diamond D in the right-half plane satisfying locality, covariance with respect to time-translation, etc.

- $\mathcal{A}$ be a net on S^1, restricted to $\mathbb{R}$.
- V: a unitary implementing a Longo-Witten endomorphism

Theorem (Longo-Witten '11)

If one defines $\mathcal{B}(D) = \mathcal{A}(I_1) \vee \mathrm{Ad}\, V(\mathcal{A}(I_2))$, then $\mathcal{B}$ is a two-dimensional net with boundary.

Hence such a unitary implementing an endomorphisms of $\mathcal{A}(\mathbb{R}_+)$ leads to a net with boundary.


t
I_2
$\mathcal{O}$
I_1
x

Borchers triple (Borchers '92)

- Local net: von Neumann algebras $\mathcal{A}(O)$ parametrized by open regions O
- Borchers triple: a **single** von Neumann algebra $\mathcal{M}$ acted on by spacetime translations

Definition

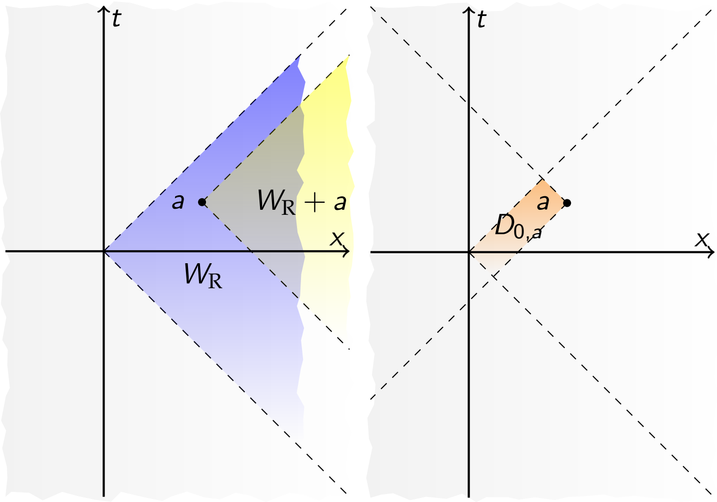
$\mathcal{M}$: vN algebra, T: positive-energy rep of $\mathbb{R}^2$, Ω: vector, is a Borchers triple if Ω is cyclic and separating for $\mathcal{M}$ and

- $\mathrm{Ad}\, T(a)(\mathcal{M}) \subset \mathcal{M}$ for $a \in W_\mathrm{R}$, $T(a)\Omega = \Omega$

Correspondence: $\mathcal{A}(W_\mathrm{R}) \Leftrightarrow \mathcal{M}$, where $W_\mathrm{R} := \{a = (a_0, a_1) : |a_0| < a_1\}$.

examples

- Factorizing S-matrix models (Lechner '06)
- Deformations (Buchholz-Lechner-Summers '10, Dybalski-T. '11, Lechner '11, etc.)

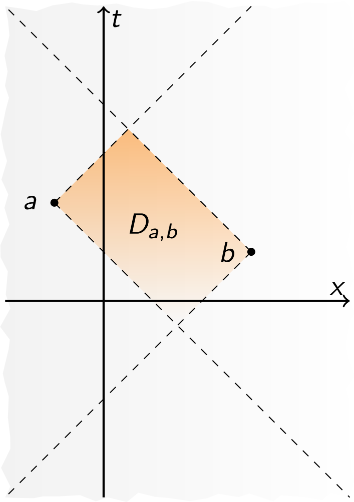

t
a
$W_{\mathrm{R}} + a$
x
W_{R}
t
a
$D_{0,a}$
x

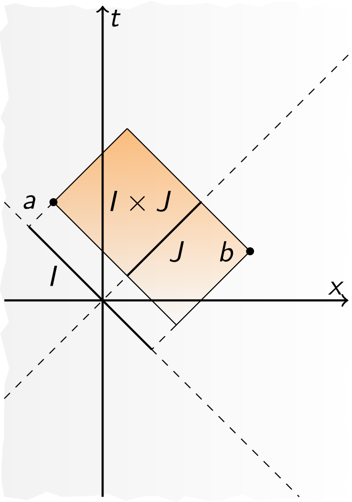
Chiral CFT in 2 dimensions

Although there exist a plenty of conformal nets on S^1, there is no notion of *interaction* for one-dimensional theory.
However, it is easy to construct a (noninteracting) two-dimensional net from a *pair* of nets on S^1.
For two nets $\mathcal{A}_+, \mathcal{A}_-$ on S^1, we define

- a **chiral** net on $\mathbb{R}^2$: $\mathcal{A}(I \times J) := \mathcal{A}_+(I) \otimes \mathcal{A}_-(J)$
- a representation $U = U_+ \otimes U_-$ of $\mathrm{PSL}(2, \mathbb{R}) \otimes \mathrm{PSL}(2, \mathbb{R}) \supset \mathcal{P}_+^\uparrow$,
- the vacuum $\Omega = \Omega_+ \otimes \Omega_-$

A chiral net $\mathcal{A}$ is *not* interacting (Dybalski-T. '11). Such nets with a simple tensor product structure can be considered as free theory in 2 dimensions.


t
a
$D_{a,b}$
b
x


t
a
$I \times J$
J
b
I
x

Some construction of wedge-local nets (T. '11)

- $\mathcal{A}_0$: a conformal net on $S^1 = \mathbb{R} \cup \{\infty\}$
- P_0: the generaor of translation $T_0(t) = e^{itP_0}$

We set, for $\kappa > 0$,

- $\mathcal{M}_\kappa := \{x \otimes \mathbb{1}, e^{i\kappa P_0 \otimes P_0}(\mathbb{1} \otimes y)e^{-i\kappa P_0 \otimes P_0} : x \in \mathcal{A}_0(\mathbb{R}_-), y \in \mathcal{A}_0(\mathbb{R}_+)\}''$
- $T := T_0 \otimes T_0$
- $\Omega := \Omega_0 \otimes \Omega_0$

Theorem (T. '11)

$(\mathcal{M}_\kappa, T, \Omega)$ *is an asymptotically complete Borchers triple with the S-matrix* $e^{i\kappa P_0 \otimes P_0}$.

Further construction of wedge-local nets (T. '11)

Theorem (T. '11)

$(\mathcal{M}_\kappa, T, \Omega)$ is an asymptotically complete Borchers triple with the S-matrix $e^{i\kappa P_0 \otimes P_0}$.

Proof) To see that it is a Borchers triple, what is nontrivial is the separating property of Ω. We set

$$\begin{aligned}\mathcal{M}_\kappa &:= \{x \otimes \mathbb{1}, e^{i\kappa P_0\otimes P_0}(\mathbb{1} \otimes y)e^{-i\kappa P_0 \otimes P_0} : x \in \mathcal{A}_0(\mathbb{R}_-), y \in \mathcal{A}_0(\mathbb{R}_+)\}'' \\ \mathcal{M}_\kappa^1 &:= \{e^{i\kappa P_0\otimes P_0}(x \otimes \mathbb{1})e^{-i\kappa P_0\otimes P_0}, \mathbb{1} \otimes y : x \in \mathcal{A}_0(\mathbb{R}_+), y \in \mathcal{A}_0(\mathbb{R}_-)\}''\end{aligned}$$

Note that $e^{i\kappa P_0 \otimes P_0} = \int e^{i\kappa p P_0} \otimes dE_0(p)$, where E_0 is the spectral measure of P_0. $\mathcal{M}_\kappa$ and $\mathcal{M}_\kappa^1$ commute since

$$e^{i\kappa P_0\otimes P_0}(x \otimes \mathbb{1})e^{-i\kappa P_0\otimes P_0} = \int e^{i\kappa p P_0}(x)e^{-i\kappa p P_0} \otimes dE_0(p).$$

Ω is cyclic for $\mathcal{M}_\kappa^1$, hence separating for $\mathcal{M}_\kappa$.

Construction of Borchers triples based on $\mathcal{A}^{(0)}$

We take again the $U(1)$-current net $\mathcal{A}^{(0)}$. For an inner symmetric function φ, set

- $\mathcal{H}^n := \mathcal{H}_1^{\otimes n}$
- $P_{i,j}^{m,n} := (\mathbb{1} \otimes \cdots \otimes \underset{i\text{-th}}{P_1} \otimes \cdots \otimes \mathbb{1}) \otimes (\mathbb{1} \otimes \cdots \otimes \underset{j\text{-th}}{P_1} \otimes \cdots \otimes \mathbb{1})$, acting on $\mathcal{H}^m \otimes \mathcal{H}^n$, $1 \leq i \leq m$ and $1 \leq j \leq n$.
- $\varphi_{i,j}^{m,n} := \varphi(P_{i,j}^{m,n})$ (functional calculus on $\mathcal{H}^m \otimes \mathcal{H}^n$).
- $S_\varphi := \bigoplus_{m,n} \prod_{i,j} \varphi_{i,j}^{m,n}$

We can take the spectral decomposition of S_φ only with respect to the right component:

$$S_\varphi = \bigoplus_n \int \prod_j \Gamma(\varphi(p_j P_1)) \otimes dE_1(p_1) \otimes \cdots \otimes dE_1(p_n)$$

Note that the integrand is a unitary operator which implements a Longo-Witten endomorphism for any value of $p_j \geq 0$.

Construction of Borchers triples

We set

- $\mathcal{M}_\varphi := \{x \otimes \mathbb{1}, S_\varphi(\mathbb{1} \otimes y)S_\varphi^* : x \in \mathcal{A}^{(0)}(\mathbb{R}_-), y \in \mathcal{A}^{(0)}(\mathbb{R}_+)\}''$
- $T := T_0 \otimes T_0$
- $\Omega := \Omega_0 \otimes \Omega_0$

Theorem (T. '11)

$(\mathcal{M}_\varphi, T, \Omega)$ *is an asymptotically complete Borchers triple with the S-matrix* S_φ*.*

Proof) To see that it is a Borchers triple, what is nontrivial is the separating property of Ω. We set

$$\mathcal{M}_\varphi^1 := \{S_\varphi(x \otimes \mathbb{1})S_\varphi^*, \mathbb{1} \otimes y : x \in \mathcal{A}^{(0)}(\mathbb{R}_+), y \in \mathcal{A}^{(0)}(\mathbb{R}_-)\}''$$

$\mathcal{M}_\varphi$ and $\mathcal{M}_\varphi^1$ commute since

$$S_\varphi(x \otimes \mathbb{1})S_\varphi^* = \bigoplus_n \int \mathrm{Ad}\left(\prod_j \Gamma(\varphi(p_j P_1))\right)(x) \otimes dE_1(p_1) \otimes \cdots dE_1(p_n).$$

Conclusion

Summary

- Interacting wedge-local nets parametrized by symmetric inner functions

Open problems

- Further examples with different asymptotic algebra
- Strict locality
- Massive analogue