Longo-Witten endomorphisms and interacting two-dimensional models

Yoh Tanimoto

Institut für Theoretische Physik, Göttingen University

December 1st 2011, Aarhus
Introduction

Historical problem

- Long standing open problem: interacting QFT models in 4-dim
- Construction of nets of von Neumann algebras

Recent progress:

- Wedge-local net in 2-dim, based on a single von Neumann algebra and the modular theory (Borchers ‘92)
- Factorizing S-matrix models (Lechner ‘08).

Present approach:

- Chiral conformal net on $S^1$: many examples
- Endomorphisms of the half-line algebra (Longo-Witten ‘11)

Main result

Interacting massless wedge-local nets in 2-dim
Fundamental notions

- $\mathcal{H}$: a Hilbert space
- $B(\mathcal{H})$: the $*$-algebra of all bounded operators on $\mathcal{H}$.

The weak operator topology on $B(\mathcal{H})$: a net of bounded operators $x_n$ is convergent to $x$ if and only if $\langle \xi, x_n \eta \rangle \to \langle \xi, x \eta \rangle$ for any $\xi, \eta \in \mathcal{H}$.

von Neumann algebras

A $*$-subalgebra $\mathcal{M}$ (closed under addition, multiplication and the $*$-operation) of $B(\mathcal{H})$ is a **von Neumann algebra** if it contains the identity operator $I$ and is closed in the weak operator topology.

Theorem (von Neumann)

A $*$-subalgebra $\mathcal{M}$ of $B(\mathcal{H})$ is a von Neumann algebra if and only if $\mathcal{M} = (\mathcal{M}')'$, where $\mathcal{M}' = \{ y \in B(\mathcal{H}) : [x, y] = 0 \text{ for } x \in \mathcal{M} \}$.

A vector $\xi$ is **cyclic** for $\mathcal{M}$ if $\overline{\mathcal{M}\xi} = \mathcal{H}$ and **separating** if $x\xi \neq 0$ for $0 \neq x \in \mathcal{M}$.
Quantum fields

Conventional quantum field

- \( \phi \): operator valued distribution on \( \mathbb{R}^d \)
- \( U \): implementation of the spacetime symmetry
- \( \Omega \) the vacuum vector

- Locality: \( x \perp y \Rightarrow [\phi(x), \phi(y)] = 0. \)
- Poincaré covariance: \( \exists U : \) positive energy rep of \( \mathcal{P}_+^\uparrow \) such that \( \text{Ad} U(g) \phi(x) = \phi(g \cdot x) \).
- Vacuum: \( \exists \Omega \) such that \( U(g) \Omega = \Omega \).

One can smear \( \phi \) by a compactly supported function \( f \) on \( \mathbb{R}^d \) to obtain an unbounded operator \( \phi(f)^" = \int \phi(x) f(x) dx \), then a bounded operator by exponentiation \( e^{i\phi(f)} \).
**Definition**

A **Poincaré covariant net** of von Neumann algebras is $\mathcal{A}(O)$: von Neumann algebras parametrized by open regions $O \subset \mathbb{R}^d$ such that

- **Isotony**: $O_1 \subset O_2 \Rightarrow \mathcal{A}(O_1) \subset \mathcal{A}(O_2)$.
- **Locality**: $O_1 \perp O_2 \Rightarrow [\mathcal{A}(O_1), \mathcal{A}(O_2)] = 0$.
- **Poincaré covariance**: $\exists U$: positive energy rep of $\mathcal{P}_+$ such that $U(g)\mathcal{A}(O)U(g)^* = \mathcal{A}(gO)$.
- **Vacuum**: $\exists \Omega$ such that $U(g)\Omega = \Omega$ and cyclic for $\mathcal{A}(O)$.

If one has a quantum field $\phi$, he can construct a net by $\mathcal{A}(O) = \{ e^{i\phi(f)} : \text{supp} f \subset O \}$''. Conversely, if a net $\mathcal{A}$ has a conformal symmetry (see below), one can reconstruct the field from $\mathcal{A}$.

**Problem**

To construct nets of von Neumann algebras.
A **conformal net** on $S^1$ is a map $\mathcal{A}$ from the set of intervals in $S^1$ into the set of von Neumann algebras on $\mathcal{H}$ which satisfies

- **Isotony:** $I \subset J \Rightarrow \mathcal{A}(I) \subset \mathcal{A}(J)$.
- **Locality:** $I \cap J \Rightarrow [\mathcal{A}(I), \mathcal{A}(J)] = 0$.
- **Möbius covariance:** $\exists U$ : positive energy rep of $\text{PSL}(2, \mathbb{R})$ such that $\text{Ad}U(g)\mathcal{A}(I) = \mathcal{A}(gI)$.
- **Vacuum:** $\exists \Omega$ such that $U(g)\Omega = \Omega$ and cyclic for $\mathcal{A}(I)$.

**Many examples:** $U(1)$-current (free massless boson), Free massless fermion, Virasoro nets (stress energy tensor), Loop group nets (noncommutative currents).

In the present work, important is the $U(1)$-current net which admits the **Fock space structure**.
Example: the $U(1)$-current net

The abelian current algebra

$$[J(f), J(g)] = i \int f(x) g'(x) \, dx$$

where $f, g \in C^\infty(\mathbb{R}, \mathbb{R})$ admits the vacuum representation $\pi_0$ on the Fock space $F(L^2(\mathbb{R}, dp))$ (there is a representation $U$ of Möb which renders $W$ covariant). One defines the $U(1)$-current net by

$$\mathcal{A}^{(0)}(I) = \{ e^{i\pi_0(J(f))} : \text{supp} f \subset I \}'' ,$$

where we identified $\mathbb{R}$ with $S^1 \setminus \{-1\}$.

- The derivative of the free massless bosonic field on 2 dimensions decomposes into the tensor product of two copies of the $U(1)$-current net.
- For a unitary $V_1$ on $L^2(\mathbb{R}, dp)$, one defines the second quantization
  $$\Gamma(V_1) = 1 \oplus V_1 \oplus (V_1 \otimes V_1) \otimes \cdots$$
Endomorphisms of $\mathcal{A}^{(0)}(\mathbb{R}_+)$

**Definition**

A Longo-Witten endomorphism of a net $\mathcal{A}$ on $S^1$ is an endomorphism of $\mathcal{A}(\mathbb{R}_+)$ implemented by a unitary $V$ commuting with translation $T(t)$.

Simplest examples: $\text{Ad} \ T(s)$ for $s \geq 0$, inner symmetry (automorphism which preserves each local algebra $\mathcal{A}(I)$ and the vacuum state).

An **inner symmetric** function $\varphi$ is the boundary value of a bounded analytic function on the upper-half plane with $|\varphi(p)| = 1$, $\varphi(p) = \overline{\varphi(-p)}$ for $p \in \mathbb{R}$. Example: $\varphi(p) = e^{i\kappa p}$ with $\kappa \geq 0$, $\frac{p-i\kappa}{p+i\kappa}$ with $\kappa > 0$.

**Theorem (Longo-Witten ‘11)**

$\mathcal{A}^{(0)}$: the $U(1)$-current net

$V_\varphi := \Gamma(\varphi(P_1))$ implements a Longo-Witten endomorphism of $\mathcal{A}^{(0)}$, where $P_1$ is the generator of the translation on the one-particle space.
Definition (Two-dimensional net with boundary)

A net $\mathcal{B}$ of von Neumann algebras in two dimensions with boundary is an assignment of von Neumann algebra $\mathcal{B}(D)$ to a diamond $D$ in the right-half plane satisfying locality, covariance with respect to time-translation, etc.

- $\mathcal{A}$ be a net on $S^1$, restricted to $\mathbb{R}$.
- $V$: a unitary implementing a Longo-Witten endomorphism

Theorem (Longo-Witten '11)

If one defines $\mathcal{B}(D) = \mathcal{A}(I_1) \vee \text{Ad} V(\mathcal{A}(I_2))$, then $\mathcal{B}$ is a two-dimensional net with boundary.

Hence such a unitary implementing an endomorphisms of $\mathcal{A}(\mathbb{R}_+)$ leads to a net with boundary.
Borchers triple (Borchers ‘92)

- Local net: von Neumann algebras $\mathcal{A}(O)$ parametrized by open regions $O$
- Borchers triple: a **single** von Neumann algebra $\mathcal{M}$ acted on by spacetime translations

**Definition**

$\mathcal{M}$: vN algebra, $T$: positive-energy rep of $\mathbb{R}^2$, $\Omega$: vector, is a Borchers triple if $\Omega$ is cyclic and separating for $\mathcal{M}$ and

- $\text{Ad} T(a)(\mathcal{M}) \subset \mathcal{M}$ for $a \in W_R$, $T(a)\Omega = \Omega$

Correspondence: $\mathcal{A}(W_R) \Leftrightarrow \mathcal{M}$, where $W_R := \{a = (a_0, a_1): |a_0| < a_1\}$.

**Examples**

- Factorizing S-matrix models (Lechner ‘06)
- Deformations (Buchholz-Lechner-Summers ‘10, Dybalski-T. ‘11, Lechner ‘11, etc.)
Although there exist a plenty of conformal nets on $S^1$, there is no notion of interaction for one-dimensional theory. However, it is easy to construct a (noninteracting) two-dimensional net from a pair of nets on $S^1$. For two nets $\mathcal{A}_+, \mathcal{A}_-$ on $S^1$, we define

- a chiral net on $\mathbb{R}^2$: $\mathcal{A}(I \times J) := \mathcal{A}_+(I) \otimes \mathcal{A}_-(J)$
- a representation $U = U_+ \otimes U_-$ of $\text{PSL}(2, \mathbb{R}) \otimes \text{PSL}(2, \mathbb{R}) \supset \mathcal{P}_+^\uparrow$,
- the vacuum $\Omega = \Omega_+ \otimes \Omega_-$

A chiral net $\mathcal{A}$ is not interacting (Dybalski-T. ‘11). Such nets with a simple tensor product structure can be considered as free theory in 2 dimensions.
Some construction of wedge-local nets (T. ‘11)

- $\mathcal{A}_0$: a conformal net on $S^1 = \mathbb{R} \cup \{\infty\}$
- $P_0$: the generator of translation $T_0(t) = e^{itP_0}$

We set, for $\kappa > 0$,

- $\mathcal{M}_\kappa := \{x \otimes 1, e^{i\kappa P_0 \otimes P_0}(1 \otimes y) e^{-i\kappa P_0 \otimes P_0} : x \in \mathcal{A}_0(\mathbb{R}_-), y \in \mathcal{A}_0(\mathbb{R}_+)\}$
- $T := T_0 \otimes T_0$
- $\Omega := \Omega_0 \otimes \Omega_0$

Theorem (T. ‘11)

$(\mathcal{M}_\kappa, T, \Omega)$ is an asymptotically complete Borchers triple with the $S$-matrix $e^{i\kappa P_0 \otimes P_0}$.
Theorem (T. ‘11)

\((\mathcal{M}_\kappa, T, \Omega)\) is an asymptotically complete Borchers triple with the S-matrix \(e^{i\kappa P_0 \otimes P_0}\).

Proof) To see that it is a Borchers triple, what is nontrivial is the separating property of \(\Omega\). We set

\[
\mathcal{M}_\kappa := \{ x \otimes 1, e^{i\kappa P_0 \otimes P_0} (1 \otimes y) e^{-i\kappa P_0 \otimes P_0} : x \in \mathcal{A}_0(\mathbb{R}_-), y \in \mathcal{A}_0(\mathbb{R}_+) \}''
\]

\[
\mathcal{M}_\kappa^1 := \{ e^{i\kappa P_0 \otimes P_0} (x \otimes 1) e^{-i\kappa P_0 \otimes P_0}, 1 \otimes y : x \in \mathcal{A}_0(\mathbb{R}_+), y \in \mathcal{A}_0(\mathbb{R}_-) \}''
\]

Note that \(e^{i\kappa P_0 \otimes P_0} = \int e^{i\kappa p P_0} \otimes dE_0(p)\), where \(E_0\) is the spectral measure of \(P_0\). \(\mathcal{M}_\kappa\) and \(\mathcal{M}_\kappa^1\) commute since

\[
e^{i\kappa P_0 \otimes P_0} (x \otimes 1) e^{-i\kappa P_0 \otimes P_0} = \int e^{i\kappa p P_0} (x) e^{-i\kappa p P_0} \otimes dE_0(p).
\]

\(\Omega\) is cyclic for \(\mathcal{M}_\kappa^1\), hence separating for \(\mathcal{M}_\kappa\).
Construction of Borchers triples based on $\mathcal{A}^{(0)}$

We take again the $U(1)$-current net $\mathcal{A}^{(0)}$. For an inner symmetric function $\varphi$, set

- $\mathcal{H}^n := \mathcal{H}_1^\otimes n$
- $P_{i,j}^{m,n} := (1 \otimes \cdots \otimes P_1 \otimes \cdots \otimes 1) \otimes (1 \otimes \cdots \otimes P_1 \otimes \cdots \otimes 1)$, acting on $\mathcal{H}^m \otimes \mathcal{H}^n$, $1 \leq i \leq m$ and $1 \leq j \leq n$.
- $\varphi_{i,j}^{m,n} := \varphi(P_{i,j}^{m,n})$ (functional calculus on $\mathcal{H}^m \otimes \mathcal{H}^n$).
- $S_\varphi := \bigoplus_{m,n} \prod_{i,j} \varphi_{i,j}^{m,n}$

We can take the spectral decomposition of $S_\varphi$ only with respect to the right component:

$$S_\varphi = \bigoplus \int \prod_n \Gamma(\varphi(p_j P_1)) \otimes dE_1(p_1) \otimes \cdots \otimes dE_1(p_n)$$

Note that the integrand is a unitary operator which implements a Longo-Witten endomorphism for any value of $p_j \geq 0$. 
Construction of Borchers triples

We set

- \( \mathcal{M}_\varphi := \{ x \otimes 1, S_\varphi(1 \otimes y)S_\varphi^*: x \in \mathcal{A}^{(0)}(\mathbb{R}_-), y \in \mathcal{A}^{(0)}(\mathbb{R}_+) \}'' \)
- \( T := T_0 \otimes T_0 \)
- \( \Omega := \Omega_0 \otimes \Omega_0 \)

**Theorem (T. ‘11)**

\((\mathcal{M}_\varphi, T, \Omega)\) is an asymptotically complete Borchers triple with the S-matrix \( S_\varphi \).

Proof) To see that it is a Borchers triple, what is nontrivial is the separating property of \( \Omega \). We set

\[ \mathcal{M}_\varphi^1 := \{ S_\varphi(x \otimes 1)S_\varphi^*, 1 \otimes y: x \in \mathcal{A}^{(0)}(\mathbb{R}_+), y \in \mathcal{A}^{(0)}(\mathbb{R}_-) \}'' \]

\( \mathcal{M}_\varphi \) and \( \mathcal{M}_\varphi^1 \) commute since

\[ S_\varphi(x \otimes 1)S_\varphi^* = \bigoplus_n \int \text{Ad} \left( \prod_j \Gamma(\varphi(p_jP_1)) \right) (x) \otimes dE_1(p_1) \otimes \cdots dE_1(p_n). \]
Conclusion

Summary

- Interacting wedge-local nets parametrized by symmetric inner functions

Open problems

- Further examples with different asymptotic algebra
- Strict locality
- Massive analogue