Wightman and Osterwalder-Schrader axioms for 2d CFT   2025.04.08 Aalto.

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1. Conformal field theory = Quantum field theory + Conformal invariance

- Statistical mechanics, critical phenomena.

(• bootstrap. "consistency" $\Rightarrow$ OPE ...)

- Axiomatic / algebraic QFT. Vertex operator algebras, Wightman, OS.

2d CFT: Virasoro symmetry, many examples, classification, ...   (cf. Liouville)

How are the axioms related?   cf. Carpi-Kawahigashi-Longo-Weiner 2018

Huang-Kong 2007 "full field algebra", Moriwaki 2024 full VOA.

Main result: (Adamo-Moriwaki-T., arXiv:2407.18822)

Unitary full VOA + technical conditions gives n-point functions $\{S_n^{(a)}\}$ satisfying the OS axioms.
The reconstructed Wightman fields are defined on $\overline{F}$, F: state space of the full VOA.

(2d) OS axioms for $\{S_n\}$. $S_n(z_1, \dots, z_n)$ $z \in \mathbb{C} \simeq \mathbb{R}^2$.

(0) S_n is a distribution on $\{(z_1, \dots, z_n) \in \mathbb{C}^n : z_j \neq z_k,\ j \neq k\}$

(1) Euclidean invariance: $S_n(z_1, \dots z_n) = S_n(\gamma(z_1), \dots, \gamma(z_n)) \cdot C(\gamma)$, $\gamma \in E(2)$

(2) Symmetry: $S_n(z_1, \dots, z_n) = S_n(z_{\sigma(1)}, \dots, z_{\sigma(n)})$, $\sigma \in \mathfrak{S}_n$.   PSL(2,C) conformal

(3) Reflection positivity: $0 \leq \sum_{m,n} S_{m+n}(f_m^* \otimes f_n)$ $f_m^*(z_1, \dots z_m) = \overline{f(\bar{z}_m, \dots \bar{z}_1)}$

(4) Clustering $\mathrm{supp} f_n \subset \{\mathrm{Im}\, z_1 > \dots > \mathrm{Im}\, z_n\}$.

(5) Linear growth. $|S_n(f)| \leq \alpha (n!)^\beta \|f\|_{kn}$.

$(\text{OS } 0\text{-}5) \overset{\text{wick}}{\Longrightarrow}$ Wightman $W_n((t_1, x_1), \dots (t_n, x_n)) \Longleftrightarrow$ Gårding-Wightman, $\phi(t,x)$, Ω

$W_n(f_1 \otimes \dots \otimes f_n) = \langle \Omega, \phi(f_1) \dots \phi(f_n) \Omega \rangle$

Vertex operator algebra $(V, Y(\cdot, z)\, \Omega)$. V: vector space, $a \in V$. $Y(a,z) = \sum_n a_{(n)} z^{-n-1}$
formal series, $a_n \in \mathrm{End}(V)$, satisfying
covariance, vacuum, spectral condition, + Virasoro, $L(z) = \sum L_n z^{-n-2}$
locality: For $a, b \in V$, there is $N \in \mathbb{N}$ s.t. $[Y(a,z), Y(b,w)](z-w)^N = 0$.
unitarity: there is a scalar product $\langle , \rangle$ on V, s.t. for quasiprimary a, $a_n^* = a_{-n}$
$\tilde{\phi}_a(z) = \sum a_n z^{-n}$. $\phi(z)^* = \phi(z^{-1})$. $a_n = a_{(n+d-1)}$

Example (Heisenberg alg.). Lie alg $\{J_n, n \in \mathbb{Z}, K\}$. $[J_m, J_n] = m\delta_{m+n} K$. K central
Fock rep $V_\alpha \ni \Omega$, $J_{-n_1} \cdots J_{-n_k}\Omega$, $J_n \Omega = 0$ for $n > 0$, $J_0 = \alpha \in \mathbb{R}$.
$Y(\Omega, z) = \mathrm{id}$, $Y(J_{-1}\Omega, z) = J(z) = \sum J_n z^{-n}$. $Y(J_{-2}\Omega, z) = \partial J(z)$,
$Y(J_{-1}J_{-1}\Omega, z) = :J(z)^2:$, $:J_m J_n: = \begin{cases} J_m J_n & m > n \\ J_n J_m & m \leq n. \end{cases}$

Let $(V, Y, \Omega, \langle , \rangle)$ be a unitary VOA. Pick a quasiprimary vector a.

$\phi_a(z) = \sum a_{(n)} z^{-n-1}$ $\quad S_n(z_1, \dots, z_n) = \langle \Omega, \phi(z_1) \cdots \phi(z_n)\Omega \rangle$.

convergent if $|z_1| > |z_2| > \cdots > |z_n|$.

Locality $\Rightarrow$ S_n extends to $\{(z_1, \dots z_n) \in \mathbb{C}^n : z_j \neq z_k, j \neq k\}$. vacuum covariance $\Rightarrow$ invariance.

<u>Thm</u> If V satisfies "polynomial energy bounds" $\quad \|a_n \Psi\| \leq (n^p+1)\|(L_0+\mathbb{1})^q \Psi\|$

then $\{S_n\}$ satisfy (OS 0–5). $\quad \Rightarrow$ Linear growth

Unitarity $\Rightarrow$ $\langle \hat\phi_a(\zeta)\Psi_1, \Psi_2 \rangle = \langle \Psi_1, \hat\phi_a(\bar\zeta^{-1})\Psi_2 \rangle$

$S_n(z_1, \dots, z_n) = \sum_k z_1^{-k_1-1} \cdots z_n^{-k_n-1} \langle \Omega, a_{k_1} \cdots a_{k_n}\Omega \rangle$

$= \sum_k \left(\frac{z_1}{z_2}\right)^{(k_2 - \cdots - k_n)} \left(\frac{z_2}{z_3}\right)^{-k_2} \cdots \left(\frac{z_{n-1}}{z_n}\right)^{k_n} \langle \Omega, a_{k_1} \cdots a_{k_n}\Omega \rangle \cdot z_1^{-1} \cdots z_n^{-1}$

Put $\Psi_n = \int f_n(z_1, \dots z_n) \cdot \hat\phi_a(z_1) \cdots \hat\phi_a(z_n)\Omega \cdot |z_1|^{-2} \cdots |z_n|^{-2} \, d\tau_1 d\xi_1 \cdots d\tau_n d\xi_n$

$0 \leq \langle \Psi_n, \Psi_n \rangle = \sum_{m,n} \int \overline{f_m(z_1, \dots z_m)} \langle \hat\phi_a(z_1) \cdots \hat\phi_a(z_m)\Omega, \hat\phi_a(z_{m+1}) \cdots \hat\phi_a(z_{m+n})\Omega \rangle |z_1|^{-2} \cdots |z|^{-2} d\mu$

$= \sum_{m,n} \int \overline{f_m(z_1, \dots z_m)} \langle \Omega, \hat\phi_a(\bar z_m^{-1}) \cdots \hat\phi_a(\bar z_1^{-1}) \hat\phi_a(z_{m+1}) \cdots \hat\phi_a(z_{m+n})\Omega \rangle |z_1|^{-2} \cdots |z|^{-2} d\mu$

$z_j \to \bar z_j^{-1} = r(z_j)$ $\quad = \sum_{m,n} \overline{f_m(rz_1, \dots rz)} \langle \Omega, \hat\phi_a(z_m) \cdots \hat\phi_a(z_1) \hat\phi_a(z_{m+1}) \cdots \hat\phi_a(z_{m+n})\Omega \rangle |z_1|^{-2} \cdots |z|^{-2} d\mu$

$J(z) = \frac{1}{|z|^4}$ $\quad$ RP with respect to r of $\hat S_n(z_1, \dots z_n) = \langle \Omega, \hat\phi_a(z_1) \cdots \hat\phi_a(z_n)\Omega \rangle$.

RP of S: use invariance w.r.t. $r(z) = \frac{1+iz}{1-iz}$. $\quad \theta z = \bar z$.

$\{S_n\} \Rightarrow$ Gårding-Wightman on $\bar V$ $\quad \phi_a(f) = \sum_n a_n \hat f_n \quad \hat f_n = \frac{1}{(2\pi)} \int e^{-in\theta} f(\tan\frac{\theta}{2}, \tan\frac{\theta}{2}) \, d\theta_+ d\theta_-$

Full VOA. (F, Y, Ω). $Y(a, z, \bar z) = \sum_{r,s} a(r,s) z^{-r-1} \bar z^{-s-1}$.

Locality: For $a, b, c, u \in F$, there is M on $\{(\zeta_1, \zeta_2) \in \mathbb{C}^2 : \zeta_1 \neq \zeta_2\}$ real analytic

s.t. $\langle u, Y(a, \zeta_1, \bar\zeta_1) Y(b, \zeta_2, \bar\zeta_2)c \rangle = M(\zeta_1, \zeta_2)$ $\quad$ for $|\zeta_1| > |\zeta_2|$.

$\langle u, Y(b, \zeta_2, \bar\zeta_2) Y(a, \zeta_1, \bar\zeta_1)c \rangle = M(\zeta_1, \zeta_2)$ $\quad |\zeta_1| < |\zeta_2|$.

$\langle u, Y(Y(a, \zeta_1, \bar\zeta_1)b, \zeta_2, \bar\zeta_2)c \rangle = M(\zeta_0 + \zeta_2, \zeta_2)$ $\quad |\zeta_2| > |\zeta_0|$

<u>Thm</u> For unitary full VOA satisfying polynomial energy bounds, polynomial spectral density, local C_1-cofiniteness, $S_n^a(\zeta_1, \dots, \zeta_n) = \langle \Omega, Y(a, \zeta_1, \bar\zeta_1) \cdots Y(a, \zeta_n, \bar\zeta_n)\Omega \rangle$ satisfy (OS 0–5). Wightman field $\phi_a(f) = \sum_{r,s} a(r,s) \hat f(s,r)$.

$\hat f(r,s) = \frac{1}{(2\pi)^2} \int e^{-i(r\theta_+ + s\theta_-)} f(\tan\frac{\theta_+}{2}, \tan\frac{\theta_-}{2}) \, d\theta_+ d\theta_-$

<u>Example</u> V: Heisenberg alg. $\quad F = \bigoplus_\alpha V_\alpha \otimes \overline{V_\alpha}$. $\quad Y_\alpha(z, \bar z) = Y_\alpha(z) \otimes \overline{Y_\alpha}(\bar z)$.

$Y_\alpha : V_\beta \to V_{\alpha+\beta}$ intertwiner

Ising?