Construction of wedge-local QFT through Longo-Witten endomorphisms

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Introduction

Construction of two-dimensional QFT

- Lagrangian, Euclidean approach
- formfactor bootstrap program
- **operator-algebraic approach**

Recent progress in operator-algebraic approach:

- wedge-local net in 2-dim, based on a single von Neumann algebra and the Tomita-Takesaki theory (Borchers '92)
- factorizing S-matrix models (Lechner '08)

Present approach:

- input: chiral conformal net on S^1 (many examples)
- endomorphisms of the half-line algebra (Longo-Witten '11)

Main result

Interacting massless/massive nets of von Neumann algebras in 2-dim

Net of von Neumann algebras

Conventional quantum field

- ϕ: operator valued distribution on $\mathbb{R}^d$, $[\phi(x), \phi(y)] = 0$ if $x \perp y$
- U: the spacetime symmetry, $U(g)\phi(x)U(g)^* = \phi(gx)$
- Ω the vacuum vector

Net of observables

$\mathcal{A}(O)$: **von Neumann algebras** (weakly closed algebras of bounded operators on a Hilbert space $\mathcal{H}$) parametrized by open regions $O \Subset \mathbb{R}^d$

- isotony: $O_1 \subset O_2 \Rightarrow \mathcal{A}(O_1) \subset \mathcal{A}(O_2)$
- locality: $O_1 \perp O_2 \Rightarrow [\mathcal{A}(O_1), \mathcal{A}(O_2)] = 0$
- Poincaré covariance: $\exists U$: positive energy rep of $\mathcal{P}_+^\uparrow$ such that $U(g)\mathcal{A}(O)U(g)^* = \mathcal{A}(gO)$
- vacuum: $\exists \Omega$ such that $U(g)\Omega = \Omega$ and cyclic for $\mathcal{A}(O)$

Correspondence: $\mathcal{A}(O) = \{e^{i\phi(f)} : \mathrm{supp} f \subset O\}''$

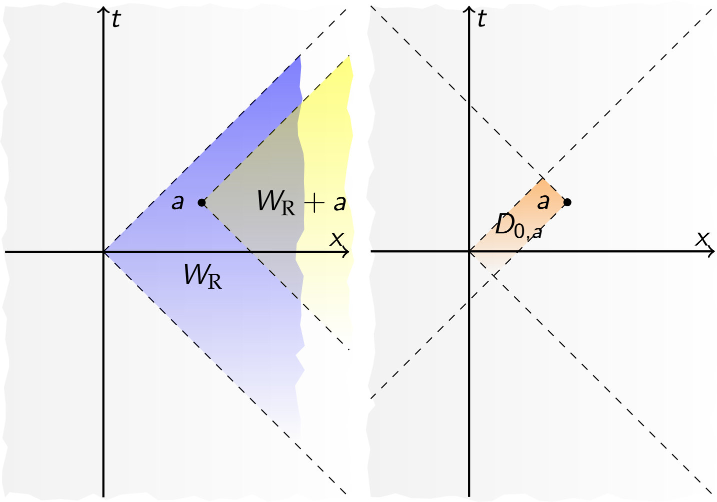
Borchers triple (Borchers '92)

Nets of von Neumann algebras are still complicated to construct directly, since they involve infinitely many von Neumann algebras. The notion of **Borchers triple** reduces the question to a single von Neumann algebra if the spacetime has **dimension 2**.

- local net: von Neumann algebras $\mathcal{A}(O)$ parametrized by open regions O
- Borchers triple: a **single** von Neumann algebra $\mathcal{M}$ acted on by spacetime translations

Idea (net $\Longrightarrow$ Borchers triple): to consider only the algebra $\mathcal{A}(W_\mathrm{R})$ corresponding to the **wedge** region $W_\mathrm{R} := \{a = (a_0, a_1) : |a_0| < a_1\}$. Any double cone $D_{a,b}$ is of the form $(W_\mathrm{R} + a) \cap (W_L + b)$ and (if the net $\mathcal{A}$ satisfies the Haag duality) $\mathcal{A}$ **can be recovered by** $\mathcal{A}(W_\mathrm{R})$, its commutant $\mathcal{A}(W_\mathrm{R})' := \{x \in B(\mathcal{H}) : [x, y] = 0 \text{ for all } y \in B(\mathcal{H})\}$ and the translation:

$$\mathcal{A}(D_{a,b}) = (U(a)\mathcal{A}(W_\mathrm{R})U(a)^*) \cap (U(b)\mathcal{A}(W_\mathrm{R})'U(b)^*).$$


t
a
$W_{\mathrm{R}} + a$
x
W_{R}
t
a
$D_{0,a}$
x

Borchers triple (Borchers '92)

Definition

$(\mathcal{M}, T, \Omega)$, where $\mathcal{M}$: vN algebra, T: positive-energy rep of $\mathbb{R}^2$, Ω: vector, is a Borchers triple if Ω is cyclic and separating for $\mathcal{M}$ and

- $\mathrm{Ad}\, T(a)(\mathcal{M}) \subset \mathcal{M}$ for $a \in W_{\mathrm{R}}$, $T(a)\Omega = \Omega$

Borchers triple $\Longrightarrow$ net

If one defines a "net" by $\mathcal{A}(D_{a,b}) := (U(a)\mathcal{M}U(a)^*) \cap (U(b)\mathcal{M}'U(b)^*)$, then T can be extended to a rep U of Poincaré group and satisfies all the axioms of local net **except the cyclicity of vacuum**.

Problem

- to construct new Borchers triples (wedge-local QFT)
- to show the cyclicity of vacuum (strict locality)

Chiral conformal net

Definition

A **conformal net** on S^1 is a map $\mathcal{A}_0$ from the set of intervals in S^1 into the set of von Neumann algebras on $\mathcal{H}_0$ which satisfies

- isotony: $I \subset J \Rightarrow \mathcal{A}_0(I) \subset \mathcal{A}_0(J)$.
- locality: $I \cap J \Rightarrow [\mathcal{A}_0(I), \mathcal{A}_0(J)] = 0$.
- Möbius covariance: $\exists U_0$: positive energy rep of $\mathrm{PSL}(2,\mathbb{R})$ such that $\mathrm{Ad}\, U_0(g)\mathcal{A}_0(I) = \mathcal{A}_0(gI)$.
- vacuum: $\exists \Omega_0$ such that $U_0(g)\Omega_0 = \Omega_0$ and cyclic for $\mathcal{A}_0(I)$.

Many examples: $U(1)$-current (free massless boson), Free massless fermion, Virasoro nets (stress energy tensor), Minimal models, Loop group nets (WZW models).

In the present work, important are the $U(1)$-**current net** and the **free massless fermion** which admit the **Fock space structure**.

Tensor product construction (trivial)

One-dimensional objects:

- $\mathcal{A}_0$: a conformal net on $S^1 = \mathbb{R} \cup \{\infty\}$
- P_0: the generator of translation $T_0(t) = e^{itP_0}$
- Ω_0: the vacuum

Two-dimensional objects:

- $\mathcal{M} := \{x \otimes \mathbb{1}, \mathbb{1} \otimes y : x \in \mathcal{A}_0(\mathbb{R}_-), y \in \mathcal{A}_0(\mathbb{R}_+)\}''$
- $T := T_0 \otimes T_0$
- $\Omega := \Omega_0 \otimes \Omega_0$

Theorem

$(\mathcal{M}, T, \Omega)$ *is a massless Borchers triple with the S-matrix* $\mathbb{1}$.

Some construction of Borchers triples (T. '11)

- $\mathcal{A}_0$: a conformal net on $S^1 = \mathbb{R} \cup \{\infty\}$
- P_0: the generator of translation $T_0(t) = e^{itP_0}$

We set, for $\kappa > 0$,

- $\mathcal{M}_\kappa := \{x \otimes \mathbb{1}, \mathrm{Ad}\, e^{i\kappa P_0 \otimes P_0}(\mathbb{1} \otimes y) : x \in \mathcal{A}_0(\mathbb{R}_-), y \in \mathcal{A}_0(\mathbb{R}_+)\}''$
- $T := T_0 \otimes T_0$
- $\Omega := \Omega_0 \otimes \Omega_0$

Theorem (T. '11)

$(\mathcal{M}_\kappa, T, \Omega)$ *is a massless Borchers triple with the S-matrix* $e^{i\kappa P_0 \otimes P_0}$.

Key of the proof: to show that Ω is separating for $\mathcal{M}_\kappa$ (wedge-locality).
Important to note: $\mathrm{Ad}\, e^{itP_0}$ for $t \geq 0$ is an **endomorphism** of $\mathcal{A}_0(\mathbb{R}_+)$, because translation by t takes $\mathbb{R}_+$ into itself.

Endomorphisms of the U(1)-current algebra $\mathcal{A}_{U(1)}(\mathbb{R}_+)$

Definition

A **Longo-Witten endomorphism** of a Möbius covariant net $\mathcal{A}_0$ is an endomorphism of $\mathcal{A}_0(\mathbb{R}_+)$ implemented by a unitary V_0 commuting with translation $T_0(t)$.

Simplest examples: $\mathrm{Ad}\, T_0(s)$ for $s \geq 0$, gauge symmetry (automorphism which preserves each local algebra $\mathcal{A}_0(I)$ and the vacuum state)

An **inner symmetric** function φ is the boundary value of a bounded analytic function on the upper-half plane with $|\varphi(p)| = 1, \varphi(p) = \overline{\varphi(-p)}$ for $p \in \mathbb{R}$. Example: $\varphi(p) = e^{i\kappa p}$ with $\kappa \geq 0$, $\frac{p - i\kappa}{p + i\kappa}$ with $\kappa > 0$

Theorem (Longo-Witten '11)

$\mathcal{A}_{U(1)}$: the $U(1)$-current net generated by the free massless current.
$V_\varphi := \Gamma(\varphi(P_1))$ implements a Longo-Witten endomorphism of $\mathcal{A}_{U(1)}$, where P_1 is the generator of the translation on the one-particle space.

We construct directly the scattering operator.

$\mathcal{A}_{U(1)}$: the $U(1)$-current net.

For an inner symmetric function φ, set

- $\mathcal{H}^n := \mathcal{H}_1^{\otimes n}$
- $P_{i,j}^{m,n} := (\mathbb{1} \otimes \cdots \otimes \underset{i\text{-th}}{P_1} \otimes \cdots \otimes \mathbb{1}) \otimes (\mathbb{1} \otimes \cdots \otimes \underset{j\text{-th}}{P_1} \otimes \cdots \otimes \mathbb{1})$, acting on $\mathcal{H}^m \otimes \mathcal{H}^n$, $1 \leq i \leq m$ and $1 \leq j \leq n$.
- $\varphi_{i,j}^{m,n} := \varphi(P_{i,j}^{m,n})$ (functional calculus on $\mathcal{H}^m \otimes \mathcal{H}^n$).
- $S_\varphi := \bigoplus_{m,n} \prod_{i,j} \varphi_{i,j}^{m,n}$

We can take the spectral decomposition of S_φ only with respect to the right component:

- $S_\varphi = \bigoplus_n \int \prod_j \Gamma(\varphi(p_j P_1)) \otimes dE_1(p_1) \otimes \cdots \otimes dE_1(p_n)$

Note that the integrand is a unitary operator which implements a Longo-Witten endomorphism for any value of $p_j \geq 0$.

Construction of massless Borchers triples

We set

- $\mathcal{M}_\varphi := \{x \otimes \mathbb{1}, S_\varphi(\mathbb{1} \otimes y)S_\varphi^* : x \in \mathcal{A}_{U(1)}(\mathbb{R}_-), y \in \mathcal{A}_{U(1)}(\mathbb{R}_+)\}''$
- $T := T_0 \otimes T_0$, $\Omega := \Omega_0 \otimes \Omega_0$

Theorem (T. '11)

$(\mathcal{M}_\varphi, T, \Omega)$ *is a massless Borchers triple with the S-matrix* S_φ.

Further examples:
There is the inclusion $\mathcal{A}_{U(1)} \subset \mathcal{F}$, where $\mathcal{F}$ is the net of the **free complex fermion** and $\mathcal{A}_{U(1)}$ is the fixed point with resect to the $U(1)$-gauge action.
The net $\mathcal{F} \otimes \mathcal{F}$ can be "twisted" by a similar procedure, and some of the twisting procedures commute with the $U(1)$-gauge action, hence give rise to twisting of $\mathcal{A}_{U(1)} \otimes \mathcal{A}_{U(1)}$.
The S-matrix **does not preserve** the subspace of one right-moving + one left-moving waves. In other words, they represent "particle production" (Bischoff-T. '11).

Construction of massive Borchers triples based on $\mathcal{A}_{U(1)}$

Note: $\mathcal{A}_{U(1)}(\mathbb{R}_+)$, together with lightlike translations $\Gamma(e^{isP_1}), \Gamma(e^{\frac{it}{P_1}})$, is equivalent to the massive free net.

- take again two copies $\mathcal{A}_{U(1)} \otimes \mathcal{A}_{U(1)}$
- $Q_{i,j}^{m,n} := (\mathbb{1} \otimes \cdots \otimes \underset{i\text{-th}}{\frac{1}{P_1}} \otimes \cdots \otimes \mathbb{1}) \otimes (\mathbb{1} \otimes \cdots \otimes \underset{j\text{-th}}{P_1} \otimes \cdots \otimes \mathbb{1})$
- $\varphi_{i,j}^{m,n} := \varphi(Q_{i,j}^{m,n})$, $R_\varphi := \bigoplus_{m,n} \prod_{i,j} \varphi_{i,j}^{m,n}$
- $\mathcal{M}_\varphi := \{x \otimes \mathbb{1}, R_\varphi(\mathbb{1} \otimes y)R_\varphi^* : x, y \in \mathcal{A}_{U(1)}(\mathbb{R}_+)\}''$
- T: $\Gamma(e^{isP_1}) \otimes \Gamma(e^{isP_1})$, $\Gamma(e^{\frac{it}{P_1}}) \otimes \Gamma(e^{\frac{it}{P_1}})$, $\Omega := \Omega_0 \otimes \Omega_0$

Theorem (T., in preparation)

$(\mathcal{M}_\varphi, T, \Omega)$ is a massive Borchers triple.

S-matrix should be **factorizing** with two species of particle. One can do a similar construction by taking the free fermion and the generator Q_0 of the gauge symmetry. This is **local** and **interacting** for some parameters due to **modular nuclearity**, and should be equivalent to the Federbush model.

Conclusion

Summary

- construction of Borchers triples through Longo-Witten endomorphisms
- wedge-local massless models with particle-production like phenomena
- local massive models

Open problems

- massive models with true particle production? $\Leftarrow$ to find non-second quantization Longo-Witten endomorphisms
- higher dimensional extension? $\Leftarrow$ understanding restriction of higher dimensional net