Construction of wedge-local QFT through Longo-Witten endomorphisms

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Introduction

Construction of two-dimensional QFT
- Lagrangian, Euclidean approach
- formfactor bootstrap program
- operator-algebraic approach

Recent progress in operator-algebraic approach:
- wedge-local net in 2-dim, based on a single von Neumann algebra and the Tomita-Takesaki theory (Borchers ‘92)
- factorizing S-matrix models (Lechner ‘08)

Present approach:
- input: chiral conformal net on $S^1$ (many examples)
- endomorphisms of the half-line algebra (Longo-Witten ‘11)

Main result
Interacting massless/massive nets of von Neumann algebras in 2-dim
Net of von Neumann algebras

Conventional quantum field

- $\phi$: operator valued distribution on $\mathbb{R}^d$, $[\phi(x), \phi(y)] = 0$ if $x \perp y$
- $U$: the spacetime symmetry, $U(g)\phi(x)U(g)^* = \phi(gx)$
- $\Omega$ the vacuum vector

Net of observables

$\mathcal{A}(O)$: von Neumann algebras (weakly closed algebras of bounded operators on a Hilbert space $\mathcal{H}$) parametrized by open regions $O \subseteq \mathbb{R}^d$

- isotony: $O_1 \subset O_2 \Rightarrow \mathcal{A}(O_1) \subset \mathcal{A}(O_2)$
- locality: $O_1 \perp O_2 \Rightarrow [\mathcal{A}(O_1), \mathcal{A}(O_2)] = 0$
- Poincaré covariance: $\exists U$: positive energy rep of $\mathcal{P}_+$ such that $U(g)\mathcal{A}(O)U(g)^* = \mathcal{A}(gO)$
- vacuum: $\exists \Omega$ such that $U(g)\Omega = \Omega$ and cyclic for $\mathcal{A}(O)$

Correspondence: $\mathcal{A}(O) = \{e^{i\phi(f)} : \text{supp} f \subset O\}$
Nets of von Neumann algebras are still complicated to construct directly, since they involve infinitely many von Neumann algebras. The notion of **Borchers triple** reduces the question to a single von Neumann algebra if the spacetime has dimension 2.

- **local net**: von Neumann algebras $\mathcal{A}(O)$ parametrized by open regions $O$

- **Borchers triple**: a **single** von Neumann algebra $\mathcal{M}$ acted on by spacetime translations

**Idea (net $\implies$ Borchers triple)**: to consider only the algebra $\mathcal{A}(W_R)$ corresponding to the **wedge** region $W_R := \{ a = (a_0, a_1) : |a_0| < a_1 \}$. Any double cone $D_{a,b}$ is of the form $(W_R + a) \cap (W_L + b)$ and (if the net $\mathcal{A}$ satisfies the Haag duality) $\mathcal{A}$ can be recovered by $\mathcal{A}(W_R)$, its commutant $\mathcal{A}(W_R)' := \{ x \in B(\mathcal{H}) : [x, y] = 0 \text{ for all } y \in B(\mathcal{H}) \}$ and the translation:

$$\mathcal{A}(D_{a,b}) = (U(a)\mathcal{A}(W_R)U(a)^*) \cap (U(b)\mathcal{A}(W_R)'U(b)^*).$$
Borchers triple (Borchers ‘92)

**Definition**

\((\mathcal{M}, T, \Omega)\), where \(\mathcal{M}\): von Neumann algebra, \(T\): positive-energy rep of \(\mathbb{R}^2\), \(\Omega\): vector, is a Borchers triple if \(\Omega\) is cyclic and separating for \(\mathcal{M}\) and

- \(\text{Ad} \, T(a)(\mathcal{M}) \subset \mathcal{M}\) for \(a \in W_\mathbb{R}\), \(T(a)\Omega = \Omega\)

**Borchers triple \implies net**

If one defines a ”net” by \(A(D_{a,b}) := (U(a)\mathcal{M}U(a)^*) \cap (U(b)\mathcal{M}'U(b)^*)\), then \(T\) can be extended to a rep \(U\) of Poincaré group and satisfies all the axioms of local net except the cyclicity of vacuum.

**Problem**

- to construct new Borchers triples (wedge-local QFT)
- to show the cyclicity of vacuum (strict locality)
Chiral conformal net

**Definition**

A **conformal net** on $S^1$ is a map $\mathcal{A}_0$ from the set of intervals in $S^1$ into the set of von Neumann algebras on $\mathcal{H}_0$ which satisfies

- **isotony:** $I \subset J \Rightarrow \mathcal{A}_0(I) \subset \mathcal{A}_0(J)$.
- **locality:** $I \cap J \Rightarrow [\mathcal{A}_0(I), \mathcal{A}_0(J)] = 0$.
- **Möbius covariance:** $\exists U_0 :$ positive energy rep of $\text{PSL}(2, \mathbb{R})$ such that $\text{Ad} U_0(g) \mathcal{A}_0(I) = \mathcal{A}_0(gI)$.
- **vacuum:** $\exists \Omega_0$ such that $U_0(g)\Omega_0 = \Omega_0$ and cyclic for $\mathcal{A}_0(I)$.

Many examples: $U(1)$-current (free massless boson), Free massless fermion, Virasoro nets (stress energy tensor), Minimal models, Loop group nets (WZW models).

In the present work, important are the $U(1)$-current net and the **free massless fermion** which admit the **Fock space structure**.
Tensor product construction (trivial)

One-dimensional objects:
- $\mathcal{A}_0$: a conformal net on $S^1 = \mathbb{R} \cup \{\infty\}
- P_0$: the generator of translation $T_0(t) = e^{itP_0}$
- $\Omega_0$: the vacuum

Two-dimensional objects:
- $\mathcal{M} := \{x \otimes 1, 1 \otimes y : x \in \mathcal{A}_0(\mathbb{R}_-), y \in \mathcal{A}_0(\mathbb{R}_+)\}$
- $T := T_0 \otimes T_0$
- $\Omega := \Omega_0 \otimes \Omega_0$

Theorem

$(\mathcal{M}, T, \Omega)$ is a massless Borchers triple with the S-matrix $\mathbb{1}$.
Some construction of Borchers triples (T. ‘11)

- \( \mathcal{A}_0 \): a conformal net on \( S^1 = \mathbb{R} \cup \{\infty\} \)
- \( P_0 \): the generator of translation \( T_0(t) = e^{itP_0} \)

We set, for \( \kappa > 0 \),
- \( \mathcal{M}_\kappa := \{ x \otimes 1, \text{Ad} e^{i\kappa P_0 \otimes P_0} (1 \otimes y) : x \in \mathcal{A}_0(\mathbb{R}_-), y \in \mathcal{A}_0(\mathbb{R}_+) \}'' \)
- \( T := T_0 \otimes T_0 \)
- \( \Omega := \Omega_0 \otimes \Omega_0 \)

**Theorem (T. ‘11)**

\( (\mathcal{M}_\kappa, T, \Omega) \) is a massless Borchers triple with the S-matrix \( e^{i\kappa P_0 \otimes P_0} \).

Key of the proof: to show that \( \Omega \) is separating for \( \mathcal{M}_\kappa \) (wedge-locality).
Important to note: \( \text{Ad} e^{itP_0} \) for \( t \geq 0 \) is an **endomorphism** of \( \mathcal{A}_0(\mathbb{R}_+) \), because translation by \( t \) takes \( \mathbb{R}_+ \) into itself.
Endomorphisms of the $U(1)$-current algebra $\mathcal{A}_{U(1)}(\mathbb{R}_+)$

**Definition**

A **Longo-Witten endomorphism** of a Möbius covariant net $\mathcal{A}_0$ is an endomorphism of $\mathcal{A}_0(\mathbb{R}_+)$ implemented by a unitary $V_0$ commuting with translation $T_0(t)$.

Simplest examples: $\text{Ad} T_0(s)$ for $s \geq 0$, gauge symmetry (automorphism which preserves each local algebra $\mathcal{A}_0(I)$ and the vacuum state)

An **inner symmetric** function $\varphi$ is the boundary value of a bounded analytic function on the upper-half plane with $|\varphi(p)| = 1, \varphi(p) = \overline{\varphi(-p)}$ for $p \in \mathbb{R}$. Example: $\varphi(p) = e^{i\kappa p}$ with $\kappa \geq 0$, $\frac{p-i\kappa}{p+i\kappa}$ with $\kappa > 0$

**Theorem (Longo-Witten ‘11)**

$\mathcal{A}_{U(1)}$: the $U(1)$-current net generated by the free massless current. $V_\varphi := \Gamma(\varphi(P_1))$ implements a Longo-Witten endomorphism of $\mathcal{A}_{U(1)}$, where $P_1$ is the generator of the translation on the one-particle space.
We construct directly the scattering operator.

\( \mathcal{A}_{U(1)} \): the \( U(1) \)-current net.

For an inner symmetric function \( \varphi \), set

- \( \mathcal{H}^n := \mathcal{H}_1^{\otimes n} \)
- \( P_{i,j}^{m,n} := (1 \otimes \cdots \otimes P_1 \otimes \cdots \otimes 1) \otimes (1 \otimes \cdots \otimes P_1 \otimes \cdots \otimes 1) \),
  acting on \( \mathcal{H}^m \otimes \mathcal{H}^n \), \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \).
- \( \varphi_{i,j}^{m,n} := \varphi(P_{i,j}^{m,n}) \) (functional calculus on \( \mathcal{H}^m \otimes \mathcal{H}^n \)).
- \( S_\varphi := \bigoplus_{m,n} \prod_{i,j} \varphi_{i,j}^{m,n} \)

We can take the spectral decomposition of \( S_\varphi \) only with respect to the right component:

\[
S_\varphi = \bigoplus_n \int \prod_j \Gamma(\varphi(p_j P_1)) \otimes dE_1(p_1) \otimes \cdots \otimes dE_1(p_n)
\]

Note that the integrand is a unitary operator which implements a Longo-Witten endomorphism for any value of \( p_j \geq 0 \).
Construction of massless Borchers triples

We set

- \( \mathcal{M}_\varphi := \{ x \otimes 1, S_\varphi(1 \otimes y)S^*_\varphi : x \in \mathcal{A}_{U(1)}(\mathbb{R}^-), y \in \mathcal{A}_{U(1)}(\mathbb{R}^+)\}'' \)
- \( T := T_0 \otimes T_0, \Omega := \Omega_0 \otimes \Omega_0 \)

**Theorem (T. ‘11)**

\((\mathcal{M}_\varphi, T, \Omega)\) is a massless Borchers triple with the S-matrix \( S_\varphi \).

**Further examples:**
There is the inclusion \( \mathcal{A}_{U(1)} \subset \mathcal{F} \), where \( \mathcal{F} \) is the net of the free complex fermion and \( \mathcal{A}_{U(1)} \) is the fixed point with respect to the \( U(1) \)-gauge action. The net \( \mathcal{F} \otimes \mathcal{F} \) can be ”twisted” by a similar procedure, and some of the twisting procedures commute with the \( U(1) \)-gauge action, hence give rise to twisting of \( \mathcal{A}_{U(1)} \otimes \mathcal{A}_{U(1)} \).

The S-matrix does not preserve the subspace of one right-moving + one left-moving waves. In other words, they represent ”particle production” (Bischoff-T. ’11).
Construction of massive Borchers triples based on $\mathcal{A}_{U(1)}$

Note: $\mathcal{A}_{U(1)}(\mathbb{R}_+)$, together with lightlike translations $\Gamma(e^{isP_1}), \Gamma(e^{itP_1})$, is equivalent to the massive free net.

- take again two copies $\mathcal{A}_{U(1)} \otimes \mathcal{A}_{U(1)}$
- $Q_{i,j}^{m,n} := (1 \otimes \cdots \otimes \frac{1}{P_1} \otimes \cdots \otimes 1) \otimes (1 \otimes \cdots \otimes P_1 \otimes \cdots \otimes 1)$
- $\varphi_{i,j}^{m,n} := \varphi(Q_{i,j}^{m,n}), R_\varphi := \bigoplus_{m,n} \prod_{i,j} \varphi_{i,j}^{m,n}$
- $\mathcal{M}_\varphi := \{ x \otimes 1, R_\varphi(1 \otimes y) R_\varphi^* : x, y \in \mathcal{A}_{U(1)}(\mathbb{R}_+) \}''$
- $\mathcal{T}: \Gamma(e^{isP_1}) \otimes \Gamma(e^{isP_1}), \Gamma(e^{itP_1}) \otimes \Gamma(e^{itP_1}), \Omega := \Omega_0 \otimes \Omega_0$

**Theorem (T., in preparation)**

$(\mathcal{M}_\varphi, \mathcal{T}, \Omega)$ is a massive Borchers triple.

S-matrix should be factorizing with two species of particle. One can do a similar construction by taking the free fermion and the generator $Q_0$ of the gauge symmetry. This is local and interacting for some parameters due to modular nuclearity, and should be equivalent to the Federbush model.
Conclusion

Summary
- construction of Borchers triples through Longo-Witten endomorphisms
- wedge-local massless models with particle-production like phenomena
- local massive models

Open problems
- massive models with true particle production? ⇐ to find non-second quantization Longo-Witten endomorphisms
- higher dimensional extension? ⇐ understanding restriction of higher dimensional net