Wightman fields for two-dimensional conformal field theory

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Mathematical Quantum Field Theory

- Quantum field theory (QFT): infinite degrees of freedom on continuum ((infrared and) ultraviolet divergences)
- Axiomatic approaches: **Wightman**, Osterwalder-Schrader, Araki-Haag-Kastler...
- Examples: free fields, $\mathcal{P}(\phi)_2$ models (and more (super)renormalizable models), some gauge theories in $d = 1+1, 1+2$, ϕ^4_3 model, integrable models in $d = 1+1$, (see Summers '12)
  conformal field theories (CFT) in $d = 1+1$ (as conformal nets).
- Physically, CFTs should capture universal properties of larger classes of QFT (being fixed points of the renormalization group flow).
- Two-dimensional CFTs have infinite-dimensional symmetries, many examples. Purely mathematical interests.

We construct Wightman fields for a class of two-dimensional CFT.

Constructive QFT, VOA and CFT

- CFT should be RG fixed points, but the constructive methods usuallly require non-zero mass. Exceptions: Ising model (Smirnov et al.), Liouville theory (Kupiainen et al.)...
- Which (traditional) axioms have been checked? (cf. rational CFT (Schweigert et al.), conformal bootstrap (Rychkov et al.))

- Algebraic approach to CFT: **Vertex Operator Algebras** (VOAs).
- Carpi-Kawahigashi-Longo-Weiner '18: Chiral **unitary** (strongly local) VOA $\Longrightarrow$ conformal net on S^1, Haag-Kastler axioms.

In this talk, we discuss

- **Wightman** fields for a class of full two-dimensional CFT.
- Schwinger functions and **Osterwalder-Schrader** axioms, including the linear growth condition (work in progress).
- Possible relations between CFT and finite-volume **massive** theories.

Two-dimensional conformal field theory

- In relativistic QFT in $d = 1 + 1$, one puts the Lorentzian metric $(x, y) = x_0 y_0 - x_1 y_1$ on $\mathbb{R}^{1+1}$.
- The conformal group (transformations of $\mathbb{R}^{1+1}$ that preserve the metric up to a function) is $\mathrm{Diff}(\mathbb{R}) \times \mathrm{Diff}(\mathbb{R})$, acting on the lightrays $x_0 \pm x_1 = 0$.
- In a quantum theory, $\mathrm{Diff}(\mathbb{R}) \times \mathrm{Diff}(\mathbb{R})$ gets a (projective) unitary representation.
- There are observables that are invariant by $\iota \times \mathrm{Diff}(\mathbb{R})$ (or $\mathrm{Diff}(\mathbb{R}) \times \iota$): chiral observables.
- **Chiral observables** are quantum fields living on the lightray $\mathbb{R}$, extending to S^1 by the conformal covariance.
- Many examples: free fields (boson/fermion), $\mathrm{Diff}(S^1)$-symmetry itself (the Virasoro algebra), the WZW models (loop groups).
- A **Full CFT** is an extension of chiral observables in such a way to keep locality, covariance etc.

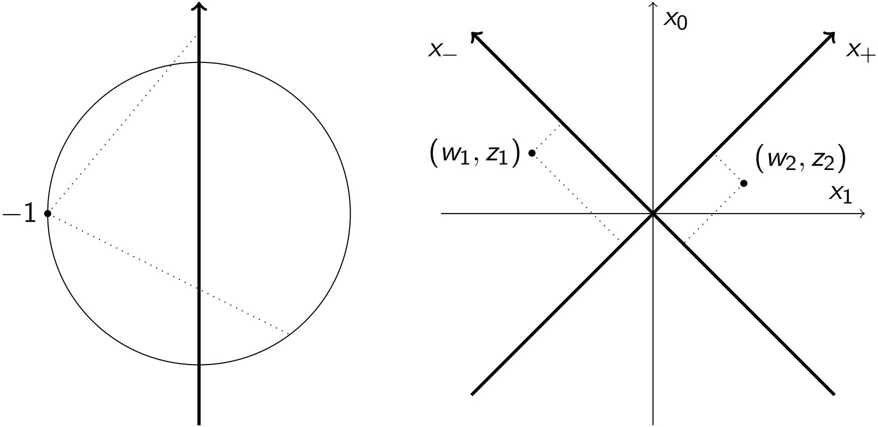

Figure: The stereographic projection of $S^1 \subset \mathbb{C}$ to $\mathbb{R}$, the lightray decomposition of $\mathbb{R}^{1+1}$.

Axiomatic approaches to 2d CFT

- Wightman fields $\{\phi_j\}$:
  - Operator-valued distributions ϕ_j on $\mathbb{R}^{1+1}$. For $f \in \mathscr{S}(\mathbb{R}^{1+1})$, $\phi_j(f)$ gives an (unbounded) operator on a common invariant dense domain in a Hibert space $\mathcal{H}$.
  - **Locality**: $[\phi_j(f), \phi_k(g)] = 0$ if $\mathrm{supp}\, f$ is spacelike to $\mathrm{supp}\, g$, **Möbius/conformal** covariance, spectrum condition, vacuum...
  - ϕ_j is **chiral** if it depends only on $x_0 \pm x_1$.
- Unitary Vertex Operator Algebra:
  - Algebra generated by formal series $Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$, V a linear space, $a \in V$ and $a_{(n)} \in \mathrm{End}(V)$, V carries a nice scalar product.
  - **Locality**: $[Y(a,w), Y(b,z)](w-z)^N = 0$ where N depends on a, b, **Möbius/Virasoro** covariance, grading, vacuum, "conformal Hamiltonian" L_0 (generator of rotations) diagonalized on V...

(Carpi-Kawahigashi-Longo-Weiner '18) For a primary vector v_j in a VOA V with "polynomial energy bounds", let $\phi_j(z) = \sum_n \phi_{j,n} z^{-d_j - n} = Y(v_j, z)$. $\mathcal{H} = \overline{V}$, $\mathscr{D} = C^\infty(L_0)$. $\phi_j(f) = \sum_n \hat{f}_n \phi_{j,n}$ is a chiral field, $f \in C^\infty(S^1)$. (thinking of $z \in S^1$)

Examples: the U(1)-current

- The derivative of the massless scalar field decomposes into the left and right chiral components: the $\mathrm{U}(1)$-current. $[J(x), J(y)] = i\delta'(x-y)$, or $[J_m, J_n] = m\delta_{m+n}, m, n \in \mathbb{Z}$. An **infinite-dimensional Lie algebra**.
- This algebra has the **vacuum representation** $\mathcal{H}_0$, where there is a distinguished vector Ω such that $J_n\Omega = 0$ for all $n \geq 0$ and the whole representation is spanned by $J_{-k_1} \cdots J_{-k_n}\Omega$, $k_j > 0$. (bosonic Fock space with the one-particle space spanned by $J_{-k}\Omega$) It admits a scalar product and $J_n^* = J_{-n}$.
- In the vacuum representation, the Heisenberg VOA is generated by $J(z) = \sum_n z^{-n-1} J_n$. This is **unitary**.
- There is a Virasoro field (stress-energy tensor) $L(z) = \sum_n L_n z^{-2-n}$, $L_n = \frac{1}{2}\sum_k : J_{n-k}J_k :$, satisfying
$$[L_m, L_n] = (m+n)L_{m+n} + \frac{1}{12}m(m^2-1)\delta_{m,-n}.$$
- $\sigma(L_0) = \mathbb{N} \cup \{0\}$. $\mathrm{Diff}(S^1)$, $\mathrm{M\ddot{o}b}$-covariance $(\{L_n\}, \{L_1, L_0, L_{-1}\})$

Examples: the U(1)-current, chiral Wightman field

- Chiral **Wightman field** on S^1: We define

$$J : C^\infty(S^1) \ni f \mapsto J(f) := \sum_n \hat{f}_n J_n$$

  on $\mathscr{D} := C^\infty(L_0)$, $f(e^{i\theta}) = \sum_n e^{in\theta}\hat{f}_n$. Temperedness from $\|J_n\Psi\| \leq C(|n|+1)\|(L_0+\mathbb{1})^{\frac{1}{2}}\Psi\|$.
  There is a positive-energy representation $U : \mathrm{Diff}(S^1) \to \mathcal{B}(\mathcal{H})$, integrating $\{L_n\}$, that makes J covariant.
- From this, a usual **Wightman field on** $\mathbb{R}^{1+1}$ can be constructed as follows:
  - Identify S^1 and $\mathbb{R} \cup \{\infty\}$ by the stereographic projection
  - Identify $\mathbb{R}$ and a lightray in $\mathbb{R}^{1+1}$
  - For $f \in \mathscr{S}(\mathbb{R}^{1+1})$, reduce it to a lightray by $f_+(x_+) = \int f(x_+, x_-)dx_-$. Define $f \mapsto J(f_+)$.
- U can be restricted to dilations and translations of $\mathbb{R}$.
- J, U, Ω satisfy the Wightman axioms, J depends only on x_+.

Superselection sectors and charged primary fields

- The $\mathrm{U}(1)$-current admits a family of representations $\mathcal{H}_\alpha$ parametrized by $\alpha \in \mathbb{R}$.
  - $\mathcal{H}_\alpha \ni \Omega_\alpha$ such that $J_n\Omega_\alpha = 0$ for $n > 0$, $J_0 = \alpha$.
  - $\mathcal{H}_\alpha = \overline{\mathcal{H}_\alpha^{\mathrm{fin}}} = \overline{\mathrm{span}\,\{J_{-k_1}\cdots J_{-k_n}\Omega_\alpha, k_j > 0\}} \cong \mathcal{H}_0$.
  - $\mathcal{H}_\alpha$ admits a scalar product and $J_n^* = J_{-n}$. Virasoro algebra L_n.
- Consider $\hat{\mathcal{H}} := \bigoplus_\alpha \mathcal{H}_\alpha$. $\hat{J}_n := \bigoplus_\alpha J_n$, $\hat{L}_n := \bigoplus_\alpha L_n$.
- For $\beta \in \mathbb{R}$, there is a primary field $Y_\beta(z)$ acting on $\bigoplus_\alpha \mathcal{H}_\alpha$, where $Y_\beta(z) : \mathcal{H}_\alpha^{\mathrm{fin}} \mapsto \mathcal{H}_{\alpha+\beta}^{\mathrm{fin}}$. Formal series in $z^s, s \in \mathbb{R}$.
- $E^\pm(\beta, z) = \exp\left(\mp \sum_{n>0} \frac{\beta \hat{J}_{\pm n}}{n} z^{\mp n}\right)$
- $Y_\beta(z) = c_\beta E^-(z)E^+(z)z^{\beta J_0}$, where c_β is the unitary shift $\mathcal{H}_\beta \to \mathcal{H}_{\alpha+\beta}$, $c_\beta\Omega_\alpha = \Omega_{\alpha+\beta}$. βJ_0 on $\mathcal{H}_\alpha$ gives $\alpha \cdot \beta$.
- As z^s is multi-valued, we cannot consider Y_β as a field on S^1. However, we can take a branch of z^s on $S^1 \setminus \{-1\}$ and Y_β can be considered as a field on it.

Full Wightman fields

- $Y_\beta(z)$ makes sense as an operator-valued distribution on $S^1 \setminus \{-1\}$:
  - $Y_\beta(z) = \sum_{s\in\mathbb{R}} Y_{\beta,s} z^{-d-s}$, $d = \frac{\beta^2}{2}$, but on each $\mathcal{H}_\alpha$ this is a countable sum.
  - For $f \in C^\infty(S^1 \setminus \{-1\})$, put $\hat{f}_s = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})e^{-is\theta}d\theta$. $Y_\beta(f) := \sum_s Y_{\beta,s}\hat{f}_s$.
  - This is convergent on $C^\infty(\hat{L}_0)$.
- It holds that $Y_\alpha(z_1)Y_\beta(z_2) = e^{i\alpha\beta}Y_\beta(z_2)Y_\alpha(z_1)$ if $\arg z_1 > \arg z_2$.
- The braiding $e^{i\alpha\beta}$: $Y_\alpha(z)$ is not local as a "field" on $\mathbb{R} \cong S^1 \setminus \{-1\}$.

- On $\hat{\mathcal{H}} \otimes \hat{\mathcal{H}}$, we consider the product field $\tilde{Y}_\beta(z,w) = Y_\beta(z) \otimes Y_\beta(w)$. Restricts to $\bigoplus_{\alpha\in\mathbb{R}} \mathcal{H}_\alpha \otimes \mathcal{H}_\alpha$.
- Identify $\mathbb{R}^{1+1} = \mathbb{R}\times\mathbb{R} \cong (S^1 \setminus \{-1\}) \times (S^1 \setminus \{-1\})$.
- If (z_1, w_1) is spacelike to (z_2, w_2), then $\arg z_1 > \arg z_2$ and $\arg w_1 < \arg z_2$ (or vice versa).
- **Locality**: $\tilde{Y}_\alpha(z_1,w_1)\tilde{Y}_\beta(z_2,w_2) = e^{i\alpha\beta - i\alpha\beta}\tilde{Y}_\beta(z_2,w_2)\tilde{Y}_\alpha(z_1,w_1) = \tilde{Y}_\beta(z_2,w_2)\tilde{Y}_\alpha(z_1,w_1)$. Covariance: $[\hat{L}_n, Y_{\beta,s}] = ((d-1)n - s)Y_{\beta,n+s}$.

Generalization to rational unitary CFT

- V: unitary VOA with "modules/representations" V_α.
- In some cases, the set of α's form an abelian group G and one has Y_α, with scalar braiding. In that case, one can construct Wightman fields in the same way (Adamo-Giorgetti-T. '23, CMP).
  Affine VOAs (WZW models) for simply laced groups with level 1 should be in this class.

- More generally, under good assumptions (see Gui '19), the modules V_α and intertwinters Y_α have unitary structures.
- It should be possible to construct Wightman fields on $\bigoplus_\alpha V_\alpha \otimes V_{\bar{\alpha}}$, where $\bar{\alpha}$ is the conjugate module (Adamo-Giorgetti-T. work in progress). cf. Huang-Kong '07, Moriwaki '23.
- Is the theory determined by the module structure $\bigoplus_\alpha V_\alpha \otimes V_{\bar{\alpha}}$? Apparently no. cf. Davydov '14, Bischoff-Kawahigashi-Longo '15.

Osterwalder-Schrader axioms

- With Wightman fields, one can associate Schwinger functions satisfying the Osterwalder-Schrader (OS) axioms, by certain analytic continuations.
- Is there a more direct construction of Schwinger functions?

- OS axioms for Schwinger "functions" $\{S_n\}$ (for simplicity, we consider only one species of field): tempered distributions on $\mathbb{R}^{2n}$ excluding coinciding points $\boldsymbol{x}_j = \boldsymbol{x}_k, j \neq k$, satisfying Euclidean/conformal covariance, reflection positivity (RP), symmetry, clustering, linear growth.
- OS reconstruction: the analytic continuation of S_n to $(x_0, x_1) \mapsto (ix_0, x_1)$ gives Wightman functions (distributional boundary values).

Schwinger functions from unitary chiral VOA

Work in progress Adamo-Moriwaki-T.

- VOA: formal series $\phi^{\mathrm{h}}(z) = \sum_n \phi_n z^{-n} (= z^d \phi(z))$.
- Assume **polynomial energy bounds** for V (Carpi-Kawahigashi-Longo-Weiner '18), valid in all known examples: $\|\phi_n \Psi\| \le C(|n|+1)^s \|(L_0 + \mathrm{id})^p \Psi\|$ for some $C, s, p > 0$.
- Schwinger functions converges for $|z_1| > \cdots > |z_n|$ and analytic:

$$\begin{aligned} &S_n(z_1, \cdots, z_n) \\ &= \langle \Omega, \phi^{\mathrm{h}}(z_1) \cdots \phi^{\mathrm{h}}(z_n) \Omega \rangle = \sum_{k_1, \cdots, k_n} \langle \Omega, \phi_{k_1} \cdots \phi_{k_n} \Omega \rangle z_1^{-k_1} \cdots z_n^{-k_n} \\ &= \sum_{k_2, \cdots, k_n} \langle \Omega, \phi_{-k_2 - \cdots - k_n} \cdots \phi_{k_n} \Omega \rangle \left(\frac{z_2}{z_1} \right)^{-k_2 - \cdots - k_n} \cdots \left(\frac{z_n}{z_{n-1}} \right)^{-k_n}, \end{aligned}$$

  Other regions by permutation and continuity.
- OS reconstruction: $z = x_1 + ix_0 = (x_1, x_0) \mapsto (x_1, ix_0)$, S_n depends on $x_1 - x_0$, giving a **chiral** correlation function.

Schwinger functions from unitary chiral/full VOA

Work in progress Adamo-Moriwaki-T.

- RP + conformal covariance:

$$0 \leq \sum_{j,k} \int \overline{f_j(\bar{z}_j^{-1}, \cdots, \bar{z}_1^{-1})} f_j(z_{j+1}, \cdots, z_{j+k}) S_{j+k}(z_1, \cdots, z_{j+k})$$
$$|J(z_1, \cdots, z_{j+k})| dx_1 dy_1 \cdots dx_{j+k} dy_{j+k},$$

  where $J(z_1, \cdots, z_n)$ is the Jacobian that makes $z \to \bar{z}^{-1}$ unitary together with the scaling factors of the fields.

- **Unitarity** $0 \leq \langle \sum_j \Psi_j, \sum_k \Psi_k \rangle$,
  $\Psi_j = \int f_j(z_1, \cdots, z_j) \phi^{\mathrm{h}}(z_1) \cdots \phi^{\mathrm{h}}(z_j) |J(z_1, \cdots, z_j)| dx_1 dy_1 \cdots dx_j dy_j \Omega$
  and $\phi^{\mathrm{h}}(z) = \phi^{\mathrm{h}}(\bar{z}^{-1})^* \implies$ Reflection positivity
- Polynomial energy bounds should imply Linear growth.

$$|\langle \Omega, \phi_{k_1} \cdots \phi_{k_n} \Omega \rangle| \leq C^n (|k_1| + 1 + \cdots + |k_n| + 1)^{np} \prod_{j=1}^{n} (|k_j| + 1)^s$$

- Full VOA: power series in z and $\bar{z}$, real analytic. OS axioms?

Some ideas from Constructive QFT

- (Glimm-Jaffe '72) Start with the free field on the **Minkowski space**, add the interaction to the Hamiltonian, define the new dynamics and take a new representation of the algebras.
- (Barata-Jäkel-Mund '23) Start with the free field on the **de Sitter space**, add the interaction to the Lorentz generators, define the new dynamics on the same Hilbert space (Haag's theorem is curcumvented by finite volume, cf. Weiner '11).
  - More abstractly (Jäkel-Mund '18), fix a Haag-Kastler net on the de Sitter space.
  - On the same Hilbert space, construct a **new representation of the Lorentz group** by adding the interaction term to the generators.
  - Under certain conditions (finite speed of propergation, existence of vacuum), one can generate a new Haag-Kastler net: keep the algebras of wedges at time zero, and the rest is defined by covariance with respect to the new representation.
  - Examples: $\mathcal{P}(\phi)_2$-models.

CFT as unperturbed theory

- Can we take the CFT as the starting point and perturb the dynamics?

- Any conformal field theory extends to the Einstein cylinder (Guido-Longo '03)
- The de Sitter space is conformally equivalent to part of the cylinder.
- By composing these maps, any CFT can be considered as a QFT on the de Sitter space.
- Lorentz transformations are contained in the Möbius group. $L_1 \otimes \mathbb{1} + \mathbb{1} \otimes L_{-1}, L_0 \otimes \mathbb{1} - \mathbb{1} \otimes L_0, L_{-1} \otimes \mathbb{1} + \mathbb{1} \otimes L_1$.

- Add a primary field to the Lorentz geneators (cf. Zamolodchikov '89, the resulting models should be integrable)

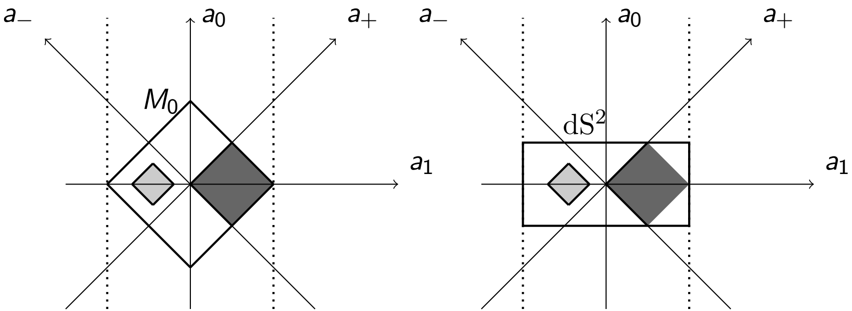

Figure: The Minkowski space M_0 and the de Sitter space dS^2 conformally embedded in $\mathbb{R}^2$. The cylinder is obtained by identifying the dotted lines. The dark grey region is a wedge W and the light grey region is a double cone.

New Lorentz generators

- (Jäkel-T. '23, LMP) Take the primary field for the $\mathrm{U}(1)$-current, let $|\beta| < \frac{1}{\sqrt{2}}$. Then $\sum_k Y_{\beta,k} \otimes Y_{\beta,n+k}$ makes sense as an operator (this is a Fourier component of the the time-zero restriction of $\tilde{Y}_\beta$).
- With $\lambda \in \mathbb{R}$, the Lorentz relations are **weakly** satisfied for

$$L_1 \otimes \mathbb{1} + \mathbb{1} \otimes L_{-1} + \lambda\Big(\sum_{\epsilon=\pm 1,k} Y_{\epsilon\beta,k} \otimes Y_{\epsilon\beta,-1+k}\Big)$$

$$L_0 \otimes \mathbb{1} - \mathbb{1} \otimes L_0$$

$$L_{-1} \otimes \mathbb{1} + \mathbb{1} \otimes L_1 + \lambda\Big(\sum_{\epsilon=\pm 1,k} Y_{\epsilon\beta,k} \otimes Y_{\epsilon\beta,k+1}\Big)$$

- The proof depends only on the primarity (diffeomorphism covariance of Y) and commutativity of the time-zero restriction. **Hopefully valid for generic CFT**.
- Open problem: extension to a group representation (domain problem). Euclidean theory + OS reconstruction might help.

Outlook/open problems

- Domain problem for the Lorentz generators? ($+$ interacting vacuum $\Longrightarrow$ new Haag-Kastler nets on dS^2, possibly massive and integrable).
- Osterwalder-Schrader axioms for unitary full VOA (work in progress with M.S. Adamo and Y. Moriwaki)
- Glimm-Jaffe axioms (functional integrals/measures on the space of distributions) for CFT?
- Maybe VOA, Wightman, Araki-Haag-Kastler are equivalent? (cf. Raymond-Tener-T. '22, CMP, Carpi-Raymond-Tener-T. work in progress, unitary VOA $\Longleftrightarrow$ Wightman fields on S^1)
- Higher dimensional CFT?