Recursive Algorithms

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Recursive Algorithms

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- Overview
  - Recursive calls
  - Proof of correctness + complexity
  - Examples
    - Greatest common divisor, power, binary search
    - Tower of Hanoi
    - Sorting: quick sort

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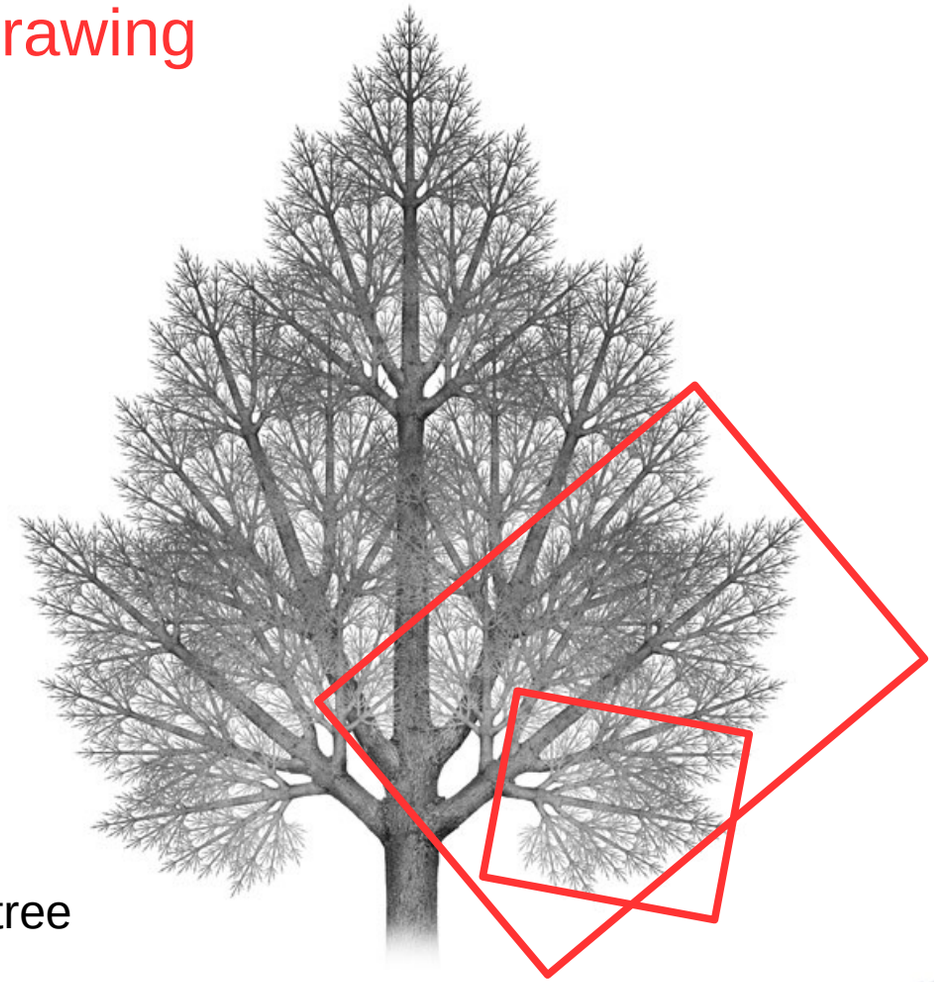
- A recursive drawing



A fractal tree

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- A recursive algorithm
  - Base case: result is determined explicitly
  - Recursive case: result is expressed in terms of same (but reduced) problem
  - The base case should be reached after a finite number of iterations
- Recursive calls
  - Direct recursion
    - A method f() of object A sends a message f() to object A
  - Indirect recursion
    - A method f() of object A sends a message g() to object A
    - And the method g() of object A sends a message f() to object A

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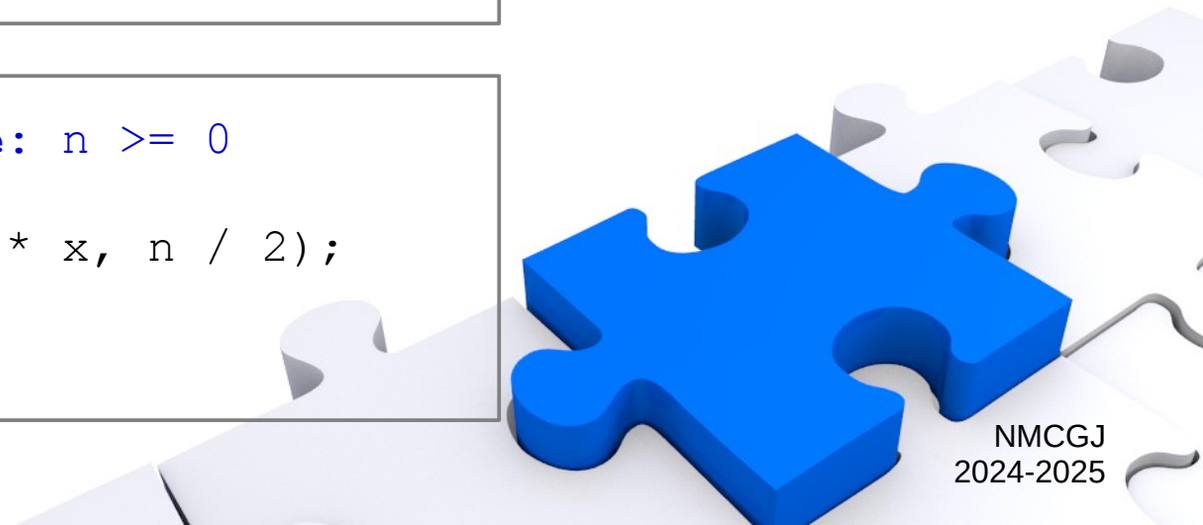
- Examples
  - Greatest common divisor (gcd) and power, version 3

```
public int gcd(int x, int y) { // pre: x > 0 && y > 0
    if (x == y) return x;
    else if (x > y) return gcd(x - y, y);
    else return gcd(x, y - x);
}
```

```
public int pow(int x, int n) { // pre: n >= 0
    if (n == 0) return 1;
    else if (n % 2 == 0) return pow(x * x, n / 2);
    else return x * pow(x, n - 1);
}
```



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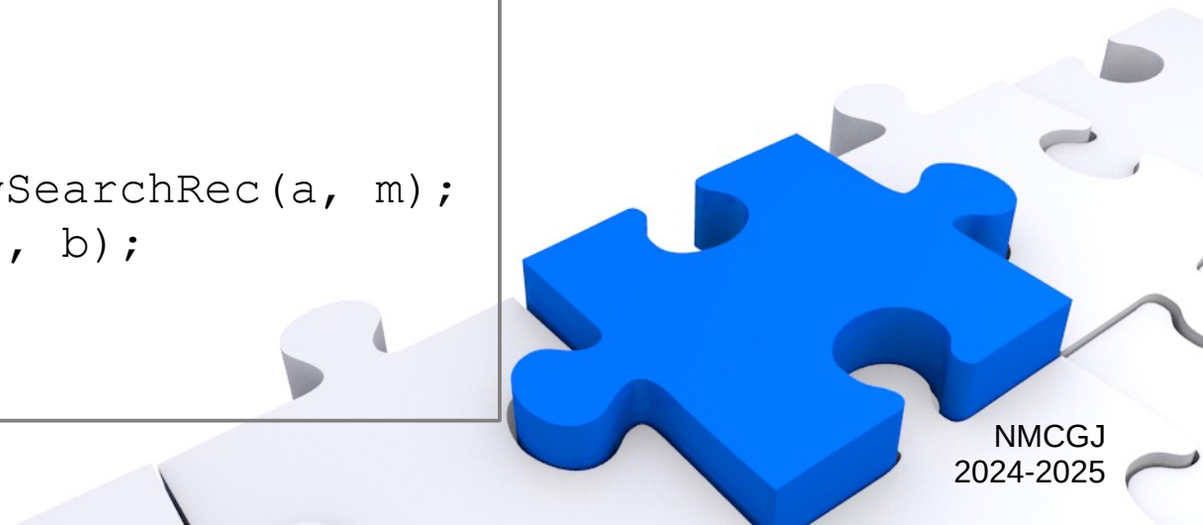
- Examples
  - Binary search: value *v* in sorted array *list* ?

```
public boolean binarySearch() { // pre: sorted list
    return binarySearchRec(0, list.length-1);
}
private boolean binarySearchRec(int a, int b) {
    // pre: sorted list && a <= b
    if (a == b) return (list[a] == v);
    else {
        int m = (a + b) / 2;
        if (v <= list[m]) return binarySearchRec(a, m);
        else return binarySearchRec(m+1, b);
    }
}
```



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- Proof of correctness
  - What we have seen before: iterative algorithm
    - Specify precondition and postcondition (result)
    - Specify invariant
    - Prove invariant is satisfied initially
    - Prove invariant is satisfied after each iteration
    - Prove “invariant && !condition” leads to postcondition
    - Termination in finite number of iterations
  - Recursive algorithm

Recursive Algorithms

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- Proof of correctness
  - What we have seen before: iterative algorithm
  - Recursive algorithm
    - Specify precondition and postcondition (result)
    - Prove postcondition is satisfied for base case
    - Prove precondition is satisfied for recursive case
    - Prove “precondition && postcondition of base case” leads to postcondition
    - Termination in finite number of iterations
  - Example: power, version 3

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- Proof of correctness

```
public int pow(int x, int n) {
    // pre: n >= 0
    // res: result == power(x,n)
    if (n == 0) {
        return 1;

    } else if (n % 2 == 0) {
        return pow(x * x, n / 2);

    } else {
        return x * pow(x, n - 1);

    }
}
```

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- Proof of correctness

```
public int pow(int x, int n) {
    // pre: n >= 0
    // res: result == power(x,n)
    if (n == 0) {
        return 1;
        // n == 0 && result == 1 == power(x,n)
    } else if (n % 2 == 0) { // n >= 0
        return pow(x * x, n / 2);
        // result == power(x*x,n/2) == power(x,n)
    } else { // n >= 1
        return x * pow(x, n - 1);
        // result == x*power(x,n-1) == power(x,n)
    }
}
```

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- Complexity
  - Example: power, version 3
    - Complexity: counting most time-consuming operations
      - Sending a message “pow” (SM_n for given n)
    - Number of messages
      - $SM_0 = 1$
      - $SM_{2m} = 1 + SM_m$
      - $SM_{2m+1} = 1 + SM_{2m} = 2 + SM_m$
    - $SM_n = O(log(n))$

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- To recurse or not to recurse …
  - So far, examples that were easy to implement as
    - Iterative algorithm
    - Recursive algorithm
  - Actually, these algorithms are called “tail recursive”
    - The last statement in the algorithm is to “restart” the algorithm
    - Tail recursive algorithms can be directly translated into loops
  - Example: Tower of Hanoi

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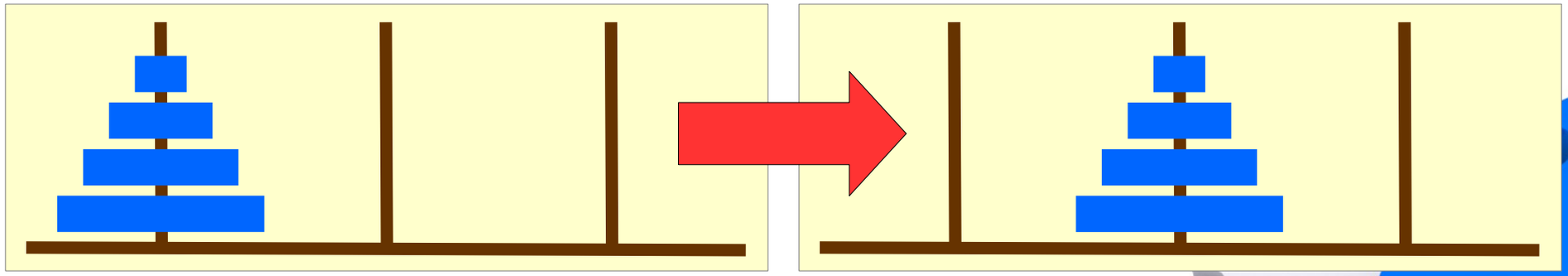
- To recurse or not to recurse …
  - Example: Tower of Hanoi
    - Objective is to move a stack of disks from one rod to another rod
    - Rules
      - One disk can be moved at a time
      - Only the upper disk from a stack can be moved on top of another stack
      - No disk may be placed on top of a smaller disk



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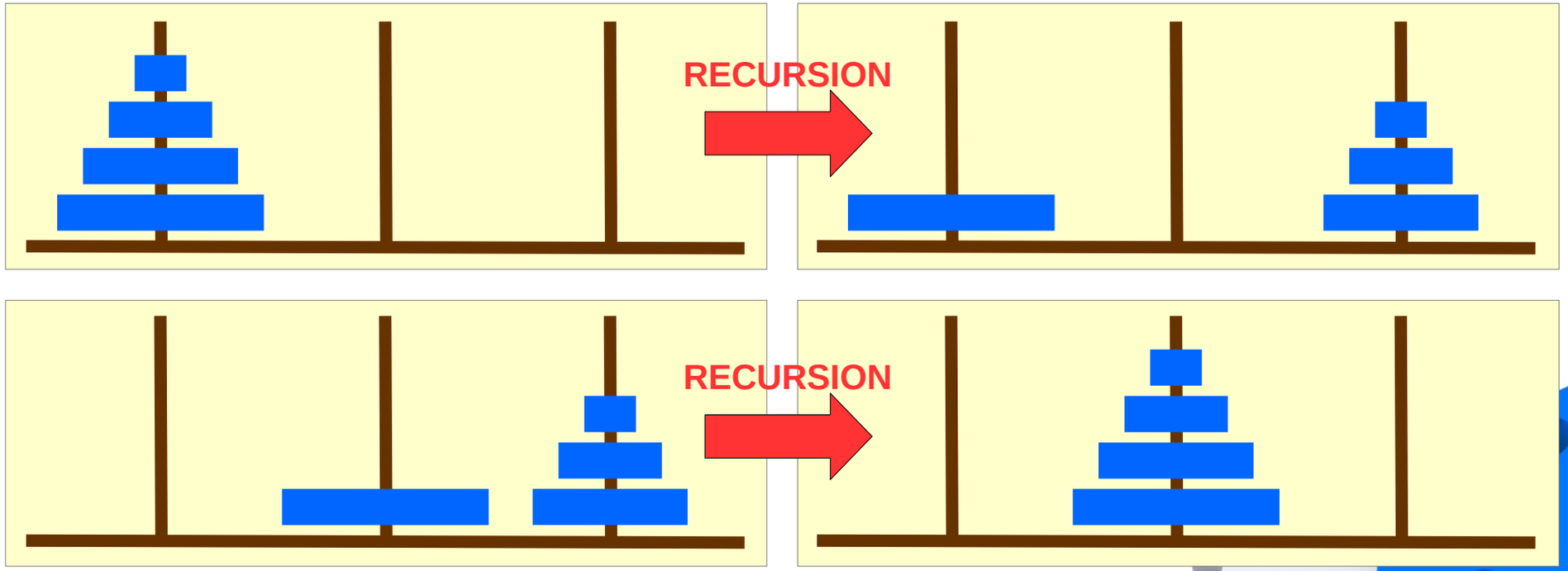
- To recurse or not to recurse …
  - Example: Tower of Hanoi



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- To recurse or not to recurse …
  - Example: Tower of Hanoi
    - Recursion ?
      - Elegant solution
      - Iterative version is more cumbersome
    - Write a Java program with output

```
Move disk from 1 to 3
Move disk from 1 to 2
Move disk from 3 to 2
Move disk from 1 to 3
...
```

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- To recurse or not to recurse …
  - Example: Tower of Hanoi

```
public void hanoi(int n, int from, int to) {
    if (n == 1) move(from, to);
    else {
        int third = 6 - from - to;
        hanoi(n - 1, from, third);
        move(from, to);
        hanoi(n - 1, third, to);
    }
}
private void move(int from, int to) {
    System.out.printf("Move disk from %d to %d\n",
        from, to);
}
```

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- To recurse or not to recurse …
  - Example: Tower of Hanoi

```
Output of hanoi(4, 1, 2)

Move disk from 1 to 3   |
Move disk from 1 to 2   |   Move disk from 3 to 2
Move disk from 3 to 2   |   Move disk from 3 to 1
Move disk from 1 to 3   |   Move disk from 2 to 1
Move disk from 2 to 1   |   Move disk from 3 to 2
Move disk from 2 to 3   |   Move disk from 1 to 3
Move disk from 1 to 3   |   Move disk from 1 to 2
Move disk from 1 to 2   |   Move disk from 3 to 2
```

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- To recurse or not to recurse …
  - Example: Tower of Hanoi
    - Complexity: counting most time-consuming operations
      - Sending a message “hanoi” (SM_n for given n)
    - Number of messages
      - $SM_1 = 1$
      - $SM_n = 1 + 2 * SM_{n-1}$
    - $SM_n = O(2^n)$

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- Sorting algorithms
  - So far, three iterative versions
    - Selection sort
    - Insertion sort
    - Bubble sort (see exercise)
  - Recursive version: quick sort
    - Divide array in two parts
      - Choose random element “pivotelement”
      - Move smaller elements in left part of array
      - Move larger elements in right part of array
      - Apply “quick sort” to both subarrays

} partitioning

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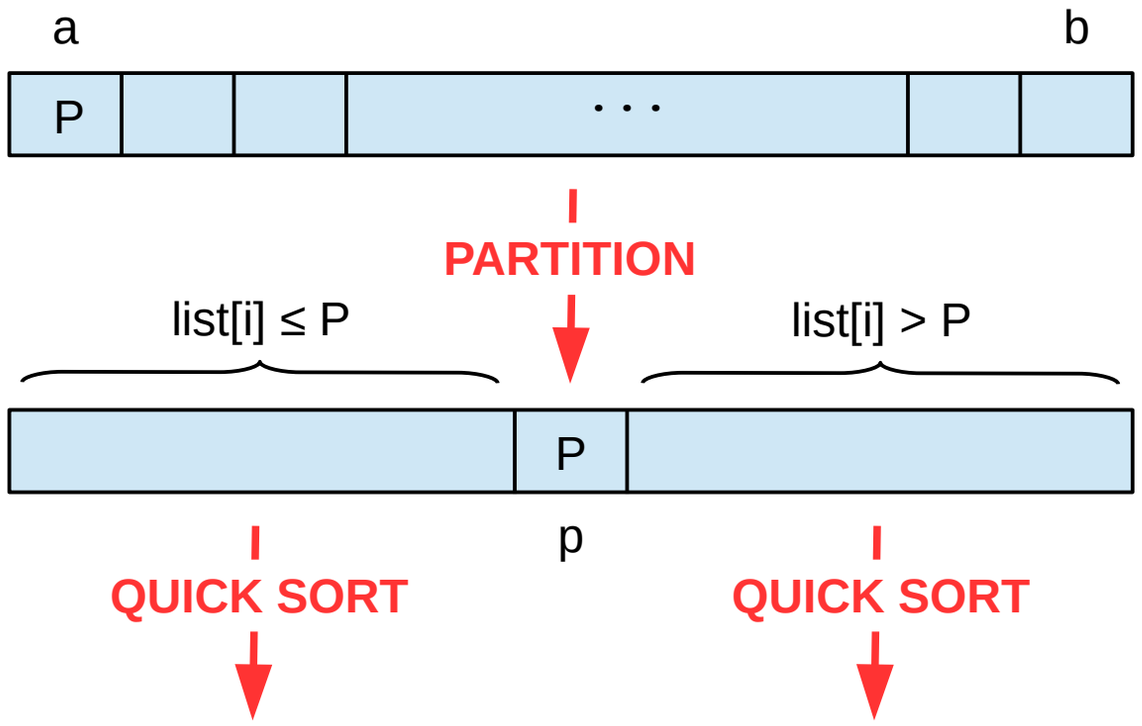
- Sorting algorithms
  - Recursive version: quick sort



Recursive Algorithms

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- Sorting algorithms
  - Recursive version: quick sort

```
public void quickSort() {
    quickSortRec(0, list.length-1);
}

private void quickSortRec(int a, int b) {
    if (a < b) {
        int p = partition(a, b); // TODO
        quickSortRec(a, p - 1);
        quickSortRec(p + 1, b);
    }
}
```

Recursive Algorithms

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- Sorting algorithms
  - Recursive version: quick sort

```
private int partition(int a, int b) {
   int c = a, d = b, pivot = list[c]; // list[c] free
   while (c < d) {
      while (c < d && list[d] > pivot) d--;
      if (c < d) {  // smaller list[d] found -> move
         list[c] = list[d];              // list[d] free
         while (c < d && list[c] <= pivot) c++;
         if (c < d) // larger list[c] found -> move
            list[d] = list[c];           // list[c] free
      }
   }
   list[c] = pivot; return c;
}
```