CIME-EMS Summer School on Splines and PDEs:
Recent Advances from Approximation Theory to
Structured Numerical Linear Algebra

Hierarchical spline spaces

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Motivation

- Splines and B-splines are of interest in a wide range of areas:
  - established tool: geometric modelling, approximation theory
  - recently: isogeometric analysis (paradigm for solving PDEs)
- In higher dimensions usually based on tensor-product topology
  - ☺ computationally efficient, geometrically intuitive
  - ☹ restriction to rectangular meshes
    $\rightarrow$ not well suited for adaptive refinement

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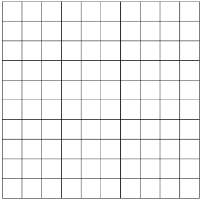

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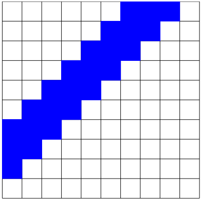

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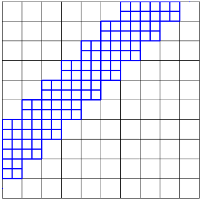

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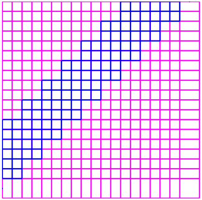

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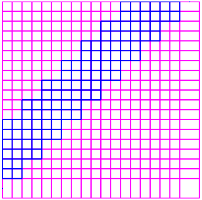

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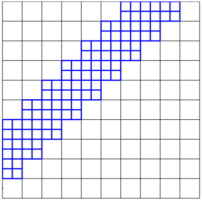
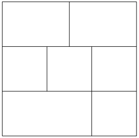


Motivation

Alternative spline spaces

- T-splines
- LR-splines
- Splines on T-meshes
- Splines on triangulations
- Splines on unstructured quad meshes
- ...
- Hierarchical splines
  - local refinement
  - any degree, any smoothness, any dimension
  - basis: linear independence, convex partition of unity, stability
  - easy construction of quasi-interpolants



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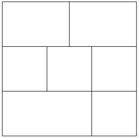

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Hierarchical model

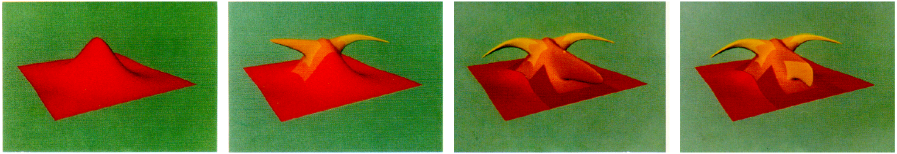
Hierarchical splines and B-splines

[Forsey, Bartels, 1988]

[Kraft, 1997]

- framework for surface fitting and modelling

  efficient construction of local features



Hierarchical model

Hierarchical B-splines: construction

[Forsey, Bartels, 1988]

[Kraft, 1997]

- sequence of n nested (tensor-product) spline spaces on Ω^0

  $\mathbb{V}^0 \subset \mathbb{V}^1 \subset \cdots \subset \mathbb{V}^{n-1}$

  each spline space $\mathbb{V}^\ell$ spanned by normalized B-spline basis

  $\mathcal{B}^\ell = \{B_{i,\ell},\ i = 1, \ldots, N_\ell\}$

- sequence of n domains

  $\Omega^0 \supseteq \Omega^1 \supseteq \cdots \supseteq \Omega^{n-1}$

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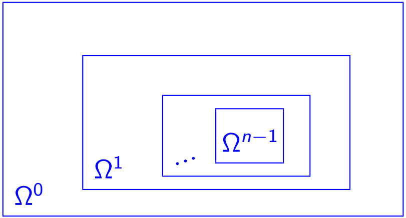
hierarchical domain



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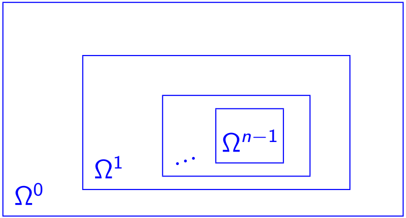

Note:

$\mathrm{supp}^0(f) = \mathrm{supp}(f) \cap \Omega^0$

Hierarchical model

Hierarchical B-splines: construction

recursive definition of $\mathcal{H}$:

(I) initialization: $\mathcal{H}^0 = \left\{ B_{i,0} \in \mathcal{B}^0 : \mathrm{supp}^0(B_{i,0}) \neq \emptyset \right\}$

(II) recursive case: construct $\mathcal{H}^{\ell+1}$ from $\mathcal{H}^{\ell}$

$$\mathcal{H}^{\ell+1} = \mathcal{H}_A^{\ell+1} \cup \mathcal{H}_B^{\ell+1}, \qquad \ell = 0, \ldots, n-2,$$

where

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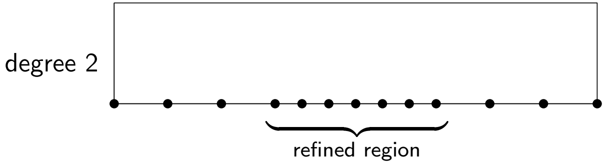
Hierarchical B-splines: 1D example



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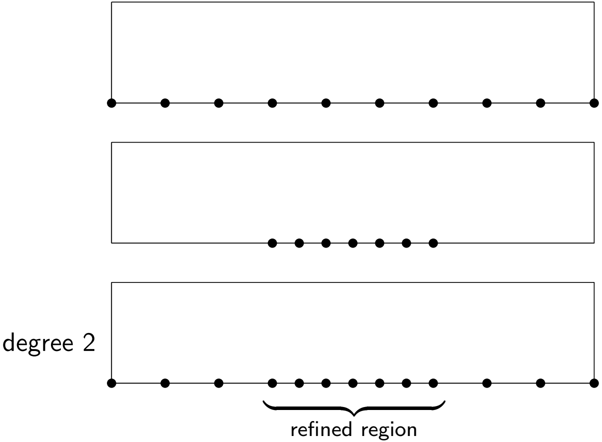

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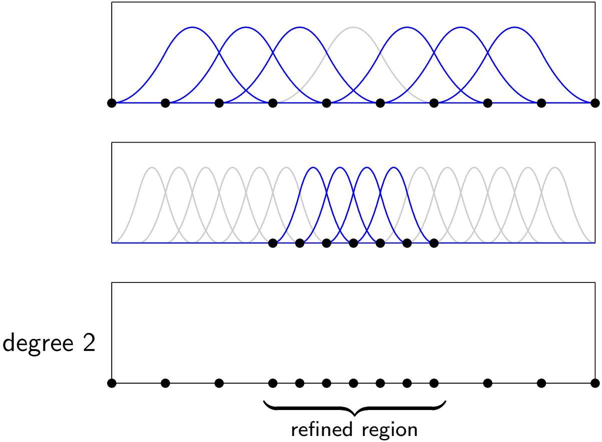

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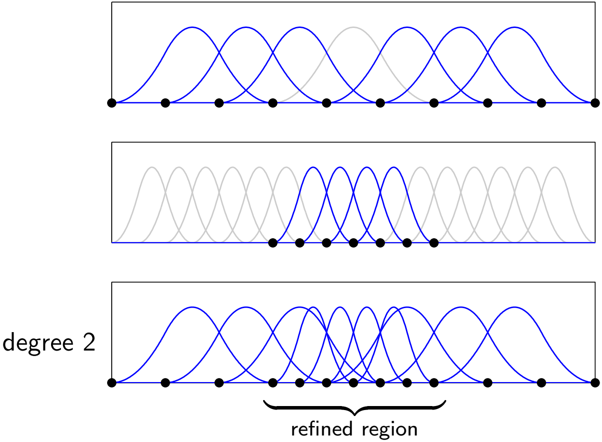

Hierarchical model

Hierarchical spline space

- ✓ local refinement
- ✓ any degree, any smoothness, any dimension

Hierarchical basis

- ✓ linearly independent
- ✓ nonnegative
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Normalization: truncated basis

- ✓ linearly independent
- ✓ nonnegative
- ✓ partition of unity
- ✓ smaller support
- ✓ improved stability

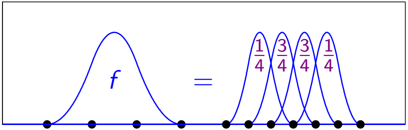
[Giannelli, Jüttler, Speleers, 2012]

Hierarchical model

From hierarchical basis $\mathcal{H}$ to truncated basis $\mathcal{T}$

- by subdivision, we can write every $f \in \mathbb{V}^\ell \subset \mathbb{V}^{\ell+1}$

$$f = \sum_{i=1}^{N_{\ell+1}} c_{i,\ell+1} \, B_{i,\ell+1}$$



- truncation mechanism:

$$\mathrm{trunc}^{\ell+1}(f) = \sum_{i\,:\,\mathrm{supp}^0(B_{i,\ell+1}) \not\subseteq \Omega^{\ell+1}} c_{i,\ell+1} \, B_{i,\ell+1}$$

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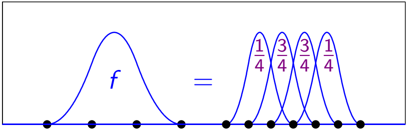

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Truncated B-splines: construction

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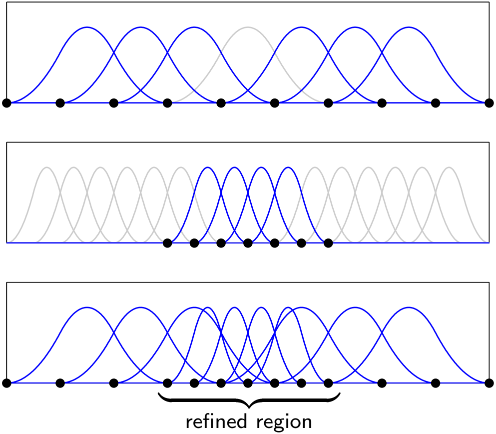
Truncated B-splines: 1D example

degree 2



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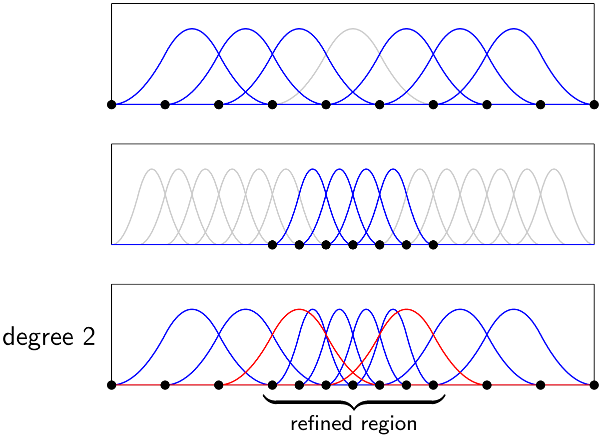

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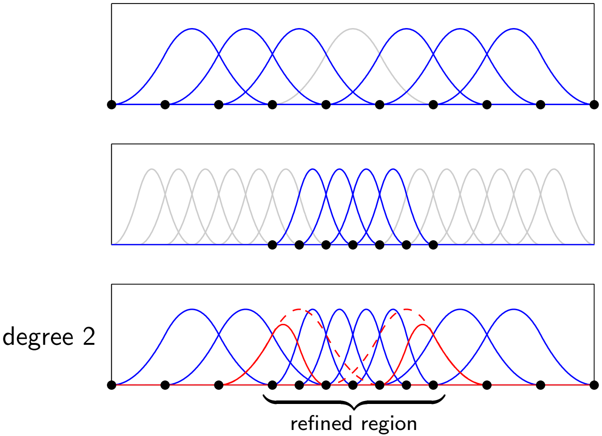

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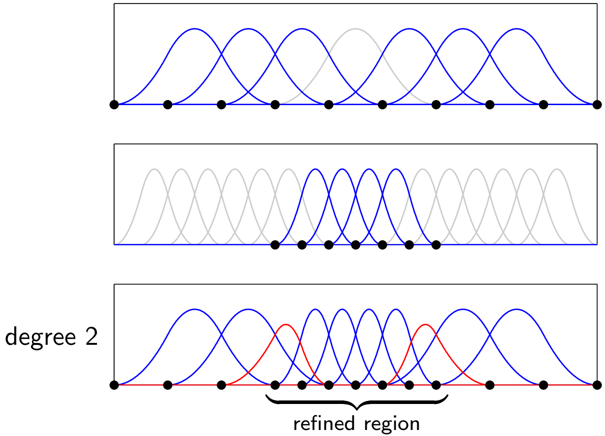


Hierarchical model

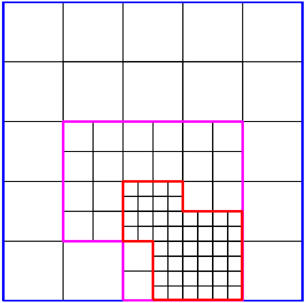
Truncated B-splines: 2D example

bidegree $[2, 2]$

level ℓ

level $\ell + 1$

level $\ell + 2$



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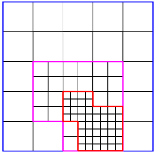

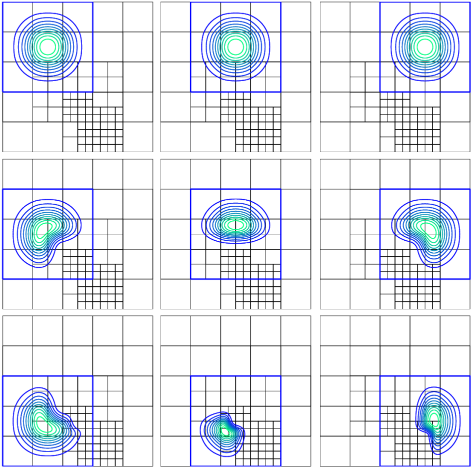

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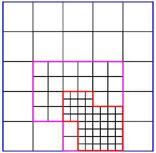

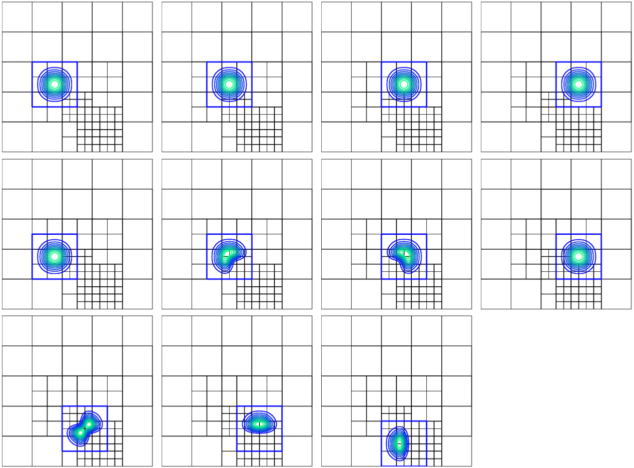

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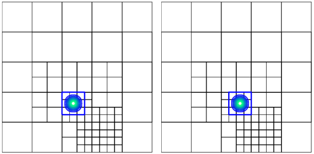

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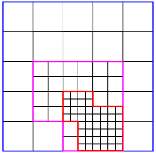

Truncated basis: properties

Truncated basis $\mathcal{T}$: another basis for hierarchical spline space

With respect to classical hierarchical basis $\mathcal{H}$:

(1) $\operatorname{span}\mathcal{H} = \operatorname{span}\mathcal{T}$

(2) the truncated basis functions are nonnegative and linearly independent

(3) reduced support of coarse basis functions;
reduced overlap of basis supports $\rightarrow$ sparser systems

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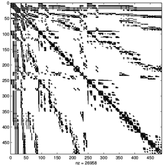
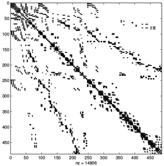
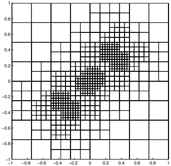
HB: 

THB: 



Truncated basis: properties

Truncated basis $\mathcal{T}$: another basis for hierarchical spline space

(4) preservation of coefficients:

let $\mathcal{I}_\ell := \{i : B_{i,\ell} \in \mathcal{B}^\ell \cap \mathcal{H}\}$, let $D^\ell := \Omega^\ell \setminus \Omega^{\ell+1}$, and let

$$f|_{D^\ell} = \sum_{k=0}^{n-1} \sum_{i \in \mathcal{I}_k} c_{i,k}^{\mathcal{T}} B_{i,k}^{\mathcal{T}}|_{D^\ell} = \sum_{j=1}^{N_\ell} c_{j,\ell} B_{j,\ell}|_{D^\ell}, \quad \forall \ell$$

then $\quad c_{i,\ell}^{\mathcal{T}} = c_{i,\ell}, \quad i \in \mathcal{I}_\ell$

Sketch of proof

(5) the truncated basis forms a partition of unity on Ω^0

$\Rightarrow$ convex partition of unity

(6) preservation of Greville abscissas

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Truncated basis $\mathcal{T}$: another basis for hierarchical spline space

(7) the truncated basis is strongly L_∞-stable:

at each level ℓ, assume $C_0 \le h^\ell_{min}/h^\ell_{max} \le C_1$ (for any direction)

let $f = \sum_{k=0}^{n-1} \sum_{i \in \mathcal{I}_k} c^{\mathcal{T}}_{i,k} B^{\mathcal{T}}_{i,k}$, then

$$K_0 \max_{i,k} |c^{\mathcal{T}}_{i,k}| \le \max_{u \in \Omega^0} |f(u)| \le K_1 \max_{i,k} |c^{\mathcal{T}}_{i,k}|$$

with K_0 and K_1 independent of the number of levels n

Sketch of proof

Quasi-interpolation

Hierarchical quasi-interpolants [Speleers, Manni, 2016]

- QI based on truncated hierarchical basis
  - convex partition of unity
  - small support
  - preservation of coefficients

- Consider a sequence of QIs in $\mathbb{V}^\ell$, $\ell = 0, \ldots, n-1$

$$Q^\ell(f) := \sum_{i=1}^{N_\ell} \lambda_{i,\ell}(f)\, B_{i,\ell}$$

then

$$Q(f) := \sum_{k=0}^{n-1} \sum_{i \in \mathcal{I}_k} \lambda_{i,k}(f)\, B^{\mathcal{T}}_{i,k}$$

Quasi-interpolation

Hierarchical quasi-interpolants [Speleers, Manni, 2016]

- QI based on truncated hierarchical basis
  - convex partition of unity → numerical stability
  - small support → local control
  - preservation of coefficients → easy construction
- Consider a sequence of QIs in $\mathbb{V}^\ell$, $\ell = 0, \ldots, n-1$

$$Q^\ell(f) := \sum_{i=1}^{N_\ell} \lambda_{i,\ell}(f)\, B_{i,\ell}$$

then

$$Q(f) := \sum_{k=0}^{n-1} \sum_{i \in \mathcal{I}_k} \lambda_{i,k}(f)\, B^{\mathcal{T}}_{i,k}$$

Quasi-interpolation

Hierarchical quasi-interpolants [Speleers, Manni, 2016]

- QI based on truncated hierarchical basis
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- Polynomial reproduction ($\mathbb{P}_d \subset \mathbb{V}^0$):

  if $Q^\ell(q) = q, \quad \forall q \in \mathbb{P}_d, \ \forall \ell$ then $Q(q) = q, \quad \forall q \in \mathbb{P}_d$

- Spline reproduction (projector):

  if $\begin{cases} Q^\ell(s) = s, \quad \forall s \in \mathbb{V}^\ell, \ \forall \ell \\ \lambda_{i,\ell} \text{ is supported in } \Omega^\ell \setminus \Omega^{\ell+1} \end{cases}$ then $Q(s) = s, \quad \forall s \in \mathbb{S}$

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- Projector $\rightarrow$ dual basis:

  $\lambda_{i,\ell}(B_{j,k}^{\mathcal{T}}) = \delta_{i,j}\delta_{\ell,k}$ ($\delta_{r,s}$: Kronecker delta)

Quasi-interpolation

Hierarchical quasi-interpolants

- Let Υ be a cell, and

$$\Lambda_\Upsilon = \mathrm{conv}\left(\bigcup_{(i,\ell):\mathrm{supp}^0(B^{\mathcal{T}}_{i,\ell}) \bigcap \Upsilon \neq \emptyset} \Lambda_{i,\ell} \cup \Upsilon \right), \quad (\Lambda_{i,\ell} : \text{support of } \lambda_{i,\ell})$$

- Full approximation order:

  if $f \in C^{p+1}(\Lambda_\Upsilon)$ and $Q(q) = q, \ \forall q \in \mathbb{P}_p$ then

$$\|f - Q(f)\|_\Upsilon \leq C \left(\mathrm{diam}(\Lambda_\Upsilon)\right)^{p+1} (1 + C_Q) \sum_{|\alpha|=p+1} \|D^\alpha f\|_{\Lambda_\Upsilon}$$

Quasi-interpolation

Numerical examples: setup

- C^1 quadratic bivariate tensor-product B-splines
- Building block $\widetilde{Q}^\ell(f) := \sum_{i=1}^{N_\ell} \widetilde{\lambda}_{i,\ell}(f)\, B_{i,\ell}$:
  - choose a cell $\Upsilon_{i,\ell}$ in support of each $B_{i,\ell}$
  - choose 3×3 points $x_{j,i,\ell} \in \Upsilon_{i,\ell}$, $j = 1, \ldots, 9$
  - solve the system $\sum_{k:\mathrm{supp}(B_{k,\ell}) \bigcap \Upsilon_{i,\ell} \neq \emptyset} c_{k,\ell} B_{k,\ell}(x_{j,i,\ell}) = f(x_{j,i,\ell})$
  - set $\widetilde{\lambda}_{i,\ell} = c_{i,\ell}$
- Set $\widetilde{Q}(f) := \sum_{k=0}^{n-1} \sum_{i \in \mathcal{I}_\ell} \widetilde{\lambda}_{i,k}(f)\, B_{i,k}^{\mathcal{T}}$

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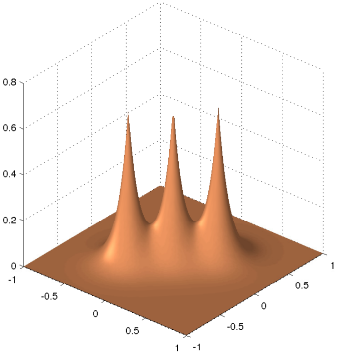
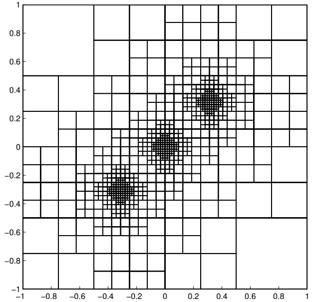
Quasi-interpolation

Numerical examples

$$f(x,y) = \frac{2}{3\exp(\sqrt{(10x-3)^2+(10y-3)^2})} + \frac{2}{3\exp(\sqrt{(10x+3)^2+(10y+3)^2})} + \frac{2}{3\exp(\sqrt{(10x)^2+(10y)^2})}$$





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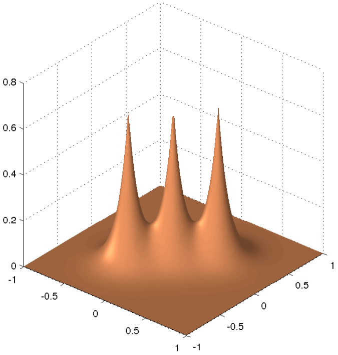

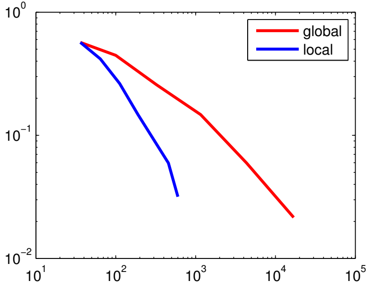

max-error versus dimension

Concluding message

Truncated hierarchical B-splines

- the hierarchical model: local refinement
- the truncation mechanism: construction of a basis for a hierarchical space with properties
  - ✓ linear independence
  - ✓ nonnegativity
  - ✓ small support
  - ✓ partition of unity
  - ✓ preservation of coefficients
  - ✓ strong L_∞-stability
  - ✓ quasi-interpolants
- Their use in adaptive isogeometric methods: see Giannelli's talk

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Truncated basis: proofs of properties

Strong L_∞-stability of truncated basis [Gianelli, Jüttler, Speleers, 2014]

let $f = \sum_{k=0}^{n-1} \sum_{i \in \mathcal{I}_k} c_{i,k}^{\mathcal{T}} B_{i,k}^{\mathcal{T}}$, then

$$K_0 \max_{i,k} |c_{i,k}^{\mathcal{T}}| \leq \max_{u \in \Omega^0} |f(u)| \leq K_1 \max_{i,k} |c_{i,k}^{\mathcal{T}}|$$

with K_0 and K_1 independent of the number of levels n

Sketch of proof.

- right: convex partition of unity $\Rightarrow$ $k_1 = 1$
- left: stability of the B-spline basis: for $cell^* \subseteq D^\ell = \Omega^\ell \setminus \Omega^{\ell+1}$

  $$c_{\beta^*}^\ell \leq K \|f\|_{\infty[cell^*]}$$

Back to properties

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Sketch of proof.

- right: convex partition of unity $\Rightarrow$ $k_1 = 1$
- left: preservation of coefficients:

$$d_{\tau^*} = c_{\beta^*}^{\ell} \leq K \|f\|_{\infty[cell^*]} \leq K \|f\|_\infty \quad \Rightarrow \quad k_0 = 1/K$$

Back to properties

Truncated basis: proofs of properties

Preservation of coefficients [Gianelli, Jüttler, Speleers, 2014]

let $D^\ell := \Omega^\ell \setminus \Omega^{\ell+1}$, and let

$$f|_{D^\ell} = \sum_{k=0}^{n-1} \sum_{i \in \mathcal{I}_k} c_{i,k}^{\mathcal{T}} B_{i,k}^{\mathcal{T}}|_{D^\ell} = \sum_{j=1}^{N_\ell} c_{j,\ell} B_{j,\ell}|_{D^\ell}, \quad \forall \ell$$

then $\quad c_{i,\ell}^{\mathcal{T}} = c_{i,\ell}, \quad i \in \mathcal{I}_\ell$

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$$f|_{D^\ell} = \sum_{k=0}^{\ell-1} \sum_{i \in \mathcal{I}_k} c_{i,k}^{\mathcal{T}} B_{i,k}^{\mathcal{T}}|_{D^\ell} + \sum_{i \in \mathcal{I}_\ell} c_{i,\ell}^{\mathcal{T}} B_{i,\ell}^{\mathcal{T}}|_{D^\ell} + \sum_{k=\ell+1}^{n-1} \sum_{i \in \mathcal{I}_k} c_{i,k}^{\mathcal{T}} B_{i,k}^{\mathcal{T}}|_{D^\ell}$$

- last sum is empty by local support of $B_{i,k}^{\mathcal{T}}$, $k > \ell$

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- central sum: no truncation effect for $k = \ell$: $B_{i,\ell}^{\mathcal{T}}|_{D^\ell} = B_{i,\ell}|_{D^\ell}$

Back to properties

Truncated basis: proofs of properties

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- first sum: representation in terms of $B_{i,\ell}$ + rearrangement of terms

Back to properties

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then $\quad c_{i,\ell}^{\mathcal{T}} = c_{i,\ell}, \quad i \in \mathcal{I}_\ell$

Sketch of proof.

$$f|_{D^\ell} = \sum_{i \notin \mathcal{I}_\ell} d_{i,\ell}^{\mathcal{T}} B_{i,\ell}|_{D^\ell} + \sum_{i \in \mathcal{I}_\ell} c_{i,\ell}^{\mathcal{T}} B_{i,\ell}|_{D^\ell}$$

- uniqueness of B-spline representation

Back to properties