CIME-EMS Summer School in Applied Mathematics

Splines and PDEs: Recent Advances
from Approximation Theory to Structured Numerical Linear Algebra

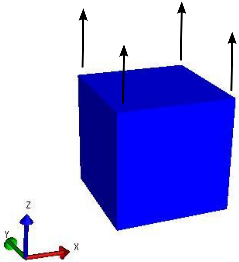
July 3 - July 7, 2017 - Cetraro

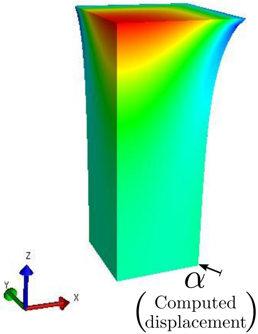
Isogeometric analysis and Locking

Giancarlo Sangalli

...with results from many colleagues: P. Antolín, A. Bressan, A. Buffa, M. Martinelli,...

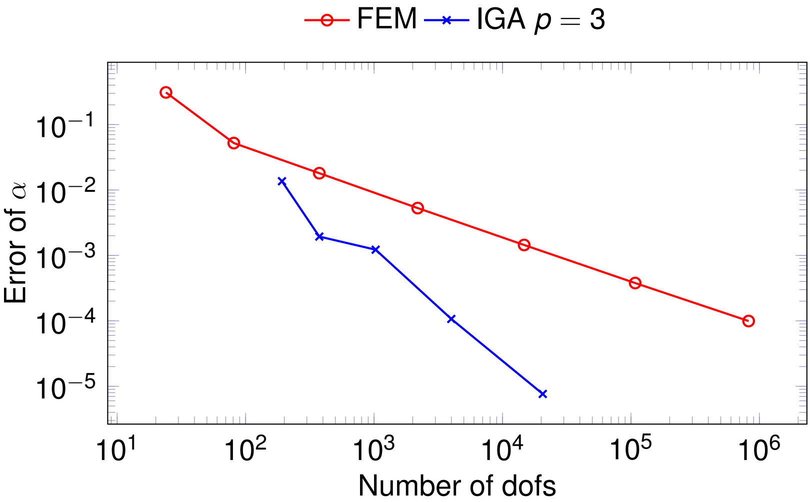
A simple stretch test



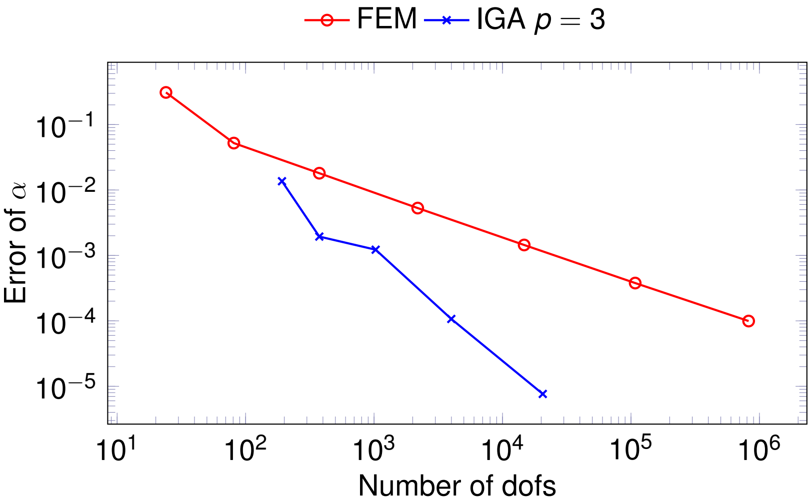


Neo–Hokean material. $E = 1.0$, $\nu = 0.3$
Symmetry conditions considered.

A simple stretch test

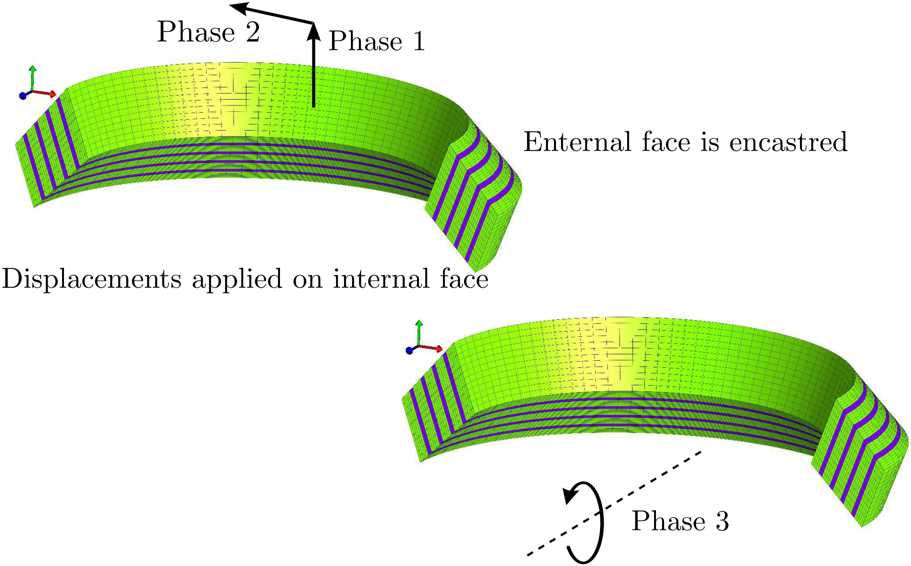


A simple stretch test



IGA works well with fewer degrees-of-freedom!

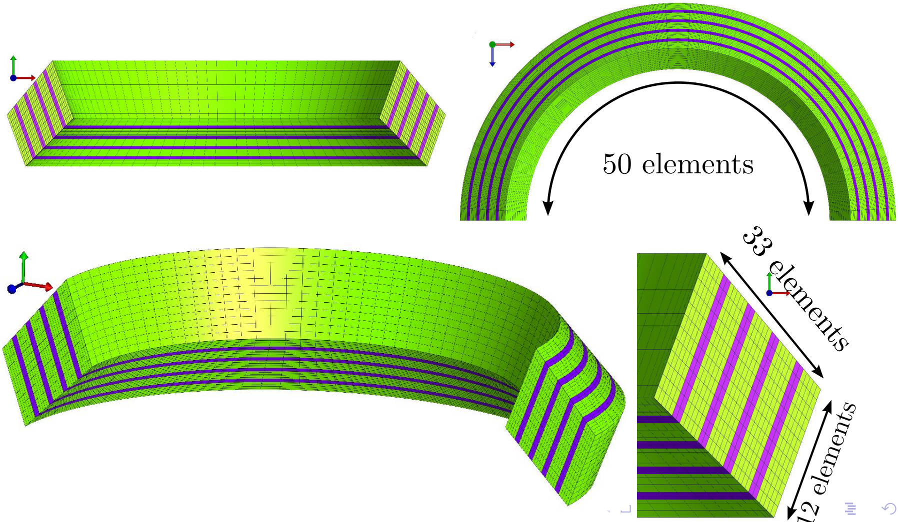
compressible + nearly incompressible test



compressible + nearly incompressible test

19800 elements, 22542 nodes → 67626 dofs

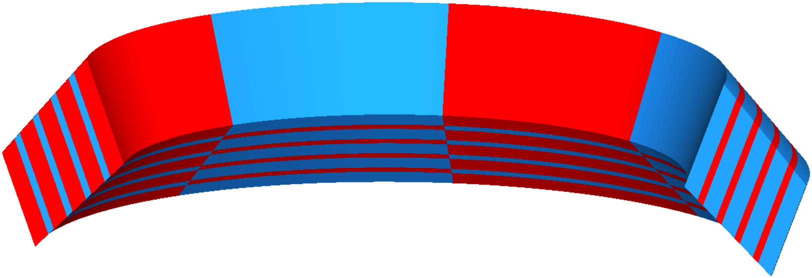
Symmetry conditions imposed



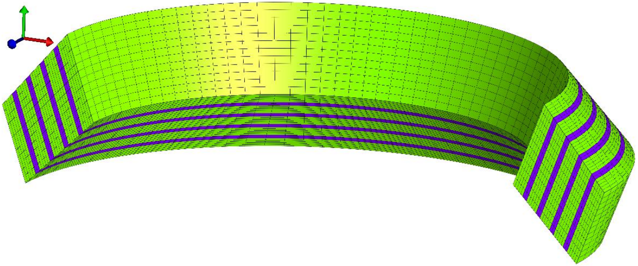
compressible + nearly incompressible test

$$36 \text{ elements } (4 \times 1 \times 9), \quad \begin{cases} p = 2 \rightarrow & 567 \text{ nodes}, & 1701 \text{ dofs} \\ p = 3 \rightarrow & 1296 \text{ nodes}, & 3888 \text{ dofs} \end{cases}$$

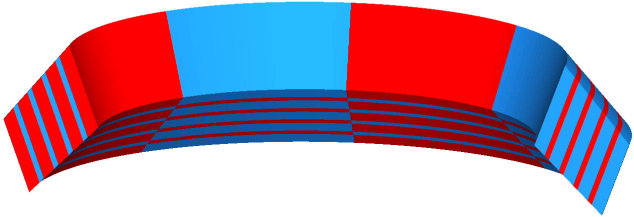
Each layer is a different NURBS patch



comp. + nearly incomp. test: FEM vs IGA mesh



19800 elements
67626 dofs



36 elements

$p = 2 \rightarrow$ 1701 dofs
$p = 3 \rightarrow$ 3888 dofs

Materials and numerical methods

(compressimble) Steel:

$E = 210\,\text{GPa}$, $\nu = 0.3$

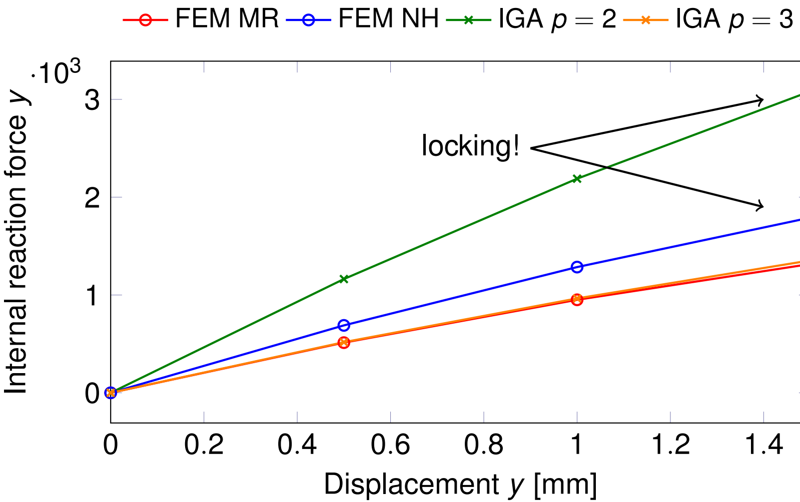
- Isotropic elasticity formulation used.

(nearly-incompressible) Rubber:

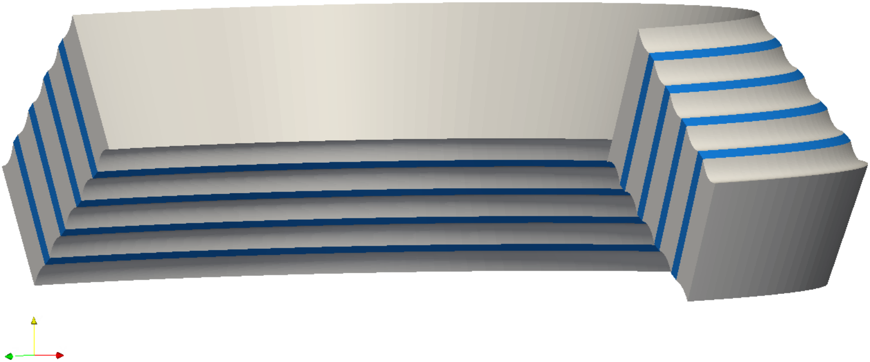
$C_{10} = 1\,\text{MPa}$, $K = 1000\,\text{MPa}$ ($E = 5.996\,\text{MPa}$, $\nu = 0.499$)

- For FEM:
  - Neo–Hokean with plain Galerkin formulation $\rightarrow$ solution locks
  - Mooney–Rivlin (three field: displacement$+$pressure$+$volume ratio) implemented with “selective-reduced integration” $\rightarrow$ suitable for incompressible materials.
- For IGA: Neo–Hokean with plain Galerkin formulation.

comp. + nearly incomp. test, phase 1

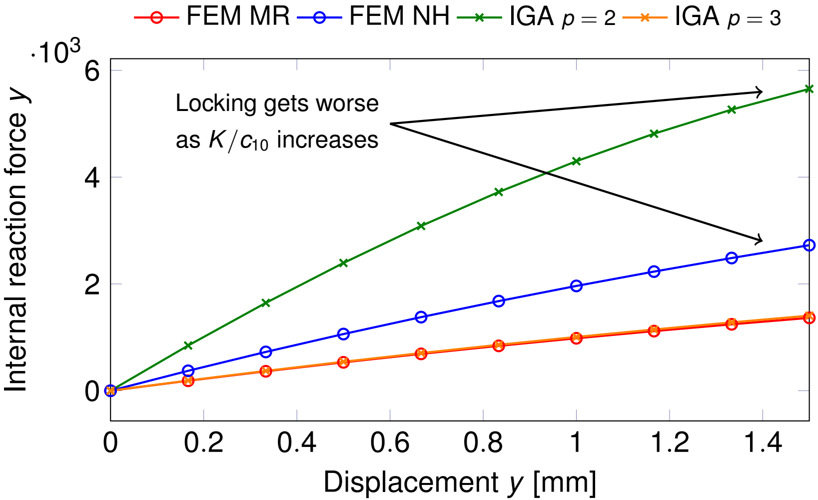


comp. + nearly incomp. test. IGA $p = 2$, phase 1

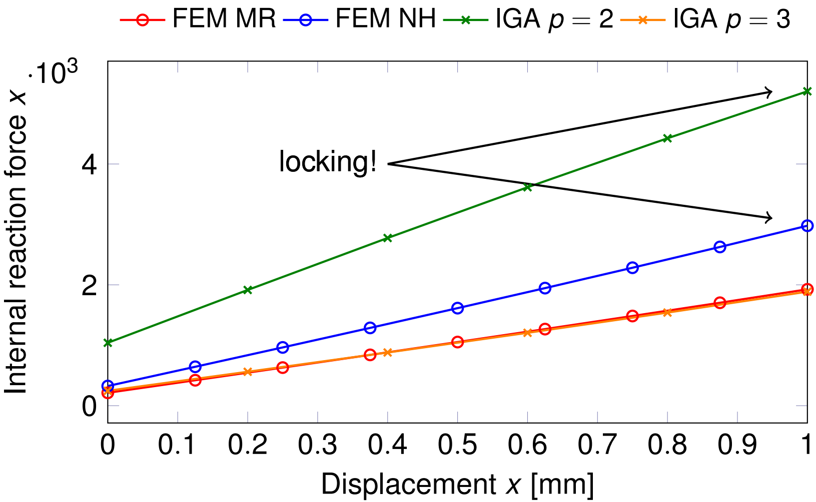


Deformation looks good, but solution is locked

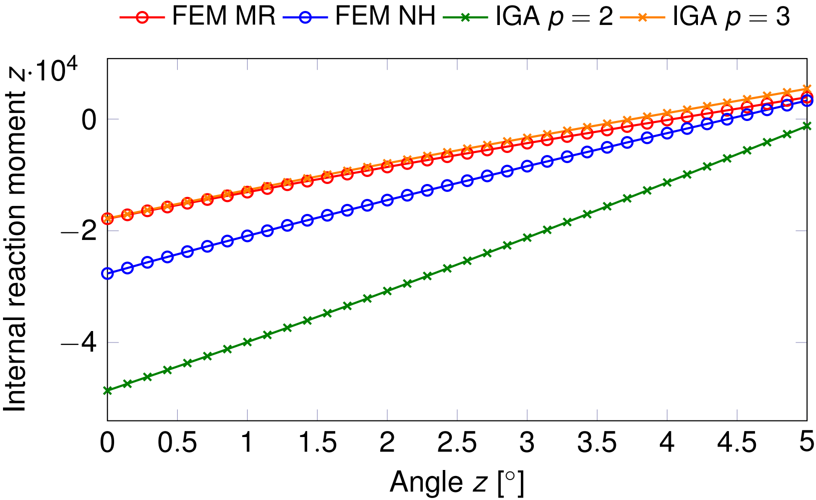
comp. + nearly incomp. test $K/c_{10} = 3000$. Phase 1



comp. + nearly incomp. test. Phase 2

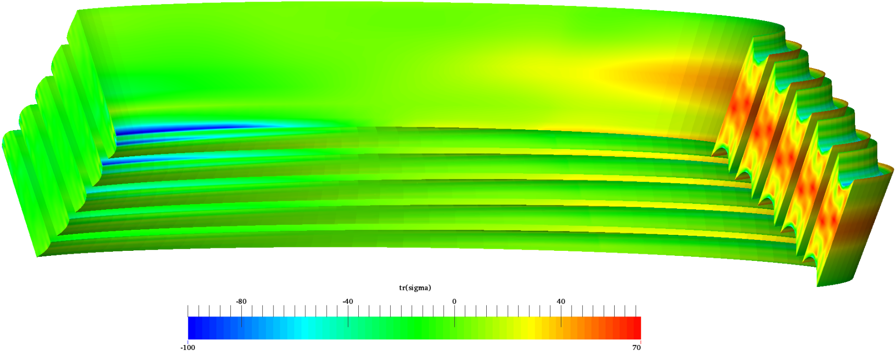


comp. + nearly incomp. test. Phase 3



Stress oscillations for IGA $p = 3$

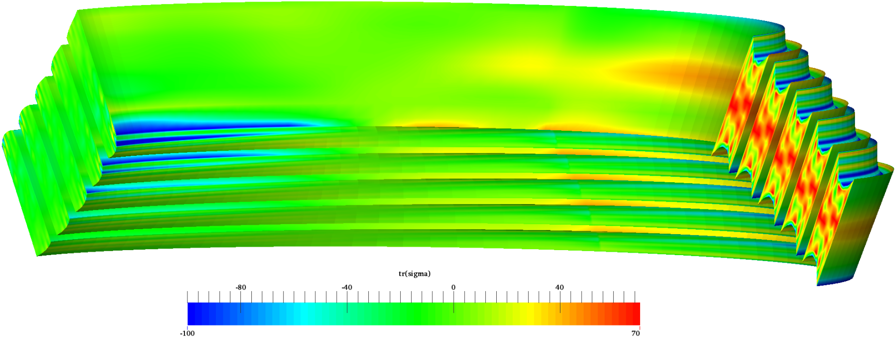
Oscillations appear in $\sigma^{\text{vol}} = \frac{1}{3}\,\mathrm{tr}(\sigma)\,\mathbf{1}$



$\frac{1}{3}\,\mathrm{tr}(\sigma)$ plotted

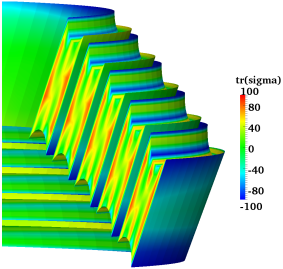
comp. + nearly incomp. test. Stress for IGA $p = 3$

$\mathbf{K/c_{10} = 3000}$

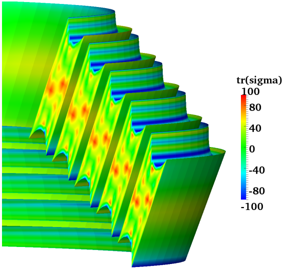


The scale is the same, the oscillations increase with K/c_{10}

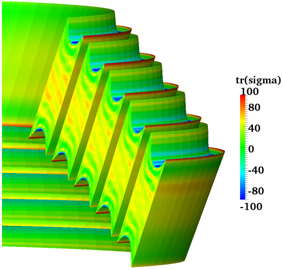
comp. + nearly incomp. Stress for IGA $p = 3$



$4 \times 2 \times 2$
elements / layer



$4 \times 4 \times 4$
elements / layer



$4 \times 8 \times 8$
elements / layer

Linear elasticity model problem

Strong form problem

$$
\begin{aligned}
\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} &= \mathbf{0} \quad \text{in}\, \Omega \\
\boldsymbol{u} &= \bar{\boldsymbol{u}} \quad \text{on}\, \Gamma_D \\
\boldsymbol{\sigma} \cdot \boldsymbol{n} &= \boldsymbol{t} \quad \text{on}\, \Gamma_N
\end{aligned}
$$

Linear elasticity model problem

Strong form problem

$$
\begin{aligned}
\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} &= \boldsymbol{0} \quad \text{in } \Omega \\
\boldsymbol{u} &= \bar{\boldsymbol{u}} \quad \text{on } \Gamma_D \\
\boldsymbol{\sigma} \cdot \boldsymbol{n} &= \boldsymbol{t} \quad \text{on } \Gamma_N
\end{aligned}
$$

Isotropic linear elasticity

$$
\begin{aligned}
\boldsymbol{\sigma} &= 2\,\mu\,\varepsilon + \lambda\,\nabla \cdot \boldsymbol{u}\,\mathbf{1} \\
\varepsilon &= \nabla^s \boldsymbol{u} \\
\lambda &= \frac{\nu\,E}{(1+\nu)\,(1-2\,\nu)} \\
\mu &= \frac{E}{2\,(1+\nu)}
\end{aligned}
$$

$$
\nu \to 1/2, \quad \lambda \to \infty
$$

Linear elasticity model problem

Strong form problem

$$
\begin{aligned}
\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} = \mathbf{0} \quad & \text{in } \Omega \\
\boldsymbol{u} = \bar{\boldsymbol{u}} \quad & \text{on } \Gamma_D \\
\boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{t} \quad & \text{on } \Gamma_N
\end{aligned}
$$

Isotropic linear elasticity

$$
\begin{aligned}
\boldsymbol{\sigma} &= 2\,\mu\,\boldsymbol{\varepsilon} + \lambda\,\nabla \cdot \boldsymbol{u}\,\mathbf{1} \\
\boldsymbol{\varepsilon} &= \nabla^s \boldsymbol{u} \\
\lambda &= \frac{\nu\,E}{(1+\nu)\,(1-2\,\nu)} \\
\mu &= \frac{E}{2\,(1+\nu)}
\end{aligned}
$$

$$\nu \to 1/2, \quad \lambda \to \infty$$

Weak form: find $\boldsymbol{u} \in (H^1(\Omega))^3$ with $\boldsymbol{u} = \bar{\boldsymbol{u}}$ on Γ_D and such that

$$\int_\Omega \nabla^s \boldsymbol{w} : \boldsymbol{\sigma}\,\mathrm{d}\Omega = \int_\Omega \boldsymbol{w} \cdot \boldsymbol{f}\,\mathrm{d}\Omega + \int_{\Gamma_N} \boldsymbol{w} \cdot \boldsymbol{t}\,\mathrm{d}\Gamma, \quad \forall \boldsymbol{w} \in (H^1_{\Gamma_D}(\Omega))^3$$

Linear elasticity model problem

Strong form problem

$$
\begin{aligned}
\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0} &\quad \text{in}\, \Omega \\
\boldsymbol{u} = \bar{\boldsymbol{u}} &\quad \text{on}\, \Gamma_D \\
\boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{t} &\quad \text{on}\, \Gamma_N
\end{aligned}
$$

Isotropic linear elasticity

$$
\begin{aligned}
\boldsymbol{\sigma} &= 2\,\mu\,\varepsilon + \lambda\,\nabla \cdot \boldsymbol{u}\,\mathbf{1} \\
\varepsilon &= \nabla^s \boldsymbol{u} \\
\lambda &= \frac{\nu\,E}{(1+\nu)\,(1-2\,\nu)} \\
\mu &= \frac{E}{2\,(1+\nu)}
\end{aligned}
$$

$$\nu \to 1/2, \quad \lambda \to \infty$$

Weak form: find $\boldsymbol{u} \in (H^1(\Omega))^3$ with $\boldsymbol{u} = \bar{\boldsymbol{u}}$ on Γ_D and such that

$$
\underbrace{\int_\Omega \nabla^s \boldsymbol{w} : \boldsymbol{\sigma}\, \mathrm{d}\Omega}_{a(\boldsymbol{w},\boldsymbol{u})} = \underbrace{\int_\Omega \boldsymbol{w} \cdot \boldsymbol{f}\, \mathrm{d}\Omega + \int_{\Gamma_N} \boldsymbol{w} \cdot \boldsymbol{t}\, \mathrm{d}\Gamma}_{L(\boldsymbol{w})}, \quad \forall \boldsymbol{w} \in (H^1_{\Gamma_D}(\Omega))^3
$$

Linear elasticity model problem

Strong form problem

$$
\begin{aligned}
\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0} \quad & \text{in } \Omega \\
\boldsymbol{u} = \bar{\boldsymbol{u}} \quad & \text{on } \Gamma_D \\
\boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{t} \quad & \text{on } \Gamma_N
\end{aligned}
$$

Isotropic linear elasticity

$$
\begin{aligned}
\boldsymbol{\sigma} &= 2\,\mu\,\varepsilon + \lambda\,\nabla \cdot \boldsymbol{u}\,\mathbf{1} \\
\varepsilon &= \nabla^s \boldsymbol{u} \\
\lambda &= \frac{\nu\,E}{(1+\nu)\,(1-2\,\nu)} \\
\mu &= \frac{E}{2\,(1+\nu)}
\end{aligned}
$$

$$\nu \to 1/2, \quad \lambda \to \infty$$

Weak form: find $\boldsymbol{u} \in (H^1(\Omega))^3$ with $\boldsymbol{u} = \bar{\boldsymbol{u}}$ on Γ_D and such that

$$\underbrace{\int_\Omega 2\mu\,\nabla^s \boldsymbol{w} : \nabla^s \boldsymbol{u} \mathrm{d}\Omega + \int_\Omega \lambda\,\nabla \cdot \boldsymbol{w}\,\nabla \cdot \boldsymbol{u}\,\mathrm{d}\Omega}_{a(\boldsymbol{w},\boldsymbol{u})} = L(\boldsymbol{w}), \quad \forall \boldsymbol{w} \in (H^1_{\Gamma_D}(\Omega))^3$$

Linear elasticity model problem

Strong form problem

$$\begin{aligned} \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0} \quad & \text{in } \Omega \\ \boldsymbol{u} = \bar{\boldsymbol{u}} \quad & \text{on } \Gamma_D \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{t} \quad & \text{on } \Gamma_N \end{aligned}$$

Isotropic linear elasticity

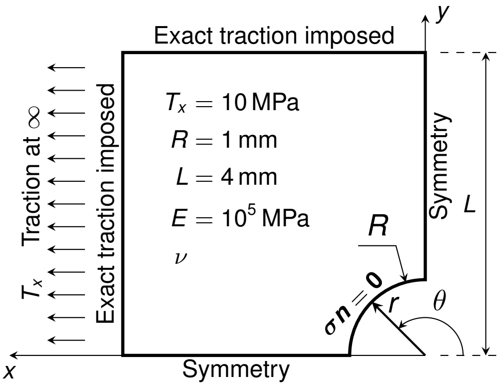
$$\begin{aligned} \boldsymbol{\sigma} &= 2\,\mu\,\varepsilon + \lambda\,\nabla \cdot \boldsymbol{u}\,\mathbf{1} \\ \varepsilon &= \nabla^s \boldsymbol{u} \\ \lambda &= \frac{\nu\,E}{(1+\nu)\,(1-2\,\nu)} \\ \mu &= \frac{E}{2\,(1+\nu)} \end{aligned}$$

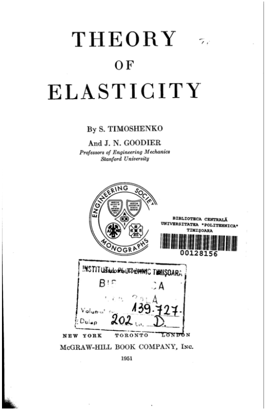
$$\nu \to 1/2, \quad \lambda \to \infty$$

Weak form: find $\boldsymbol{u} \in (H^1(\Omega))^3$ with $\boldsymbol{u} = \bar{\boldsymbol{u}}$ on Γ_D and such that

$$\underbrace{\int_\Omega 2\mu\,\nabla^s \boldsymbol{w} : \nabla^s \boldsymbol{u} d\Omega + \int_\Omega \lambda\,\nabla \cdot \boldsymbol{w}\,\nabla \cdot \boldsymbol{u}\,d\Omega}_{a(\boldsymbol{w},\boldsymbol{u})} = L(\boldsymbol{w}), \quad \forall \boldsymbol{w} \in (H^1_{\Gamma_D}(\Omega))^3$$

Classical benchmark: plate with a hole





THEORY
OF
ELASTICITY

By S. TIMOSHENKO
And J. N. GOODIER
Professors of Engineering Mechanics
Stanford University

ENGINEERING SOCIETIES MONOGRAPHS

NEW YORK TORONTO LONDON
McGRAW-HILL BOOK COMPANY, INC.
1951

$$\sigma_{rr}(r,\ \theta) = \frac{T_x}{2}\left[1 - \frac{R^2}{r^2} + \left(1 - 4\frac{R^2}{r^2} + 3\frac{R^4}{r^4}\right)\cos 2\theta\right] \ ,$$

$$\sigma_{\theta\theta}(r,\ \theta) = \frac{T_x}{2}\left[1 + \frac{R^2}{r^2} - \left(1 + 3\frac{R^4}{r^4}\right)\cos 2\theta\right] \ ,$$

$$\sigma_{r\theta}(r,\ \theta) = -\frac{T_x}{2}\left(1 + 2\frac{R^2}{r^2} - 3\frac{R^4}{r^4}\right)\sin 2\theta \ ,$$

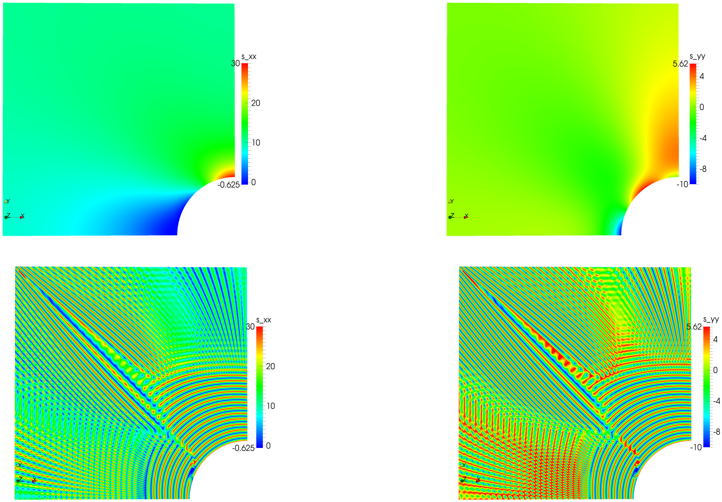
Standard formulation

Plain Galerkin (displacement) formulation

$$a(\boldsymbol{w}, \boldsymbol{u}) = \int_\Omega 2\mu \nabla^s \boldsymbol{w} : \nabla^s \boldsymbol{u} \mathrm{d}\Omega + \int_\Omega \lambda \nabla \cdot \boldsymbol{w} \nabla \cdot \boldsymbol{u} \, \mathrm{d}\Omega$$

- Stress oscillations and locking
- Symmetric ✓
- Sparse ✓
- Definite positive ✓

Exact vs plain formulation for $\nu = 0.49999$



Possible solution: $\bar{B}$ projection technique

projection technique

$$a(\boldsymbol{w}, \boldsymbol{u}) = \int_\Omega 2\mu \nabla^s \boldsymbol{w} : \nabla^s \boldsymbol{u} \mathrm{d}\Omega + \int_\Omega \lambda \, \pi(\nabla \cdot \boldsymbol{w}) \, \pi(\nabla \cdot \boldsymbol{u}) \, \mathrm{d}\Omega$$

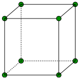
$$\pi(\phi)(\boldsymbol{x}) = \sum_i \tilde{B}_i(\boldsymbol{x}) \, c_i(\phi) \in \text{"coarser space"}$$

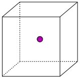
Possible solution: $\bar{B}$ projection technique

projection technique

$$a(\boldsymbol{w}, \boldsymbol{u}) = \int_\Omega 2\mu \nabla^s \boldsymbol{w} : \nabla^s \boldsymbol{u} \mathrm{d}\Omega + \int_\Omega \lambda \pi(\nabla \cdot \boldsymbol{w}) \pi(\nabla \cdot \boldsymbol{u}) \mathrm{d}\Omega$$

$$\pi(\phi)(\boldsymbol{x}) = \sum_i \tilde{B}_i(\boldsymbol{x}) c_i(\phi) \in \text{"coarser space"}$$

$\boldsymbol{w}, \boldsymbol{u} \in$ 

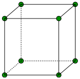
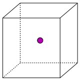
$\pi(\nabla \cdot \boldsymbol{w}), \pi(\nabla \cdot \boldsymbol{u}) \in$ 

Possible solution: $\bar{B}$ projection technique

projection technique

$$a(\boldsymbol{w},\, \boldsymbol{u}) = \int_\Omega 2\mu\, \nabla^s \boldsymbol{w} : \nabla^s \boldsymbol{u} \mathrm{d}\Omega + \int_\Omega \lambda\, \pi\left(\nabla \cdot \boldsymbol{w}\right)\, \pi\left(\nabla \cdot \boldsymbol{u}\right)\, \mathrm{d}\Omega$$

$$\pi(\phi)(\boldsymbol{x}) = \sum_i \tilde{B}_i(\boldsymbol{x})\, c_i(\phi) \in \text{"coarser space"}$$

$\boldsymbol{w},\, \boldsymbol{u} \in$  $\pi\left(\nabla \cdot \boldsymbol{w}\right), \pi\left(\nabla \cdot \boldsymbol{u}\right) \in$ 

... then perform degree-elevation and knot-insertion

Possible solution: $\bar{B}$ projection technique

projection technique

$$a(\boldsymbol{w},\, \boldsymbol{u}) = \int_\Omega 2\mu\, \nabla^s \boldsymbol{w} : \nabla^s \boldsymbol{u} \mathrm{d}\Omega + \int_\Omega \lambda\, \pi\left(\nabla \cdot \boldsymbol{w}\right)\, \pi\left(\nabla \cdot \boldsymbol{u}\right)\, \mathrm{d}\Omega$$

$$\pi(\phi)(\boldsymbol{x}) = \sum_i \tilde{B}_i(\boldsymbol{x})\, c_i(\phi) \in \text{"coarser space"}$$

$\bar{B}$-method [Elguedj, Bazilevs, Calo, and Hughes, 2008]

$\boldsymbol{u} \in \mathcal{S}_p \times \mathcal{S}_p \times \mathcal{S}_p$ and $\pi(\cdot) \in \mathcal{S}_{p-1}$

Possible solution: $\bar{B}$ projection technique

projection technique

$$a(\boldsymbol{w}, \boldsymbol{u}) = \int_\Omega 2\mu \nabla^s \boldsymbol{w} : \nabla^s \boldsymbol{u} \mathrm{d}\Omega + \int_\Omega \lambda \, \pi(\nabla \cdot \boldsymbol{w}) \, \pi(\nabla \cdot \boldsymbol{u}) \, \mathrm{d}\Omega$$

$$\pi(\phi)(\boldsymbol{x}) = \sum_i \tilde{B}_i(\boldsymbol{x}) \, c_i(\phi) \in \text{"coarser space"}$$

$$c_i(\phi) = \tilde{\mathbf{M}}_{ij}^{-1} \int_\Omega \tilde{N}_j(\boldsymbol{x}) \, \phi(\boldsymbol{x}) \mathrm{d}\Omega$$

Possible solution: $\bar{B}$ projection technique

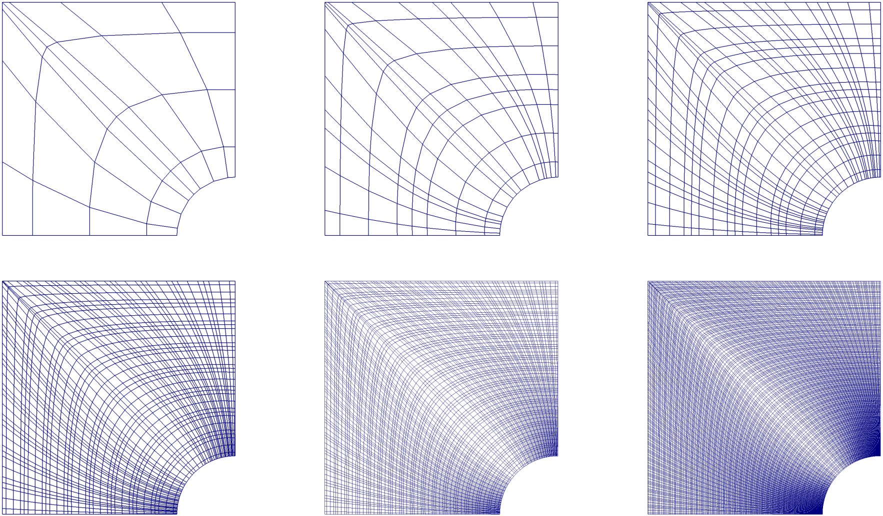
projection technique

$$a(\boldsymbol{w}, \boldsymbol{u}) = \int_\Omega 2\mu \nabla^s \boldsymbol{w} : \nabla^s \boldsymbol{u} \mathrm{d}\Omega + \int_\Omega \lambda \, \pi(\nabla \cdot \boldsymbol{w}) \, \pi(\nabla \cdot \boldsymbol{u}) \, \mathrm{d}\Omega$$

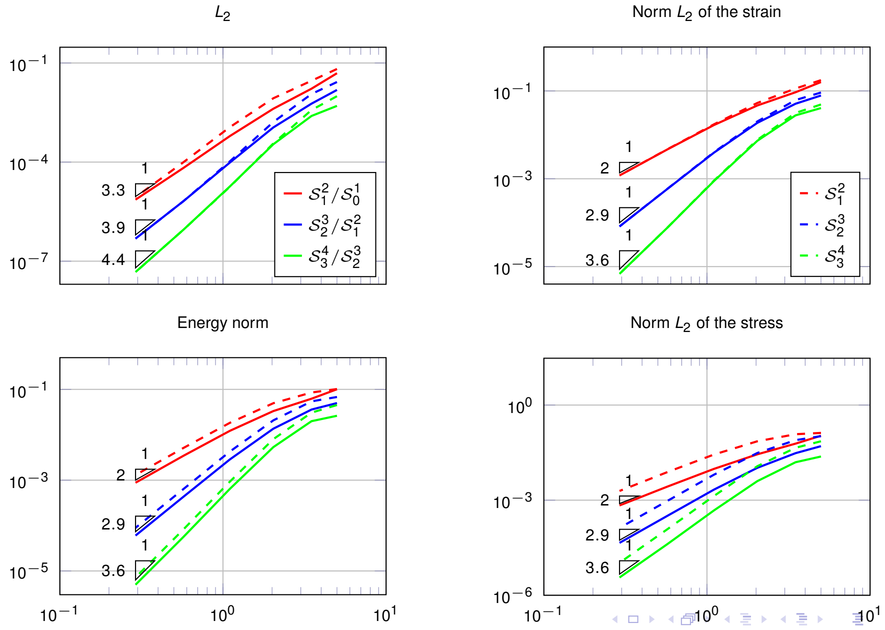
$$\pi(\phi)(\boldsymbol{x}) = \sum_i \tilde{B}_i(\boldsymbol{x}) \, c_i(\phi) \in \text{"coarser space"}$$

- Unlocked solution ✓
- Symmetric ✓
- full matrix
- Definite positive ✓

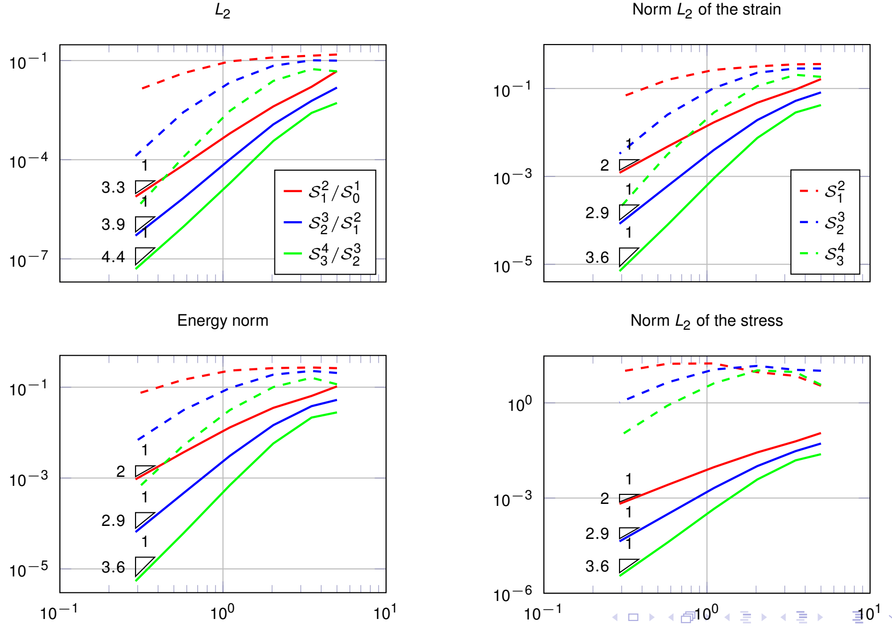
mesh refinement in our next tests



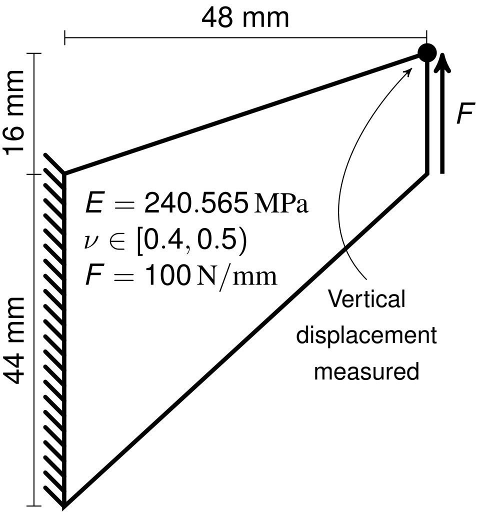
$\bar{B}$ for $\nu = 0.4$



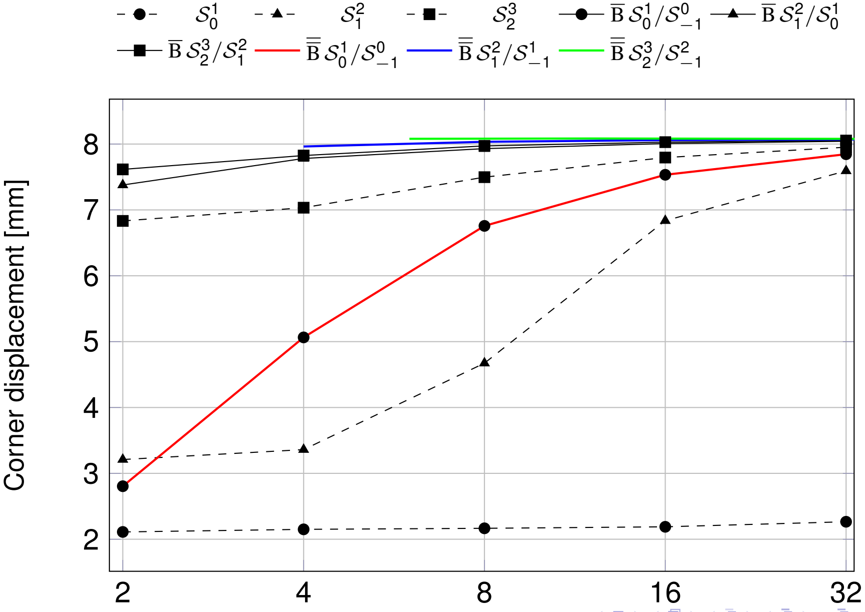
$\bar{B}$ for $\nu = 0.49999$



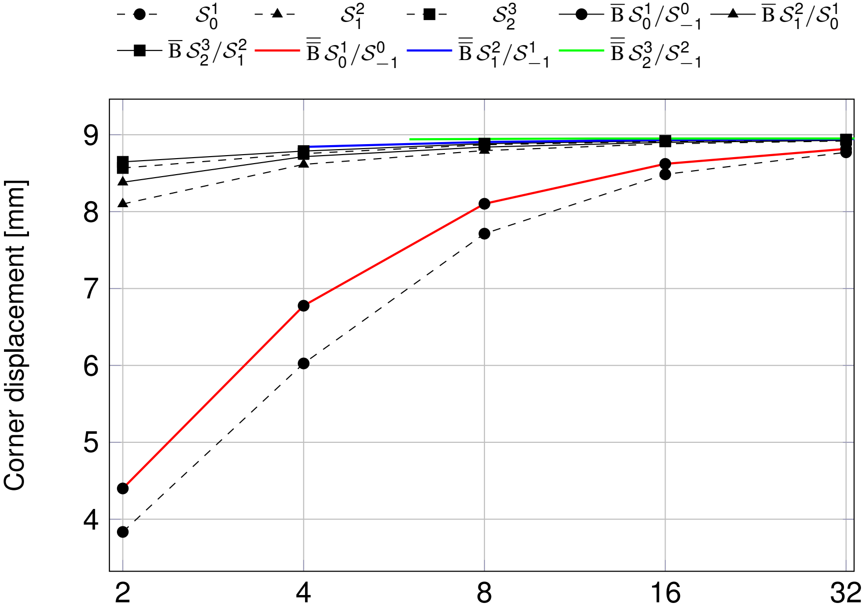
Cook Membrane



Cook Membrane $\nu = \mathbf{0.49999}$



Cook Membrane $\nu = \mathbf{0.4}$



Projection formulation as a mixed formulation

projection technique

$$\int_\Omega 2\mu \nabla^s \boldsymbol{w} : \nabla^s \boldsymbol{u} + \int_\Omega \lambda \, \pi \, (\nabla \cdot w) \; \pi \, (\nabla \cdot u) = \int_\Omega \boldsymbol{w} \cdot \boldsymbol{f}, \quad \forall \boldsymbol{w}$$

Mixed formulation (for the unknowns *u* and *p*)

$$\int_\Omega 2\mu \nabla^s \boldsymbol{w} : \nabla^s \boldsymbol{u} + \int_\Omega \nabla \cdot \boldsymbol{w} \, \wp = \int_\Omega \boldsymbol{w} \cdot \boldsymbol{f}, \qquad \forall \boldsymbol{w}$$

$$\int_\Omega (\nabla \cdot \boldsymbol{u}) \, q - \lambda^{-1} \int_\Omega \wp \, q = 0, \quad \forall q$$

where the second equation states $\lambda \, \pi(\nabla \cdot \boldsymbol{u}) = \wp$

Projection formulation as a mixed formulation

projection technique

$$\int_\Omega 2\mu \nabla^s \boldsymbol{w} : \nabla^s \boldsymbol{u} + \int_\Omega \lambda \pi (\nabla \cdot w)\, \pi (\nabla \cdot u) = \int_\Omega \boldsymbol{w} \cdot \boldsymbol{f}, \quad \forall \boldsymbol{w}$$

Mixed formulation (for the unknowns $\boldsymbol{u}$ and p)

$$\int_\Omega 2\mu \nabla^s \boldsymbol{w} : \nabla^s \boldsymbol{u} + \int_\Omega \nabla \cdot \boldsymbol{w}\, \wp = \int_\Omega \boldsymbol{w} \cdot \boldsymbol{f}, \qquad \forall \boldsymbol{w}$$

$$\int_\Omega (\nabla \cdot \boldsymbol{u})\, q - \lambda^{-1} \int_\Omega \wp\, q = 0, \quad \forall q$$

where the second equation states $\lambda\, \pi(\nabla \cdot \boldsymbol{u}) = \wp$

$\bar{B}$ method: $\boldsymbol{u}, \boldsymbol{w} \in \mathcal{S}^p_{p-1} \times \mathcal{S}^p_{p-1} \times \mathcal{S}^p_{p-1}$ and $\wp, q \in \mathcal{S}^{p-1}_{p-2}$.

Well-posedness of mixed formulation

Ladyzenskaya-Babuška-Brezzi (LBB) inf-sup condition

The discrete displacement space $\boldsymbol{V}_h \subset (H^1_{\Gamma_D}(\Omega))^2$ and the discrete pressure space $Q_h \subset L^2(\Omega)$ have to fulfill

$$\inf_{q \in Q_h} \sup_{\boldsymbol{v} \in \boldsymbol{V}_h} \frac{\int_\Omega \nabla \cdot \boldsymbol{v}\, q \, \mathrm{d}\Omega}{\|q\|_{L^2} \|\boldsymbol{v}\|_{(H^1)^2}} \geq C_{is} > 0 \quad \text{(uniformly w.r.t. } h\text{)}.$$

The **inf-sup condition** above holds if “locally” and “on average” there are more knot lines of the displacement field than knot lines of the pressure field, roughly speaking... [Bressan and Sangalli, 2013] ▸ quick proof

Well-posedness of $\boldsymbol{u}$ component only

However "some elements... are definitely known not to satisfy the well-known LBB stability condition but nevertheless are favored and widely utilized in engineering applications, the prime example being the mean dilatation, bilinear quadrilateral element $Q1/P0$" [Elguedj, Bazilevs, Calo, and Hughes, 2008]

Pitkäranta-Stenberg weak LBB inf-sup stability

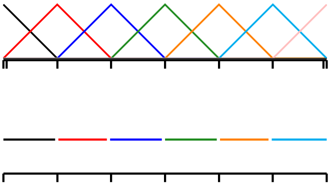
[Pitkäranta and Stenberg, 1984]

"The analysis of the $Q1/P0$ element relies on a weak BB-type stability condition..."

A special case for the mixed formulation

$\boldsymbol{u} \in$

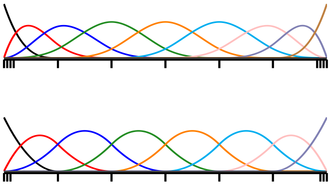
$\wp \in$



Q^1/P^0 method

$\boldsymbol{u} \in$

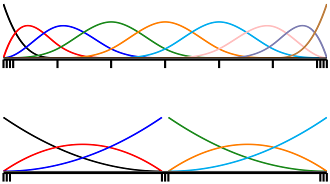
$\wp \in$



$\bar{\boldsymbol{B}}$ method

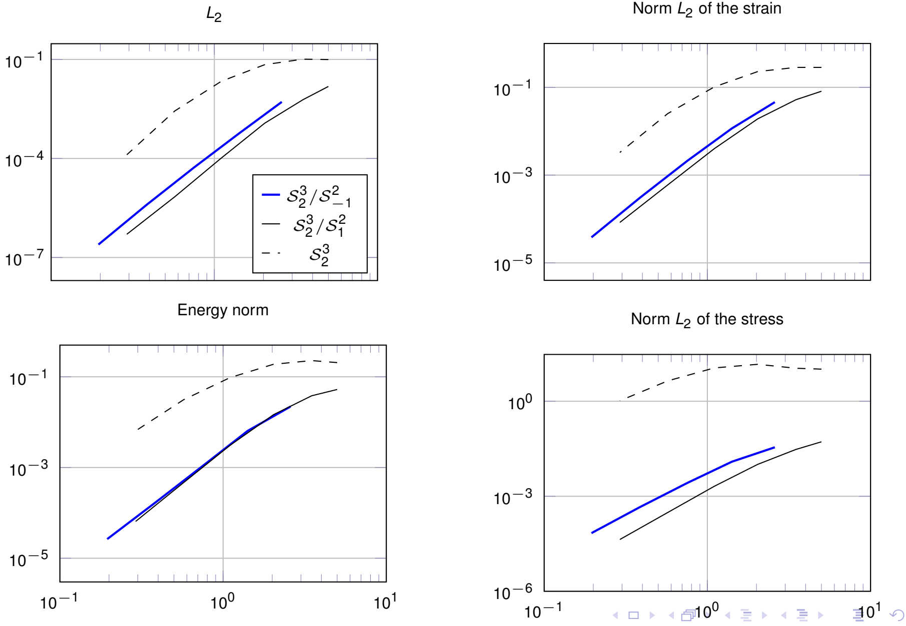
$\boldsymbol{u} \in$

$\wp \in$

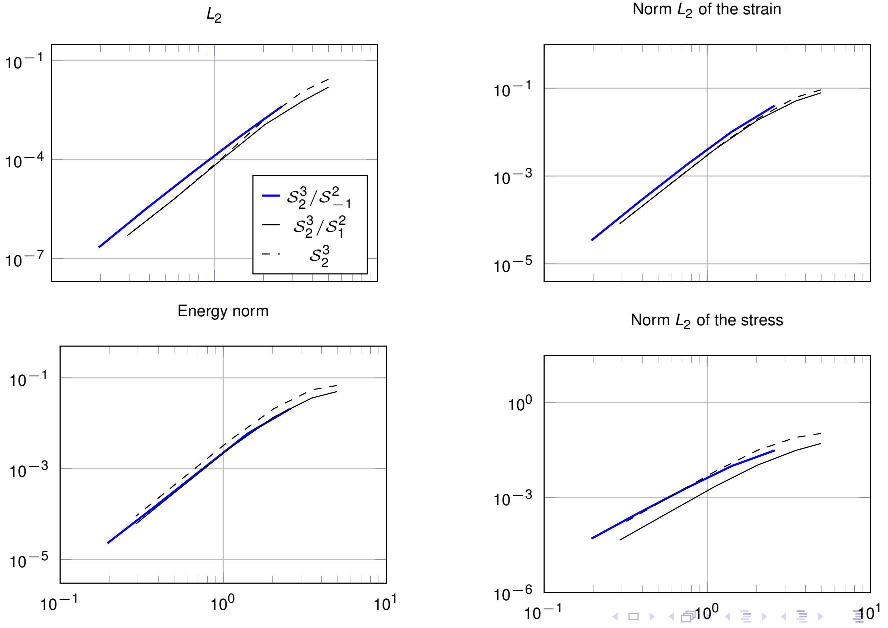


$\bar{\bar{\boldsymbol{B}}}$ method

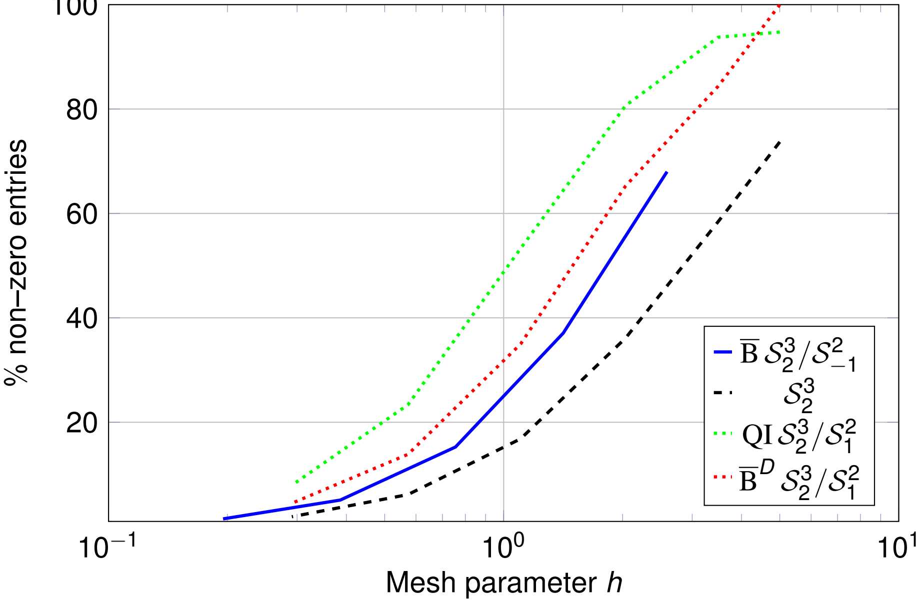
Macroelement projection π_M for $\nu = 0.49999$



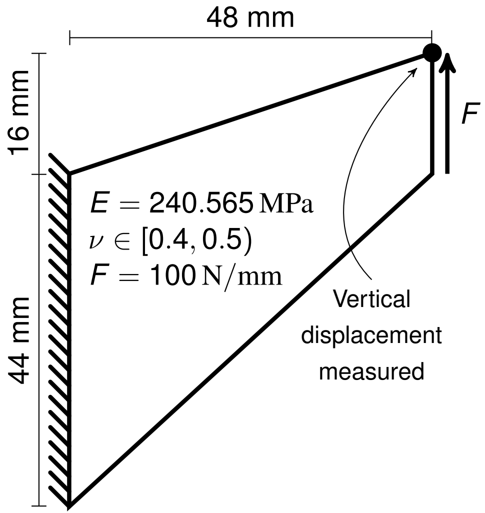
Macroelement projection π_M for $\nu = 0.4$



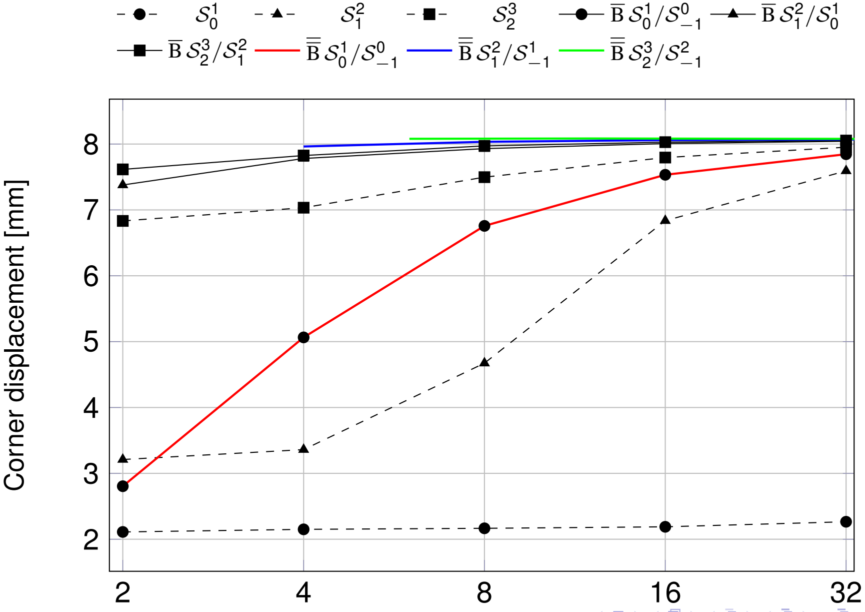
Macroelement projection π_M: sparsity



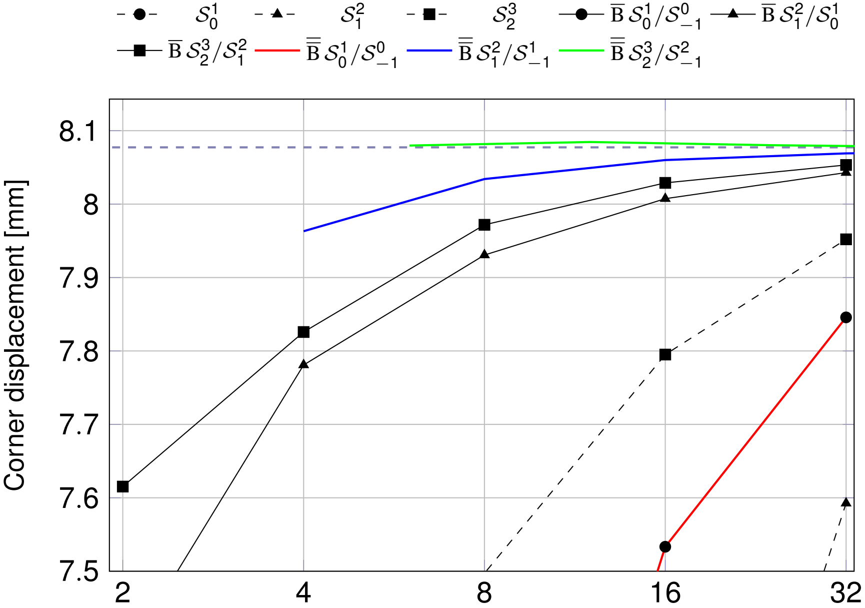
Cook Membrane



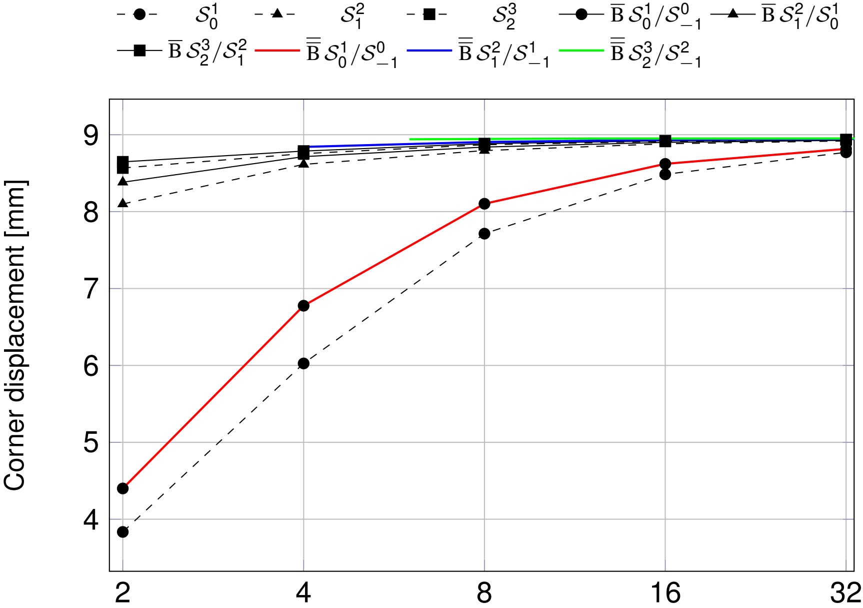
Cook Membrane $\nu = \mathbf{0.49999}$



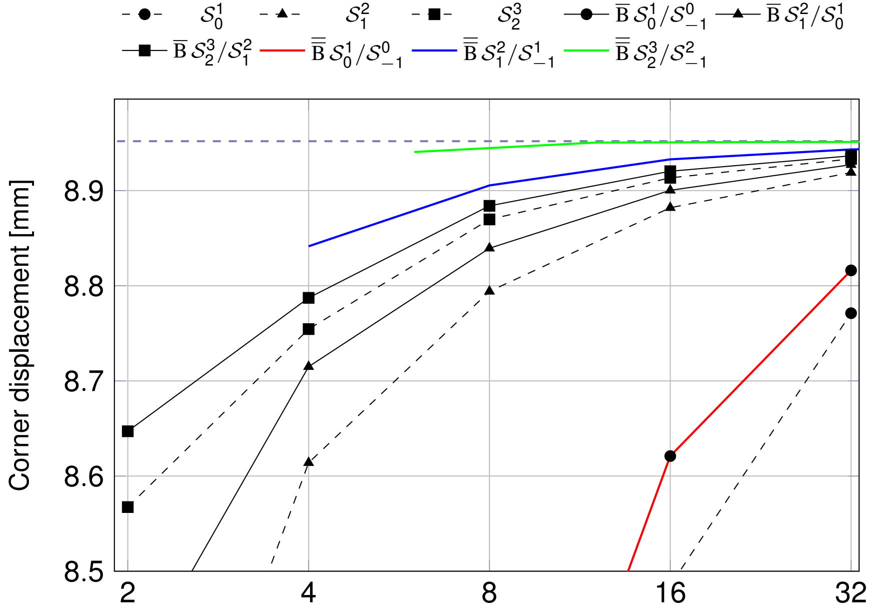
Cook Membrane $\nu = \mathbf{0.49999}$



Cook Membrane $\nu = \mathbf{0.4}$

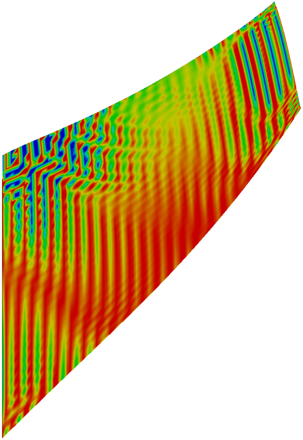


Cook Membrane $\nu = \mathbf{0.4}$

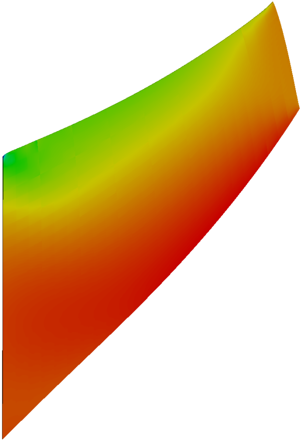


Cook Membrane σ_{xx} for $\nu = \mathbf{0.49999}$ $p = \mathbf{3}$

Standard formulation

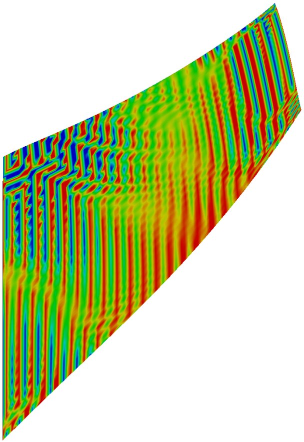


Discontinuous proj. $\overline{\overline{\mathrm{B}}}$

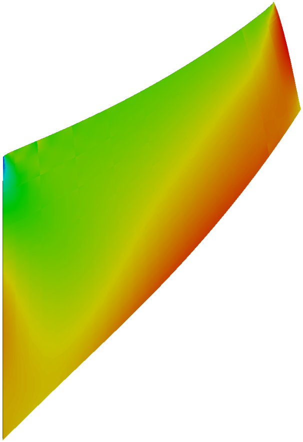


Cook Membrane σ_{yy} for $\nu = \mathbf{0.49999}$ $p = \mathbf{3}$

Standard formulation



Discontinuous proj. $\overline{\overline{\mathrm{B}}}$



References

Bressan, A. and G. Sangalli (2013). Isogeometric discretizations of the Stokes problem: stability analysis by the macroelement technique. *IMA J. Numer. Anal. 33*(2), 629–651.

Elguedj, T., Y. Bazilevs, V. Calo, and T. J. R. Hughes (2008). $\bar{B}$ and $\bar{F}$ projection methods for nearly incompressible linear and non-linear elasticity and plasticity using higher-order NURBS elements. *Comput. Methods Appl. Mech. Engrg. 197*, 2732–2762.

Pitkäranta, J. and R. Stenberg (1984). Error bounds for the approximation of the stokes problem using bilinear/constant elements on irregular quadrilateral meshes.