CIME-EMS Summer School in Applied Mathematics

Splines and PDEs: Recent Advances
from Approximation Theory to Structured Numerical Linear Algebra

July 3 - July 7, 2017 - Cetraro

Approximation properties of isogeometric spaces

Giancarlo Sangalli

...with results from many colleagues: L. Beirao da Veiga, A. Buffa, A. Collin, T. Takacs, R. Vázquez,...

How accurate is IGA vs FEA?

Let us discuss a 1D toy problem...

1D toy problem

Find u such that

$$-u''(x) = f(x), \quad x \in (0,1) \qquad u(0) = u(1) = 0$$

Toy problem in variational form

Find $u_h \in V_h$ such that

$$\int_0^1 u_h'(x)\, v_h'(x) = \int_0^1 f(x)\, v_h(x), \qquad \forall v_h \in V_h$$

- $V_h = S_k^p = \text{span}\{B_i^p\}$
- $h =$ mesh size of the knot vector Ξ
- $p =$ degree
- $\Xi =$ knot vector with interior multiplicity $p-k$ $\rightsquigarrow$ C^k regularity.

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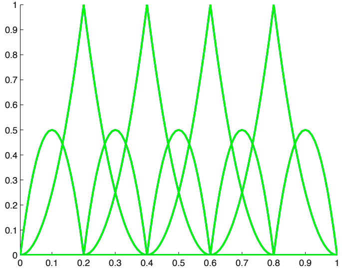
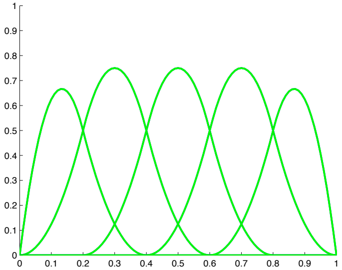
Examples of V_h for p=2 and 5 Bézier elements:



$\Xi = [0, 0, .2, .2, .4, .4, .6, .6, .8, .8, 1, 1]$

FEM

splines with C^0 continuity



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k-method

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Observe: more d.o.f's for lower continuity

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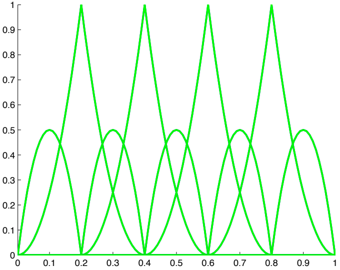

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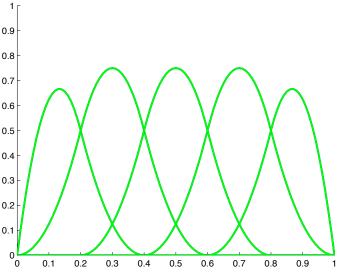

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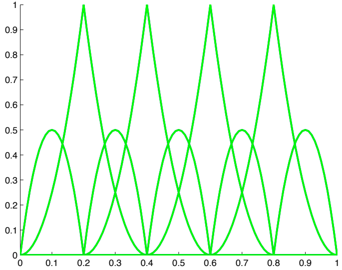

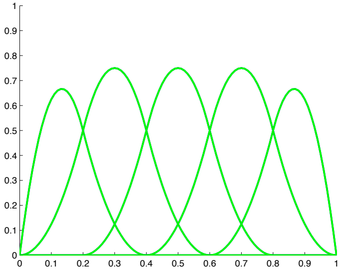

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Introducing the B-spline basis $\{B_i^p\}$:

Toy problem: linear system

Find $u_h(x) = \sum_j u_j B_j^p(x)$ such that

$$\sum_j \underbrace{\left[\int_0^1 (B_j^p)'(x)\,(B_i^p)'(x) dx\right]}_{\boldsymbol{A}_{i,j}} [u_j] = \int_0^1 f(x)\, B_i^p(x) dx, \quad \forall i = 1, \dots, \dim(V_h).$$

The compact support of B-splines means $\boldsymbol{A}$ is sparse...

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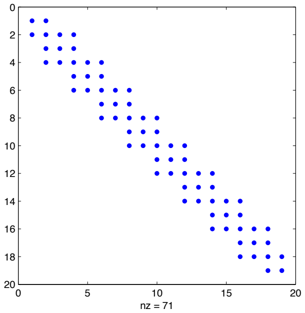
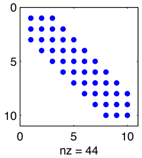
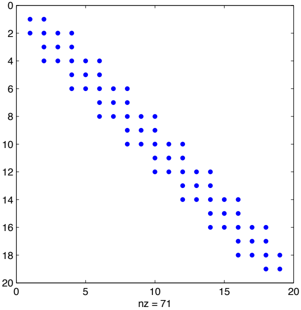
Sparsity pattern of $\boldsymbol{A}$ for p=2 and 10 Bézier elements:

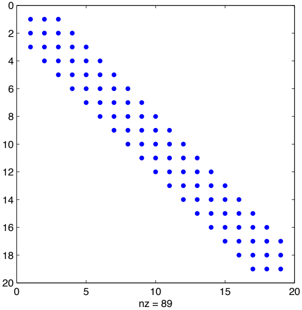




Left: C^0 vs right: C^1.

Sparsity pattern of $\boldsymbol{A}$ for p=2 and 19 d.o.f.'s:





Left: C^0 with $h = 1/10$ vs right: C^1 with $h \approx 1/20$.

Observe: the left linear system has same bandwidth and is more sparse, but it is also associated to a coarser mesh.

How do they compare in terms of accuracy?

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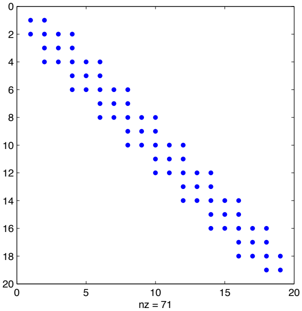

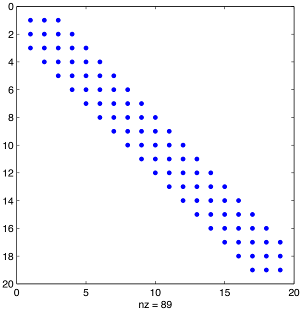

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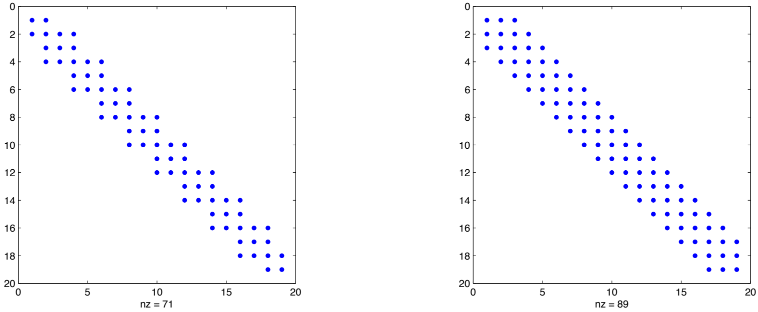

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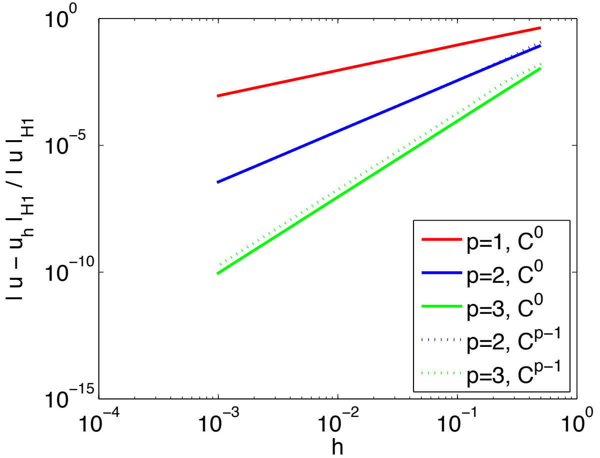
1D accuracy test ($u = \sin(\pi x)$)



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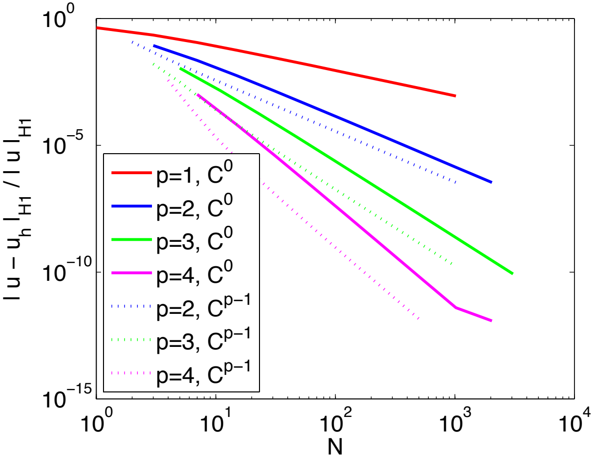

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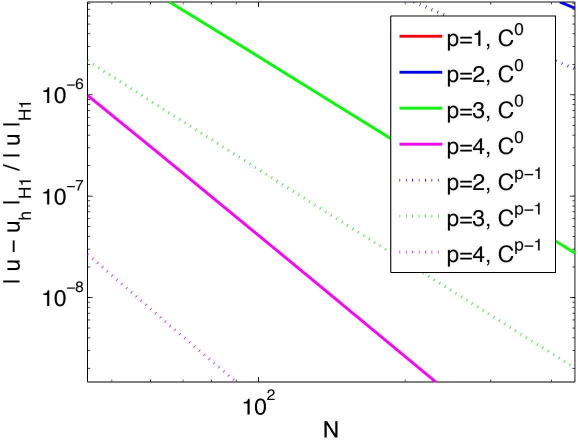

Higher accuracy per degree-of-freedom for k-method (C^{p-1} splines)

1D accuracy estimates

The previous plots show the known behaviour [Schumaker, 2007]

$$\|u - u_h\|_{H^1} \leq C(p, k) N^{-p} |u|_{H^{p+1}}$$

The role of the degree p and the regularity k is more difficult to study.

There is numerical evidence that C^{p-1} splines are an optimal approximating space:

- N-width [Evans, Bazilevs, Babuška, and Hughes, 2009]
- exponential convergence [Buffa, Sangalli, and Schwab, 2014]
- spectrum convergence [Hughes, Reali, and Sangalli, 2008]

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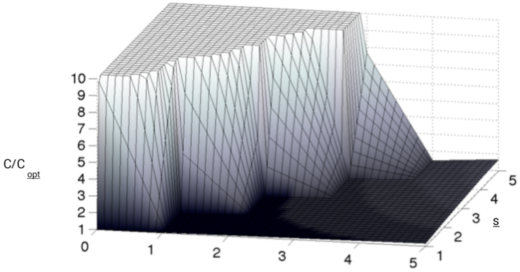
C^{p-1} splines: $\|u - u_h\|_{L^2} \leq CN^{-s}|u|_{H^s}$

best N-dim. space: $\|u - u_h\|_{L^2} \leq C_{opt}N^{-s}|u|_{H^s}$



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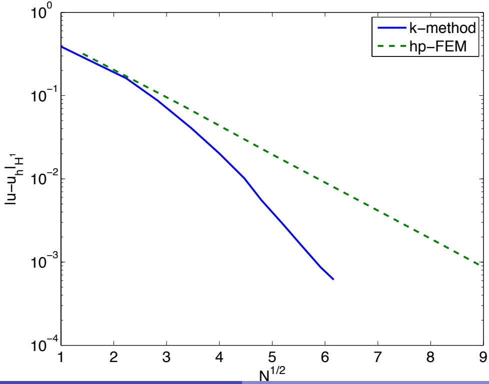
Combined local *h*-refinement and degree elevation allows "exponential" accuracy:



$$u(x) = x^{0.7} - x$$

$$\|u-u_h\|_{H^1} \leq Ce^{-b\sqrt{N}}$$

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Find the eigenfunctions $u_n \neq 0$ and frequencies $\omega_n > 0$ such that

$$-u''(x) = \omega_n^2 u(x), \quad x \in (0,1) \qquad u(0) = u(1) = 0$$

Find the discrete eigenfunctions $u_{n,h} \in V_h$ and $\omega_{n,h} > 0$ such that

$$\int_0^1 u'_{n,h}(x)\, v'_h(x) = \omega_{n,h}^2 \int_0^1 u_{n,h}(x)\, v_h(x), \qquad \forall v_h \in V_h$$

We are interested in $|\omega_{n,h} - \omega_n|/\omega_n$ ...

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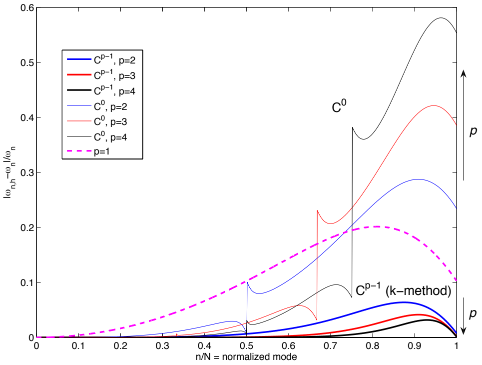

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C^{p-1} splines approximation properties

- N-width:
  C^{p-1} splines optimally approximate low-order Sobolev spaces
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- spectrum convergence:
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Extension to multi-D

Multivariate B-splines

Univariate B-splines

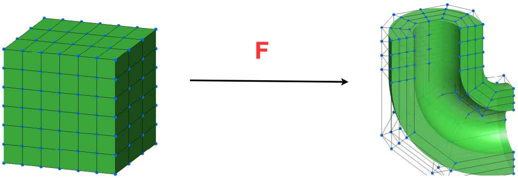
S^p = space of B-splines of degree p and regularity C^k, up to $k = p - 1$

Multivariate B-splines

B-splines are generalized by tensor products on a d-dimensional parametric patch $[0,1]^d$. E.g., for $d = 3$

$$S^{p_1,p_2,p_3} := S^{p_1} \otimes S^{p_2} \otimes S^{p_3}.$$

same for NURBS...



$\mathbf{F}(\xi) = \sum_{\mathbf{i}} \mathbf{C_i} N_{\mathbf{i}}(\xi)$

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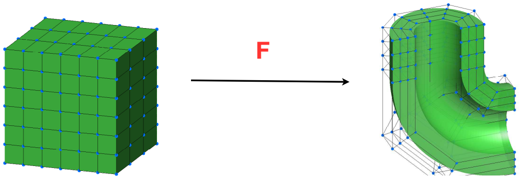

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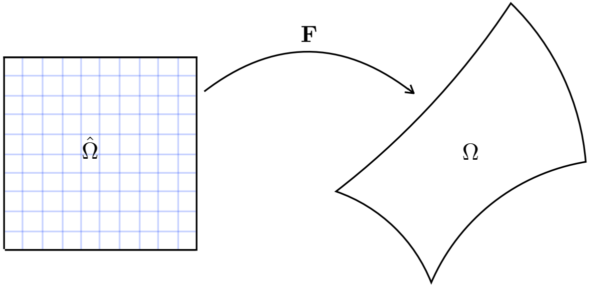
IGA on a NURBS-mapped geometry



Ω is described by a CAD parametrisation $\mathbf{F} \in \mathcal{S}_r^p(\widehat{\Omega}) \times \mathcal{S}_r^p(\widehat{\Omega})$

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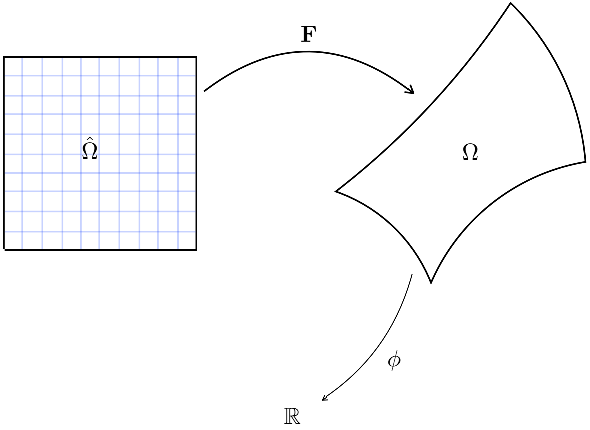

$\phi : \Omega \to \mathbb{R}$ is an "approximate solution" of the PDE

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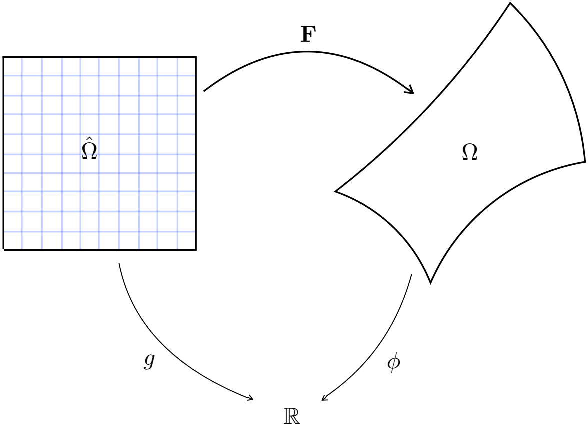

$\phi : \Omega \to \mathbb{R}$ is an *isogeometric function* if $g = \phi \circ \mathbf{F} \in \mathcal{S}^p_r(\widehat{\Omega})$

[Cottrell, Hughes, and Bazilevs, 2009]

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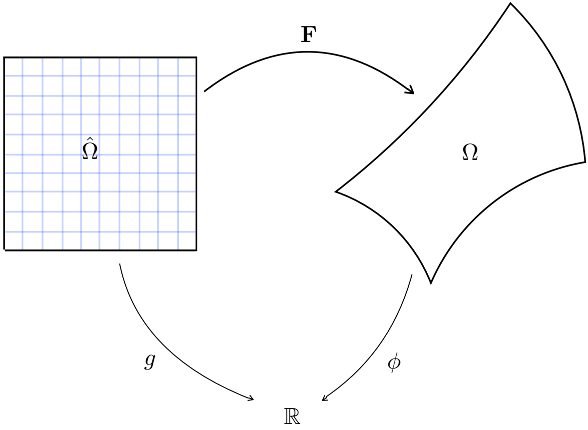
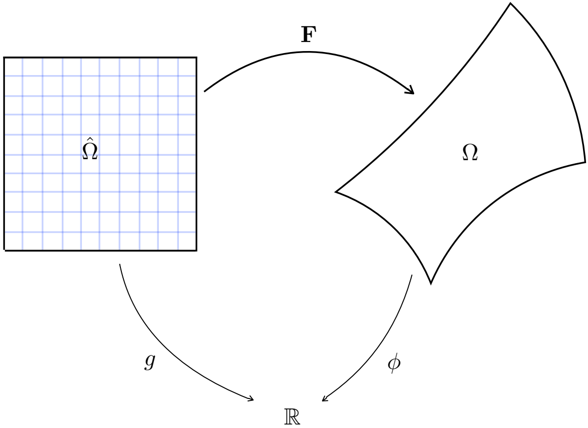


Isogeometric space

$$\mathcal{V} = \left\{ \phi : \Omega \to \mathbb{R} \text{ such that } g = \phi \circ \mathbf{F} \in \mathcal{S}_r^p(\widehat{\Omega}) \right\}$$

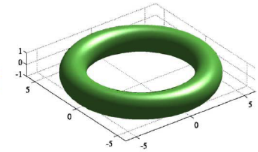
IGA on a NURBS-mapped geometry

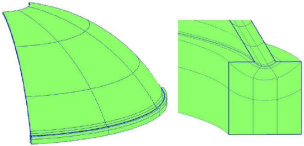


$$\begin{bmatrix} \mathbf{F} \\ g \end{bmatrix} \text{ parametrizes of } \Sigma = \{(x, y, \phi(x, y)) : (x, y) \in \Omega\}$$

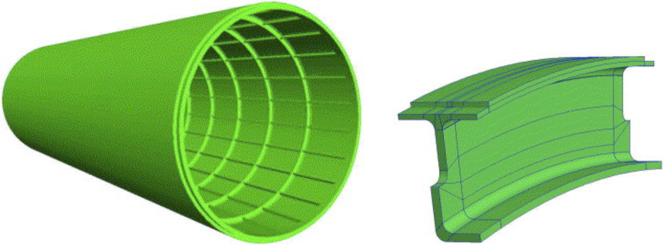
single patch vs. multipatch geometry

- single patch:





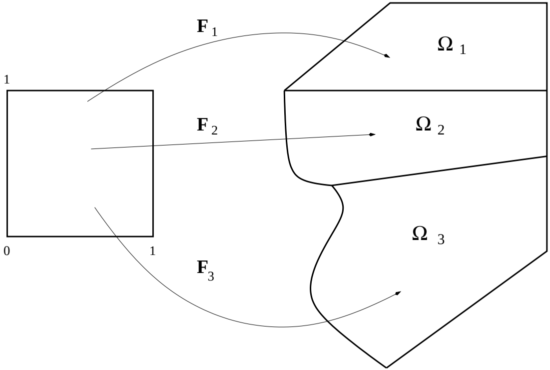
- multi-patch:



Courtesy by [Cottrell, Reali, Bazilevs, and Hughes, 2006]

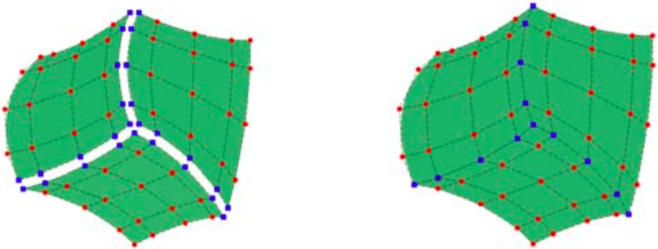
Multipatch domains: the framework

In CADs, geometries are described by mappings of several patches.



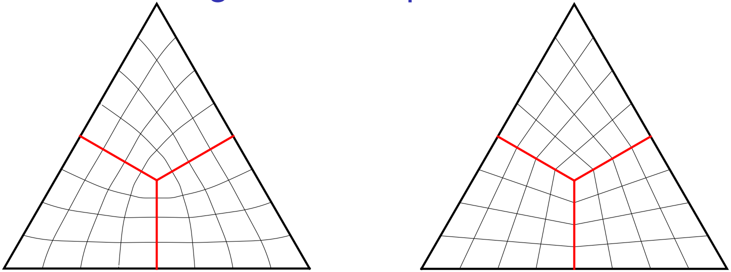
- C^0 gluing (easy for conforming meshes)
- C^1 gluing (challenge at extraordinary vertices)
- weak (C^0, C^1, ...) gluing

Multipatch domains: C^0 conforming case



C^0 conforming meshes just requires identification of control points and control variables

Multipatch: C^1 isogeometric spaces



- Left: C^1 parametrization (possibly with few exceptions) [Nguyen, Karciauskas, and Peters, 2014; D. Toshniwal, H. Speleers, T.J.R. Hughes Hughes, 2017] ...
- Right: (only) C^0 parametrization [Kapl, Buchegger, Bercovier, and Jüttler, 2016; Collin, Sangalli, and Takacs, 2015] ...

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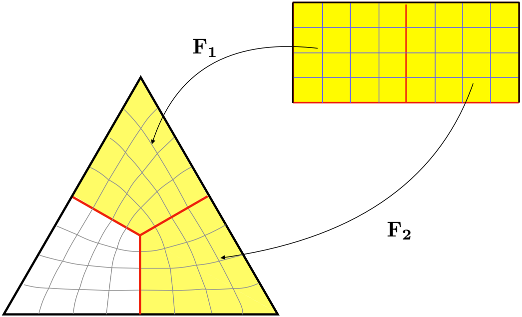

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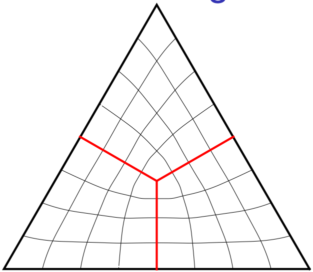

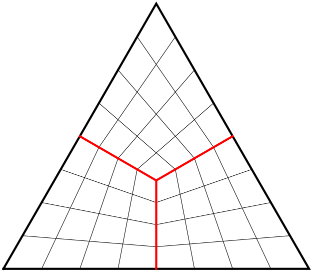

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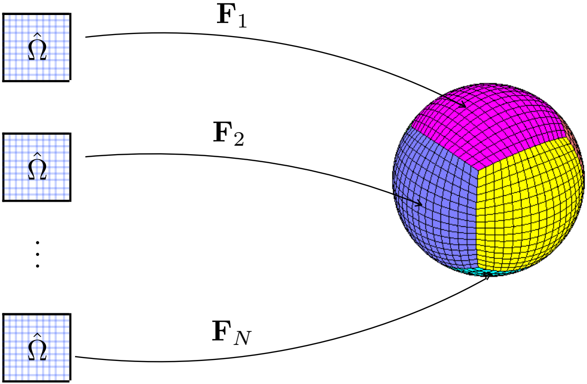
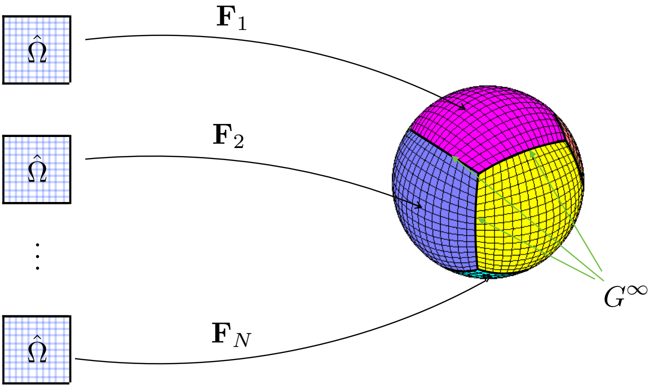
Isogeometric spaces on multipach geometries



- Can we construct C^1 isogeometric spaces on the geometry above?
- How do they perform in term of approximation properties? [Collin, Sangalli, and Takacs, 2015] [Bercovier and Matskewich, 2014] [Kapl, Vitrih, Jüttler, and Birner, 2015]

Isogeometric spaces on multipach geometries



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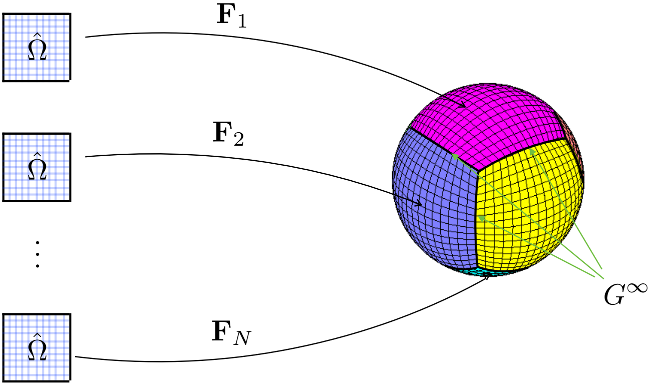

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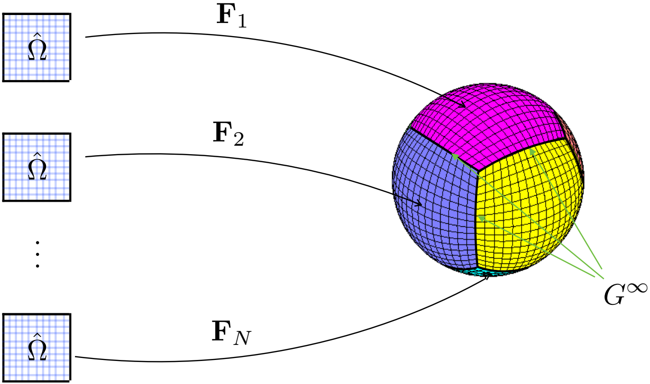

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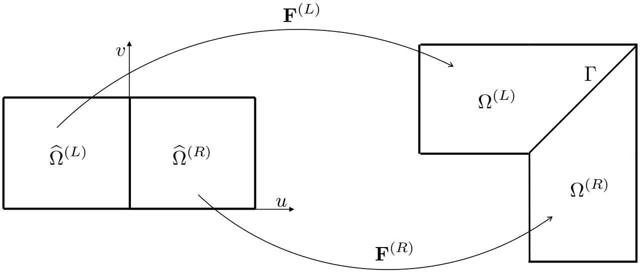
L-shape parametrization

$$\mathbf{F}^{(L)} : [-1, 0] \times [0, 1] = \widehat{\Omega}^{(L)} \to \Omega^{(L)},$$
$$\mathbf{F}^{(R)} : [0, 1] \times [0, 1] = \widehat{\Omega}^{(R)} \to \Omega^{(R)},$$
$$\Gamma = \{\mathbf{F}(0, v) \equiv \mathbf{F}^{(L)}(0, v) \equiv \mathbf{F}^{(R)}(0, v), v \in [0, 1]\}$$



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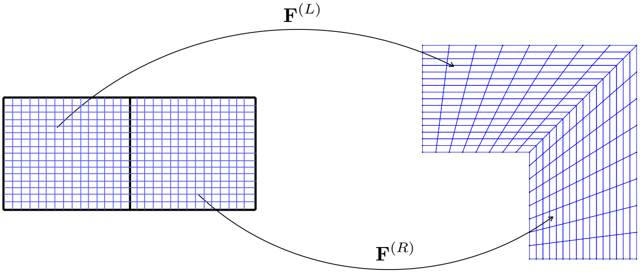
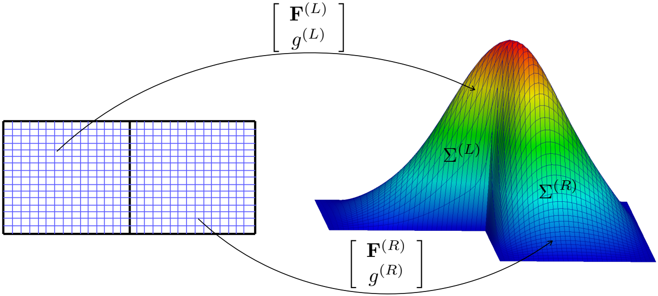


C^1 isogeometric functions

Given $\phi : \Omega \to \mathbb{R}$, let $g^{(S)} = \phi \circ \mathbf{F}^{(S)} \in \mathcal{S}_r^p(\widehat{\Omega}^{(S)})$, $S \in \{L, R\}$, and

$$\begin{bmatrix} \mathbf{F}^{(L)} \\ g^{(L)} \end{bmatrix} : \widehat{\Omega}^{(L)} \to \Sigma^{(L)}, \qquad \begin{bmatrix} \mathbf{F}^{(R)} \\ g^{(R)} \end{bmatrix} : \widehat{\Omega}^{(R)} \to \Sigma^{(R)} \qquad (*)$$

$\phi \in C^1(\Omega) \Leftrightarrow$ the graph parametrization (*) is G^1

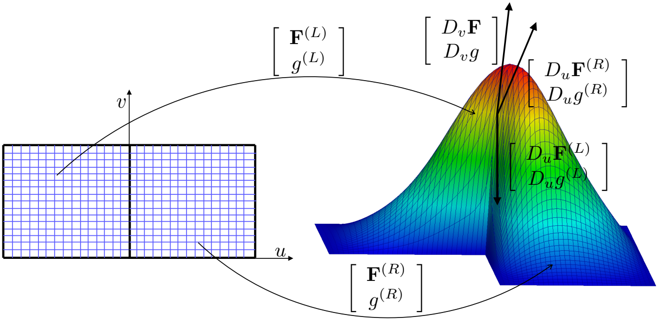


G^1 graph condition

$\phi \in C^1(\Omega) \Leftrightarrow$ the tangent planes formed by

$$\begin{bmatrix} D_u\mathbf{F}^{(L)}(0,v) \\ D_u g^{(L)}(0,v) \end{bmatrix}, \begin{bmatrix} D_v\mathbf{F}(0,v) \\ D_v g(0,v) \end{bmatrix} \text{ and } \begin{bmatrix} D_u\mathbf{F}^{(R)}(0,v) \\ D_u g^{(R)}(0,v) \end{bmatrix}, \begin{bmatrix} D_v\mathbf{F}(0,v) \\ D_v g(0,v) \end{bmatrix}$$

do coincide:

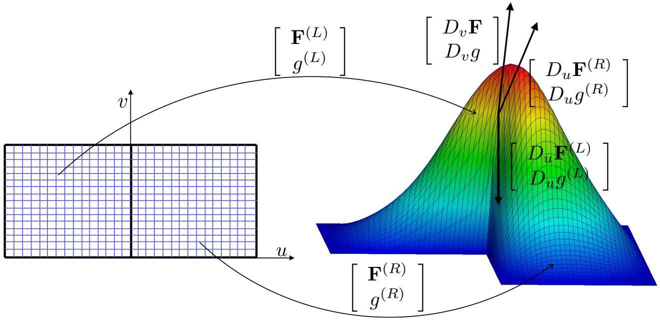


G^1 graph condition

$\phi \in C^1(\Omega) \Leftrightarrow \exists \alpha^{(L)}, \alpha^{(R)}, \beta : [0, 1] \to \mathbb{R}$, s.t. $\alpha^{(L)}(v)\alpha^{(R)}(v) < 0$ and

$$\alpha^{(L)}(v) \begin{bmatrix} D_u\mathbf{F}^{(L)}(0, v) \\ D_u g^{(L)}(0, v) \end{bmatrix} + \alpha^{(R)}(v) \begin{bmatrix} D_u\mathbf{F}^{(R)}(0, v) \\ D_u g^{(R)}(0, v) \end{bmatrix} + \beta(v) \begin{bmatrix} D_v\mathbf{F}(0, v) \\ D_v g(0, v) \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \end{bmatrix}.$$

$(\alpha^{(L)}(v), \alpha^{(R)}(v), \beta(v)$ are determined up to a common multiple $\gamma(v))$.



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($\alpha^{(L)}(v), \alpha^{(R)}(v), \beta(v)$ are determined up to a common multiple $\gamma(v)$).

Steps of the isogeometric method

- the geometry parametrizations $\mathbf{F}^{(L)}, \mathbf{F}^{(R)}$ are first given
- $\alpha^{(L)}, \alpha^{(R)}, \beta$ are determined by $\mathbf{F}^{(L)}, \mathbf{F}^{(R)}$:

$$\alpha^{(L)}(v)D_u\mathbf{F}^{(L)}(0,v) + \alpha^{(R)}(v)D_u\mathbf{F}^{(R)}(0,v) + \beta(v)D_v\mathbf{F}(0,v) = \mathbf{0}.$$

- the C^1 isogeometric space is given by the condition on $g^{(L)}, g^{(R)}$:

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About $\alpha^{(L)}, \alpha^{(R)}, \beta$ in the G^1 condition

Given $\mathbf{F}^{(L)}$, $\mathbf{F}^{(R)}$, then (see [?Bercovier and Matskewich, 2014])

$$\begin{aligned}
\alpha^{(L)}(v) &= \gamma(v) \det \left[\begin{array}{cc} D_u\mathbf{F}^{(R)}(0,v) & D_v\mathbf{F}^{(R)}(0,v) \end{array} \right], \\
\alpha^{(R)}(v) &= -\gamma(v) \det \left[\begin{array}{cc} D_u\mathbf{F}^{(L)}(0,v) & D_v\mathbf{F}^{(L)}(0,v) \end{array} \right], \\
\beta(v) &= \gamma(v) \det \left[\begin{array}{cc} D_u\mathbf{F}^{(L)}(0,v) & D_u\mathbf{F}^{(R)}(0,v) \end{array} \right].
\end{aligned}$$

Moreover it holds

$$\beta(v) = -(\alpha^{(R)}(v)\beta^{(R)}(v) + \alpha^{(L)}(v)\beta^{(L)}(v)),$$

where

$$\beta^{(S)}(v) = \frac{D_u\mathbf{F}^{(S)}(0,v) \cdot D_v\mathbf{F}^{(S)}(0,v)}{\|D_v\mathbf{F}^{(S)}(0,v)\|^2}, \quad S \in \{L, R\}.$$

An example of C^1-*locking*: when $g^{(S)} \in \mathcal{S}^p_{p-1}(\widehat{\Omega}^{(S)})$

For any gluing data $\alpha^{(S)}$ and $\beta^{(S)}$, when $g^{(S)} \in \mathcal{S}^p_{p-1}(\widehat{\Omega}^{(S)})$ *h*-convergence fails in general at the interface:

$$
\begin{aligned}
D_u g^{(L)} &\in (\mathcal{S}^{p-1}_{p-2}[-1,0] \otimes \mathcal{S}^p_{p-1}[0,1]) \\
D_v g^{(L)} &\in (\mathcal{S}^{p}_{p-1}[-1,0] \otimes \mathcal{S}^{p-1}_{p-2}[0,1]) \\
D_u g^{(R)} &\in (\mathcal{S}^{p-1}_{p-2}[0,1] \otimes \mathcal{S}^p_{p-1}[0,1]) \\
D_v g^{(R)} &\in (\mathcal{S}^{p}_{p-1}[0,1] \otimes \mathcal{S}^{p-1}_{p-2}[0,1])
\end{aligned}
$$

Since the C^1-condition is

$$
\alpha^{(L)}(v)\underbrace{D_u g^{(L)}(0,v)}_{\in \mathcal{S}^p_{p-1}} + \alpha^{(R)}(v)\underbrace{D_u g^{(R)}(0,v)}_{\in \mathcal{S}^p_{p-1}} + \beta(v)\underbrace{D_v g(0,v)}_{\in \mathcal{S}^{p-1}_{p-2}} = 0,
$$

then $D_v g(0,v) \in \mathcal{S}^{p-1}_{p-1}$, i.e., $g(0,v)$ is a global p-degree polynomial!

AS G^1 parametrization

Definition (Analysis-suitable G^1 geometry parametrization)

The parametrizations $\mathbf{F}^{(L)}$ and $\mathbf{F}^{(R)}$ are *Analysis-Suitable* (AS) G^1 at Γ if there exist $\alpha^{(L)}, \alpha^{(R)}, \beta^{(L)}, \beta^{(R)}$ *linear functions* such that $\forall v \in [0,1]$,

$$\alpha^{(L)}(v) D_u \mathbf{F}^{(L)}(0,v) + \alpha^{(R)}(v) D_u \mathbf{F}^{(R)}(0,v) + \beta(v) D_v \mathbf{F}(0,v) = \mathbf{0},$$

where $\beta(v) = -(\alpha^{(R)}(v)\beta^{(R)}(v) + \alpha^{(L)}(v)\beta^{(L)}(v))$.

Examples:

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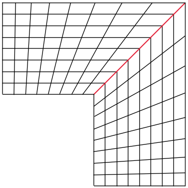

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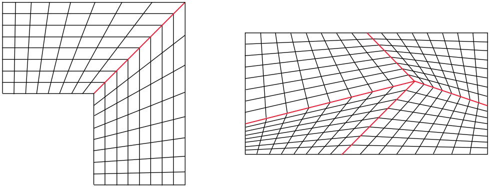

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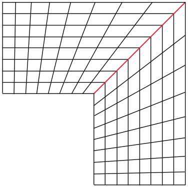
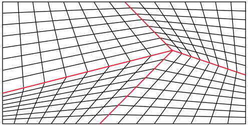
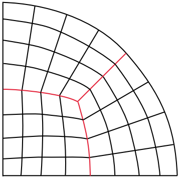
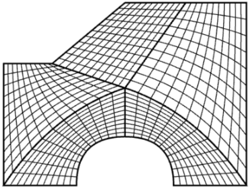


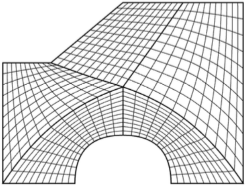




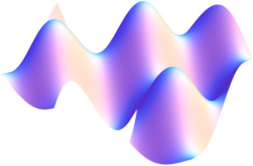
Non-AS-G^1 vs. AS-G^1 convergence



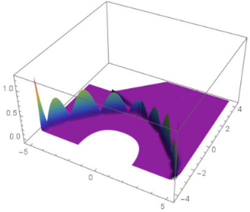
Initial (non-AS-G^1) parameterization



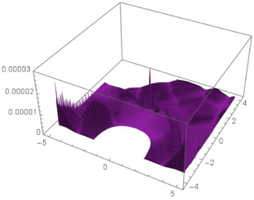
AS-G^1 parameterization



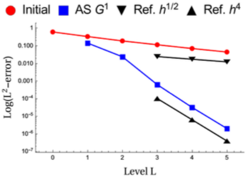
Exact solution



Abs. error initial parametr.



Abs. error AS-G^1 parametr.



Convergence rates

Approximation of "exact solution" by S_1^3 isogeometric functions.

Dual basis [Lyche, Manni, Speleers, 2017]

Given a 1D knot vector $[\xi_1, \ldots, \xi_{n+p+1}]$ there exists a *stable* dual basis

$\left\{\lambda[\xi_i, \ldots, \xi_{i+p+1}](\cdot)\right\}_{i=1,\ldots,n}$ to the spline basis $\left\{B[\xi_i, \ldots, \xi_{i+p+1}]\right\}_{i=1,\ldots,n}$

that is

$$\underbrace{\lambda[\xi_i, \ldots, \xi_{i+p+1}]}_{\lambda_{i,p}}(\underbrace{B[\xi_j, \ldots, \xi_{j+p+1}]}_{B_{j,p}})$$

$$\equiv \int \lambda[\xi_i, \ldots, \xi_{i+p+1}]B[\xi_j, \ldots, \xi_{j+p+1}] = \delta_{ij}$$

The existence of a stable dual basis guarantees:

- linear independence of the B-spline basis
- existence of a projection operator (quasi-interpolant) with optimal approximation properties w.r.t. h.

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A spline projection operator [Lyche, Manni, Speleers, 2017]

$$\mathcal{Q}_{p,\Xi} : L^2([0,1]) \to S_p(\Xi), \qquad \mathcal{Q}_{p,\Xi}(f) = \sum_{j=1}^{n} \lambda_{j,p}(f) B_{j,p},$$

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$$\lambda_{i,p}(f) = \mathcal{L}_{i,p,\Xi} := \frac{1}{h} \int_{\xi_{m_i}}^{\xi_{m_i+1}} \Big(\sum_j \alpha_j \phi_{j,i}\Big) f$$

A spline projection operator

- $\mathcal{Q}_{p,\Xi}$ preserves splines, that is $\mathcal{Q}_{p,\Xi}(f) = f, \quad \forall f \in S_p(\Xi)$.
- Stability holds:

$$|\lambda_{j,p}(f)| \leq C(\xi_{j+p+1} - \xi_j)^{-1/2} \|f\|_{L^2(\xi_j,\xi_{j+p+1})},$$

  where the constant C depends on the polynomial degree p with the upperbound
- For any non empty knot span $I_i = (\zeta_i, \zeta_{i+1})$ it holds

$$\|\mathcal{Q}_{p,\Xi}(f)\|_{L^2(I_i)} \leq C\|f\|_{L^2(\widetilde{I}_i)},$$

  where the constant C depends only upon the degree p, and $\widetilde{I}_i$ is the support extension.

A spline projection operator with BC

The operator $\mathcal{Q}_{p,\Xi}$ can be modified in order to match boundary conditions. We can define, for all $f \in C^0([0,1])$:

$$\widetilde{\mathcal{Q}}_{p,\Xi}(f) = \sum_{j=1}^{n} \widetilde{\lambda}_{j,p}(f) B_{j,p} \quad \text{with}$$

$$\widetilde{\lambda}_{1,p}(f) = f(0)\,,\ \widetilde{\lambda}_{n,p}(f) = f(1)\,,\ \widetilde{\lambda}_{j,p}(f) = \lambda_{j,p}(f), \quad j = 2, \ldots, n-1.$$

L^2 stability stated for $\mathcal{Q}_{p,\Xi}$ cannot be valid for $\widetilde{\mathcal{Q}}_{p,\Xi}$, but:
For any non empty knot span $I_i = (\zeta_i, \zeta_{i+1})$ it holds

$$\|\widetilde{\mathcal{Q}}_{p,\Xi}(f)\|_{L^2(I_i)} \leq C\big(\|f\|_{L^2(\widetilde{I}_i)} + \widetilde{h}_i |f|_{H^1(\widetilde{I}_i)}\big)$$

$$|\widetilde{\mathcal{Q}}_{p,\Xi}(f)|_{H^1(I_i)} \leq C\|f\|_{H^1(\widetilde{I}_i)}$$

Bound on the projection error in Sobolev spaces

Spline approximation in standard Sobolev spaces

$\exists C$ depending only on p (and the local mesh ratio) such that for all $r, s \in \mathbb{N}$, $0 \leq r \leq s \leq p+1$, and all $f \in H^s(I)$

$$|f - \mathcal{Q}_{p,\Xi}(f)|_{H^r(I_i)} \leq C(\widetilde{h}_i)^{s-r}|f|_{H^s(\widetilde{I}_i)} \quad \forall i = 1, \ldots, N-1.$$

Let q be any polynomial of degree p living on $[0,1]$. Noting that, since $q \in S_p(\Xi)$, it holds $\mathcal{Q}_{p,\Xi}(q) = q$ and using stabily

$$\|f - \mathcal{Q}_{p,\Xi}(f)\|_{L^2(I_i)} \leq \|f - q\|_{L^2(I_i)} + \|\mathcal{Q}_{p,\Xi}(q-f)\|_{L^2(I_i)} \leq C\|f-q\|_{L^2(\widetilde{I}_i)}.$$

The term above is bounded by polynomial approximation, leading to the bound for $r = 0$. By inverse estimates:

$$\begin{aligned} |f - \mathcal{Q}_{p,\Xi}(f)|_{H^r(I_i)} &\leq |f-q|_{H^r(I_i)} + |\mathcal{Q}_{p,\Xi}(q-f)|_{H^r(I_i)} \\ &\leq |f-q|_{H^r(I_i)} + Ch_i^{-r}\|\mathcal{Q}_{p,\Xi}(q-f)\|_{L^2(I_i)} \\ &\leq |f-q|_{H^r(I_i)} + Ch_i^{-r}\|f-q\|_{L^2(\widetilde{I}_i)} \end{aligned}$$

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Bent Sobolev spaces

Given $q \in \mathbb{N}$ the piecewise polynomial space

$\mathcal{P}_q(\Xi) = \{v \in L^2(I) : v|_{I_i} \text{ is a } q\text{-deg. pol.}, \forall i = 1, \ldots, N-1, I_i = (\zeta_i, \zeta_{i+1})\}.$

Define the bent Sobolev space (depends on the knot vector Ξ):

$$\mathcal{H}^s(I) = \left\{ \begin{array}{l} f \in L^2(I) \text{ such that } f|_{I_i} \in H^s(I_i) \; \forall \; i = 1, \ldots, N-1, \text{ and} \\ D_-^k f(\zeta_i) = D_+^k f(\zeta_i), \forall k \le \min\{s-1, k_i\}, \forall i = 2, \ldots, N-1, \end{array} \right\}$$

$$\|f\|^2_{\mathcal{H}^s(I)} = \sum_{j=0}^{s} |f|^2_{\mathcal{H}^j(I)}, \quad |f|^2_{\mathcal{H}^j(I)} = \sum_{i=1}^{N-1} |f|^2_{H^j(I_i)} \quad \forall j = 0, 1, \ldots, s,$$

Given an integer s such that $0 \le s \le p$, we define the space

$$\widetilde{S}_{s,p}(\Xi) = \mathcal{P}_s(\Xi) \cap S_p(\Xi).$$

$S_p(\Xi) = \widetilde{S}_{p,p}(\Xi)$, and, for any $s < p$, $\widetilde{S}_{s,p}(\Xi)$ is still a spline space but associated to a knot vector different from Ξ. Moreover

$$S_p(\Xi) = \mathcal{P}_p(\Xi) \cap \mathcal{H}^p(I).$$

a link between bent and standard Sobolev spaces

Let $s \in \mathbb{N}$, $s \leq p+1$. There exists a projector $\Gamma : \mathcal{H}^s(I) \to \widetilde{S}_{s-1,p}(\Xi)$ such that $f - \Gamma(f) \in H^s(I)$. If $s \geq 2$, $\Gamma(f)(\zeta_1) = \Gamma(f)(\zeta_N) = 0$.

Assume $s = p+1$, and construct $\Gamma : \mathcal{H}^{p+1}(I) \to S_p(\Xi)$. We need

$$D_+^k(f - \Gamma(f))(\zeta_i) = D_-^k(f - \Gamma(f))(\zeta_i), \quad \forall i = 2, \dots, N-1,\ k = 0, 1, \dots, p.$$

Then we need $D_+^k(\Gamma(f))(\zeta_i) - D_-^k(\Gamma(f))(\zeta_i) = D_+^k(f)(\zeta_i) - D_-^k(f)(\zeta_i)$ for all $i = 2, \dots, N-1$ and $k = k_i + 1, \dots, p$. Let $\varphi_i^k : I \to \mathbb{R}$ be defined as

$$S_p(\Xi) \ni \varphi_i^k(x) = \left(\max\{0, x - \zeta_i\}\right)^k \qquad \forall\, x \in I.$$

We then have and define

$$D_-^l \varphi_i^k(\zeta_i) = 0\,, \qquad D_+^l \varphi_i^k(\zeta_i) = k!\, \delta_{lk},$$

$$\Gamma(f)(x) = \sum_{i=2}^{N-1} \sum_{q=k_i+1}^{p} \frac{D_+^q f(\zeta_i) - D_-^q f(\zeta_i)}{q!}\, \varphi_i^q(x), .$$

Bound on the projection error in bent Sobolev spaces

Spline approximation in bent Sobolev spaces

[Beirão da Veiga, Cho , Sangalli, 2014]

$\exists C$, only dependent on p (and the local mesh ratio) such that for all $r, s \in B$, $0 \leq r \leq s \leq p+1$, and all $f \in \mathcal{H}^s(I)$

$$|f - \mathcal{Q}_{p,\Xi}(f)|_{H^r(I_i)} \leq C(\widetilde{h}_i)^{s-r}|f|_{\mathcal{H}^s(\widetilde{I}_i)} \quad \forall i = 1, \dots, N-1.$$

$\Gamma(f) \in \widetilde{S}_{s-1,p}(\Xi) \subset S_p(\Xi) \Rightarrow f - \mathcal{Q}_{p,\Xi}(f) = (f - \Gamma(f)) - \mathcal{Q}_{p,\Xi}(f - \Gamma(f))$.
Since $(f - \Gamma(f)) \in H^s(I)$, we apply the classical error bound

$$\begin{aligned}
|f - \mathcal{Q}_{p,\Xi}(f)|_{H^r(I_i)} &= |f - \Gamma(f) - \mathcal{Q}_{p,\Xi}(f - \Gamma(f))|_{H^r(I_i)} \\
&\leq C(\widetilde{h}_i)^{s-r}|f - \Gamma(f)|_{H^s(\widetilde{I}_i)} = C(\widetilde{h}_i)^{s-r}\sqrt{\sum_{I_j \subset \widetilde{I}_i} |f - \Gamma(f)|^2_{H^s(I_j)}} \\
&= C(\widetilde{h}_i)^{s-r}\sqrt{\sum_{I_j \subset \widetilde{I}_i} |f|^2_{H^s(I_j)}} = C(\widetilde{h}_i)^{s-r}|f|_{\mathcal{H}^s(\widetilde{I}_i)}.
\end{aligned}$$

Approximation of isogeometric spaces

From the univariate spline projector $\mathcal{Q}_{p,\Xi}$, we tensorize and construct $\Pi_{\mathbf{p},\Xi}$, then define $\Pi_{V_h} : L^2(\Omega) \to V_h$

$$\Pi_{V_h} f = \frac{\Pi_{\mathbf{p},\Xi}(W(f \circ \mathbf{F}))}{W} \circ \mathbf{F}^{-1} \qquad f \in L^2(\Omega).$$

Approximation error (h-refinement) of isogeometric spaces

[Beirão da Veiga, Cho , Sangalli, 2014]

Given $0 \leq r \leq s \leq \min(p_1, \dots, p_d) + 1$, there exists a constant C depending only on $\mathbf{p}, \mathbf{F}, W$ (and the local mesh ratio) such that

$$\|f - \Pi_{V_h} f\|_{H^r(K_{\mathbf{i}})} \leq C(h_{\widetilde{K}_{\mathbf{i}}})^{s-r} \|f\|_{H^s(\widetilde{K}_{\mathbf{i}})} \quad \forall K_{\mathbf{i}} \in \mathcal{M},$$

$$\|f - \Pi_{V_h} f\|_{H^r(\Omega)} \leq C h^{s-r} \|f\|_{H^s(\Omega)},$$

for all f in $H^s(\Omega)$.

Non-tensor-product splines for IGA

Three approaches have emerged in the isogeometric community:

- T-splines [Sederberg, Zheng, Bakenov, and Nasri, 2003; Bazilevs, Calo, Cottrell, Evans, Hughes, Lipton, Scott, and Sederberg, 2010; Scott, Li, Sederberg, and Hughes, 2012]
  - Analysis suitable (AS) T-splines $\subset$ T-spline [Li, Zheng, Sederberg, Hughes, and Scott, 2012; Beirão da Veiga, Buffa, Sangalli, and Vázquez, 2013] .
- locally-refinable (LR) splines [Dokken, Lyche, and Pettersen, 2013]
- hierarchical splines [Vuong, Giannelli, Jüttler, and Simeon, 2011; Giannelli, Jüttler, and Speleers, 2012]
- subdivision surfaces [Cirak et al.; Nguyen, Karciauskas, Peters; Barendrecht; . . .]

Tensor product B-splines easily inherit the maths properties from the univariate B-splines because of the tensor-product construction.

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Definition of PB-splines [Sederberg, Zheng, Bakenov, and Nasri, 2003]

Consider a set of multivariate B-splines

$$\{N_{\mathbf{A},\mathbf{p}},\ \mathbf{A} \in \mathcal{A}\}$$

- $\mathcal{A}$ is a set of indices, i.e., $\mathbf{A} \in \mathcal{A} \leftrightarrow N_{\mathbf{A},\mathbf{p}}$ is one-to-one,
- the $N_{\mathbf{A},\mathbf{p}}$ have the structure

$$N_{\mathbf{A},\mathbf{p}}(\zeta) = N[\Xi_{\mathbf{A},1,p_1}](\zeta_1)N[\Xi_{\mathbf{A},2,p_2}](\zeta_2) \text{ or}$$

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The space spanned is

$$S_{\mathbf{p}}(\mathcal{A}) = \operatorname{span}\{N_{\mathbf{A},\mathbf{p}},\ \mathbf{A} \in \mathcal{A}\}.$$

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$$S_{\mathbf{p}}(\mathcal{A}) = \mathrm{span}\left\{ N_{\mathbf{A},\mathbf{p}},\ \mathbf{A} \in \mathcal{A} \right\}.$$

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$$S_{\mathbf{p}}(\mathcal{A}) = \operatorname{span}\{N_{\mathbf{A},\mathbf{p}},\ \mathbf{A} \in \mathcal{A}\}.$$

Definition of DC-splines [Beirão da Veiga, Buffa, Sangalli, and Vázquez, 2014]

Overlapping knot vectors

Two local knot vectors $\Xi' = \{\xi'_1, \ldots \xi'_{p+2}\}$ and $\Xi'' = \{\xi''_1, \ldots \xi''_{p+2}\}$ overlap if formed by $p + 2$ neighbouring knots of the same knot vector.

For example:

- overlapping local knot vector

- non-overlapping local knot vector

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Overlapping and partially-overlapping splines

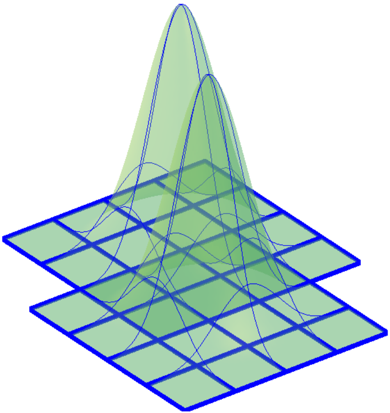
Two B-splines $N_{\mathbf{A}',\mathbf{p}}$ $N_{\mathbf{A}'',\mathbf{p}}$ overlap if the local knot vectors $\Xi_{\mathbf{A}',\ell,p_\ell}$ and $\Xi_{\mathbf{A}'',\ell,p_\ell}$ in each direction $\ell = 1, \dots, d$ overlap.

Two B-splines $N_{\mathbf{A}',\mathbf{p}}$ $N_{\mathbf{A}'',\mathbf{p}}$ partially overlap if, when $\mathbf{A}' \neq \mathbf{A}''$, there exists a direction ℓ such that $\Xi_{\mathbf{A}',\ell,p_\ell} \neq \Xi_{\mathbf{A}'',\ell,p_\ell}$ and overlap.

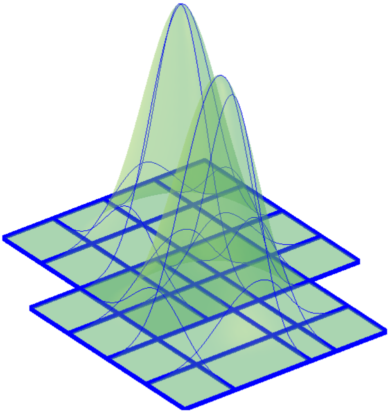
Dual-Compatible (DC) set and space

$\{N_{\mathbf{A},\mathbf{p}},\ \mathbf{A} \in \mathcal{A}\}$ is a DC set of B-splines if each pair of B-splines in it partially overlap. Its span $S_{\mathbf{p}}(\mathcal{A})$ is a DC spline space.

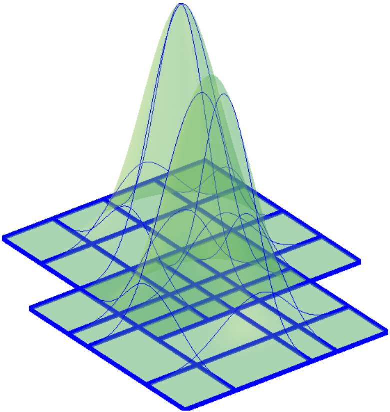
Example of overlapping B-splines



Example of partially overlapping B-splines



Example of not partially overlapping B-splines



Dual basis [Schumaker, 2007; Beirão da Veiga, Buffa, Sangalli, and Vázquez, 2014]

Given a 1D knot vector $[\xi_1, \ldots, \xi_{n+p+1}]$ there exists a dual basis

$\left\{\lambda[\xi_i, \ldots, \xi_{i+p+1}](\cdot)\right\}_{i=1,\ldots,n}$ to the spline basis $\left\{N[\xi_i, \ldots, \xi_{i+p+1}]\right\}_{i=1,\ldots,n}$

that is $\lambda[\xi_i, \ldots, \xi_{i+p+1}](N[\xi_j, \ldots, \xi_{j+p+1}]) = \delta_{ij}$.

Dual basis to a dual-compatible (DC) set

Assume that $\{N_{\mathbf{A},\mathbf{p}},\ \mathbf{A} \in \mathcal{A}\}$ is a DC set. The set $\{\lambda_{\mathbf{A},\mathbf{p}},\ \mathbf{A} \in \mathcal{A}\}$ where

$$\lambda_{\mathbf{A},\mathbf{p}} = \lambda[\Xi_{\mathbf{A},1,p_1}] \otimes \ldots \otimes \lambda[\Xi_{\mathbf{A},d,p_d}]$$

is a dual basis for $\{N_{\mathbf{A},\mathbf{p}},\ \mathbf{A} \in \mathcal{A}\}$.

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Dual basis to a dual-compatible (DC) set

The set $\{\lambda_{\mathbf{A},\mathbf{p}},\ \mathbf{A} \in \mathcal{A}\}$ where $\lambda_{\mathbf{A},\mathbf{p}} = \lambda[\Xi_{\mathbf{A},1,p_1}] \otimes \ldots \otimes \lambda[\Xi_{\mathbf{A},d,p_d}]$ is a dual basis for the DC set $\{N_{\mathbf{A},\mathbf{p}},\ \mathbf{A} \in \mathcal{A}\}$.

Consider any $N_{\mathbf{A}',\mathbf{p}}$ and $\lambda_{\mathbf{A}'',\mathbf{p}}$, with $\mathbf{A}', \mathbf{A}'' \in \mathcal{A}$. We then need to show

$$\lambda_{\mathbf{A}'',\mathbf{p}}(N_{\mathbf{A}',\mathbf{p}}) = \begin{cases} 1 & \text{if } \mathbf{A}'' = \mathbf{A}', \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathbf{A}' = \mathbf{A}''$, then we have $\lambda_{\mathbf{A}'',\mathbf{p}}(N_{\mathbf{A}',\mathbf{p}}) = 1$.
If $\mathbf{A}' \neq \mathbf{A}''$, for the partial overlap assumption there is $\bar{\ell}$ such that $\Xi_{\mathbf{A}',\ell,p_\ell} \neq \Xi_{\mathbf{A}'',\ell,p_\ell}$ and overlap, then $\lambda[\Xi_{\mathbf{A}'',\ell,p_\ell}](N[\Xi_{\mathbf{A}',\ell,p_\ell}]) = 0$ and then

$$\lambda_{\mathbf{A}'',\mathbf{p}}(N_{\mathbf{A}',\mathbf{p}}) = \prod_{\ell=1}^{d} \lambda[\Xi_{\mathbf{A}'',\ell,p_\ell}](N[\Xi_{\mathbf{A}',\ell,p_\ell}]) = 0.$$

Properties of DC-splines: local linear independence

Linear independence [Beirão da Veiga, Buffa, Sangalli, and Vázquez, 2014]

The B-splines in a DC set $\{N_{\mathbf{A},\mathbf{p}},\ \mathbf{A} \in \mathcal{A}\}$ are linearly independent.

Assume

$$\sum_{\mathbf{A}\in\mathcal{A}} C_{\mathbf{A}} N_{\mathbf{A},\mathbf{p}} = 0$$

for some coefficients $C_{\mathbf{A}}$; for any $\mathbf{A}' \in \mathcal{A}$, applying $\lambda_{\mathbf{A}',\mathbf{p}}$ we get

$$C_{\mathbf{A}'} = \lambda_{\mathbf{A}',\mathbf{p}} \left(\sum_{\mathbf{A}\in\mathcal{A}} C_{\mathbf{A}} N_{\mathbf{A}',\mathbf{p}} \right) = 0.$$

LLI

In a DC set there are at most $(p_1 + 1) \cdot \ldots \cdot (p_d + 1)$ B-splines that are non-null in each Bézier element. If the space of polynomials of degree $\mathbf{p}$ is in $S_{\mathbf{p}}(\mathcal{A})$, the B-splines are locally linearly independent.

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Properties of DC-splines: partition of unity

Partition of unity [Beirão da Veiga, Buffa, Sangalli, and Vázquez, 2014]

The B-splines in a DC set $\{N_{\mathbf{A},\mathbf{p}},\ \mathbf{A} \in \mathcal{A}\}$ form a partition of the unity, if the constant function belongs to the space.

Let

$$\sum_{\mathbf{A}\in\mathcal{A}} C_{\mathbf{A}} N_{\mathbf{A},\mathbf{p}} = 1$$

for some coefficients $C_{\mathbf{A}}$. For any $\mathbf{A}' \in \mathcal{A}$, applying $\lambda_{\mathbf{A}',\mathbf{p}}$ we get

$$C_{\mathbf{A}'} = \lambda_{\mathbf{A}',\mathbf{p}} \left(\sum_{\mathbf{A}\in\mathcal{A}} C_{\mathbf{A}} N_{\mathbf{A},\mathbf{p}} \right) = 1.$$

Approximation properties of DC-splines

L^2–stability of the projector

Given a DC set of B-splines, $\Pi_{\mathbf{p}}(f)(\zeta) = \sum_{\mathbf{A}\in\mathcal{A}} \lambda_{\mathbf{A},\mathbf{p}}(f) N_{\mathbf{A},\mathbf{p}}(\zeta)$ realizes:

$$||\Pi_{\mathbf{p}}(f)||_{L^2(Q)} \leq C||f||_{L^2(\widetilde{Q})} \quad \forall Q \subset \Omega,\ \forall f \in L^2(\Omega).$$

We use of the notion of support extension $\widetilde{Q}$ associated to an element $Q \subset \Omega$:

$$\widetilde{Q} = \bigcup_{\substack{\mathbf{A}\in\mathcal{A} \\ \mathrm{supp}(N_{\mathbf{A},\mathbf{p}})\cap Q \neq \emptyset}} \mathrm{supp}(N_{\mathbf{A},\mathbf{p}}),$$

and recall positivity and partition of unity property:

$$\sum_{\mathbf{A}\in\mathcal{A}} \left|N_{\mathbf{A},\mathbf{p}}(\zeta)\right| \leq C.$$

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L^2–stability of the projector

Given a DC set of B-splines, $\Pi_{\mathbf{p}}(f)(\zeta) = \sum_{\mathbf{A}\in\mathcal{A}} \lambda_{\mathbf{A},\mathbf{p}}(f) N_{\mathbf{A},\mathbf{p}}(\zeta)$ realizes:

$$||\Pi_{\mathbf{p}}(f)||_{L^2(Q)} \leq C||f||_{L^2(\widetilde{Q})} \quad \forall Q \subset \Omega,\ \forall f \in L^2(\Omega).$$

Denote by $\mathcal{A}(\zeta)$ the set of $\mathbf{A} \in \mathcal{A}$ such that $N_{\mathbf{A},\mathbf{p}}(\zeta) > 0$, by $Q_{\mathbf{A}}$ the common support of $N_{\mathbf{A},\mathbf{p}}$ and $\lambda_{\mathbf{A},\mathbf{p}}$, it follows that

$$\begin{aligned} \left|\Pi_{\mathbf{p}}(f)(\zeta)\right|^2 &= \Big|\sum_{\mathbf{A}\in\mathcal{A}(\zeta)} \lambda_{\mathbf{A},\mathbf{p}}(f) N_{\mathbf{A},\mathbf{p}}(\zeta)\Big|^2 \leq C \max_{\mathbf{A}\in\mathcal{A}(\zeta)} \left|\lambda_{\mathbf{A},\mathbf{p}}(f)\right|^2 \\ &\leq C \max_{\mathbf{A}\in\mathcal{A}(\zeta)} |Q_{\mathbf{A}}|^{-1} ||f||^2_{L^2(Q_{\mathbf{A}})} \leq C|Q|^{-1}||f||^2_{L^2(\widetilde{Q})}, \end{aligned}$$

where we have used that $\forall \mathbf{A} \in \mathcal{A}(\zeta)$, $Q \subset Q_{\mathbf{A}}$ (and therefore $|Q| \leq |Q_{\mathbf{A}}|$) and that $Q_{\mathbf{A}} \subset \widetilde{Q}$. Integrating over Q yields $||\Pi_{\mathbf{p}}(f)||^2_{L^2(Q)} \leq C||f||^2_{L^2(\widetilde{Q})}$.

Approximation properties of DC-splines

L^2–stability of the projector

Given a DC set of B-splines, $\Pi_{\mathbf{p}}(f)(\zeta) = \sum_{\mathbf{A}\in\mathcal{A}} \lambda_{\mathbf{A},\mathbf{p}}(f) N_{\mathbf{A},\mathbf{p}}(\zeta)$ realizes:

$$||\Pi_{\mathbf{p}}(f)||_{L^2(Q)} \leq C||f||_{L^2(\widetilde{Q})} \quad \forall Q \subset \Omega,\ \forall f \in L^2(\Omega).$$

Optimal approximation of the projector

Assume that the space of polynomials of degree $p = \min_{1\leq\ell\leq d}\{p_\ell\}$ is included into the space $S_{\mathbf{p}}(\mathcal{A})$. Then there exists a constant C only dependent on $\mathbf{p}$ such that for $0 \leq s \leq p+1$

$$||f - \Pi_{\mathbf{p}}(f)||_{L^2(Q)} \leq C(h_{\bar{Q}})^s |f|_{H^s(\bar{Q})} \quad \forall Q \subset \Omega,\ \forall f \in H^s(\Omega),$$

where $h_{\bar{Q}}$ is the diameter of $\bar{Q}$ (smallest d-rectangle containing $\widetilde{Q}$).

Greville sites for DC-splines

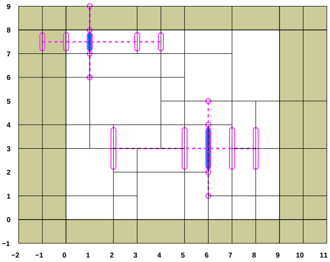
Greville sites

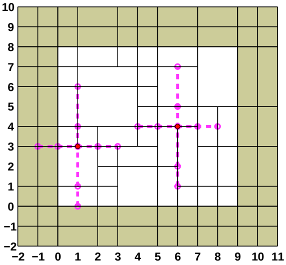
Assume that the linear polynomials belong to the space $S_{\mathbf{p}}(\mathcal{A})$. Then we have that

$$\zeta = \sum_{\mathbf{A}\in\mathcal{A}} \begin{bmatrix} \gamma[\Xi_{\mathbf{A},1,p_1}] \\ \vdots \\ \gamma[\Xi_{\mathbf{A},d,p_d}] \end{bmatrix} N_{\mathbf{A},\mathbf{p}}(\zeta), \quad \forall \zeta \in \Omega,$$

where $\gamma[\Xi_{\mathbf{A},\ell,p_\ell}]$ denotes the Greville average of $\Xi_{\mathbf{A},\ell,p_\ell}$, that is, the average of the p_ℓ internal knots of $\Xi_{\mathbf{A},\ell,p_\ell}$.

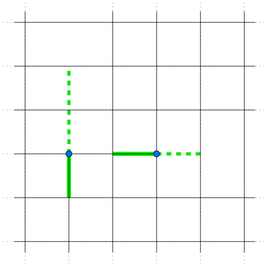
T-splines: $\mathbf{p} = (3,2)$ and $\mathbf{p} = (3,3)$





Construction of the horizontal and vertical index vector (red crosses), for some values of $\mathbf{p} = (p_1, p_2)$, and for the anchors marked in blue.

AS T-splines

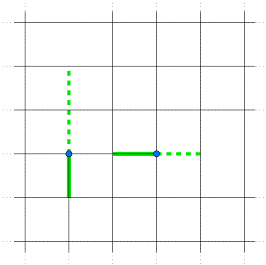


Extensions for degree $p_1 = 2$ (horizontal) and $p_2 = 3$ (vertical). The dashed lines represent the face extensions.

AS $\Leftrightarrow$ DC [Beirão da Veiga, Buffa, Sangalli, and Vázquez, 2013]

Given an "admissible" T-mesh, it is AS (analysis-suitable) if and only if the set of T-spline blending functions is DC (dual-compatible).

AS T-splines



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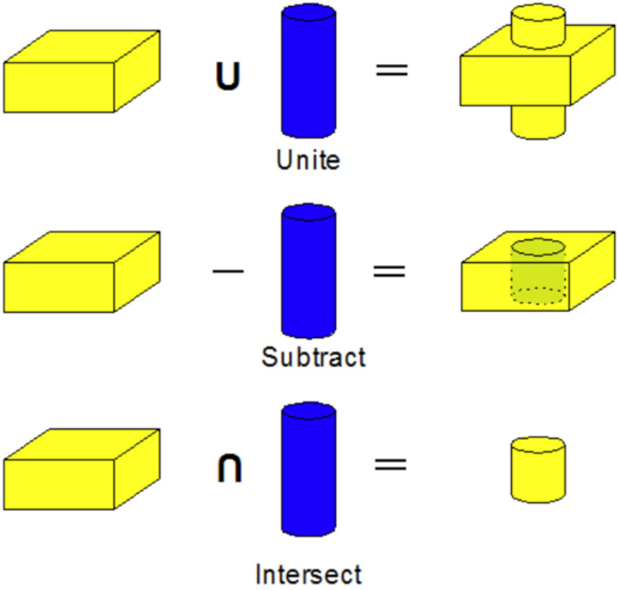
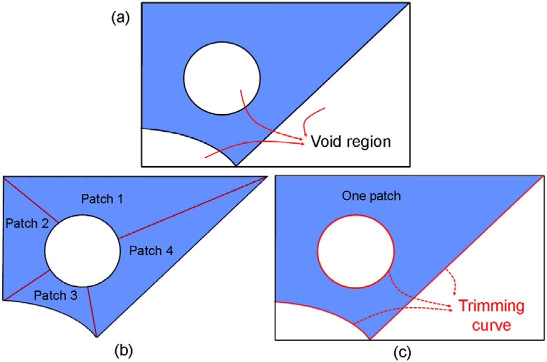
Geometries generated by boolean operations



Geometries generated by boolean operations

Possible approaches to handle trimmed domains are:

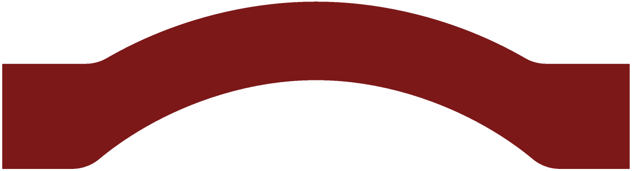
- multipatch reparametrization (b)
- dealing directly with trimmed domain (c)



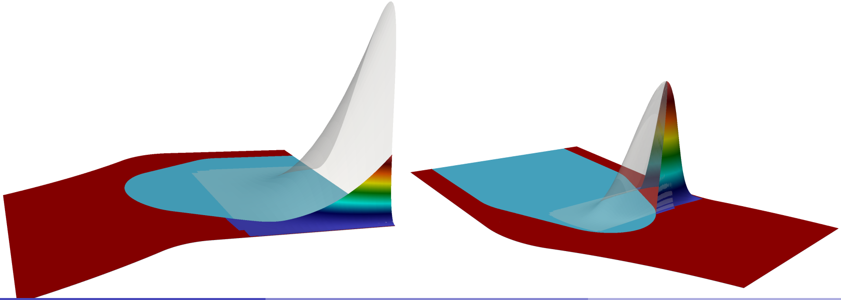
Picture from "*Isogeometric analysis for trimmed CAD surfaces*", Hyun-Jung Kim, Yu-Deok Seo, Sung-Kie Youn, Comput. Methods Appl. Mech. Engrg. 198 (2009) 2982-2995.

Trimming

Trimming is a basic operation in CAD.



Solve in the red part, cutting out the blue part.
The trimming also "cuts" the basis functions of the method.



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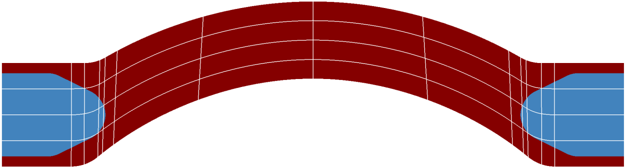

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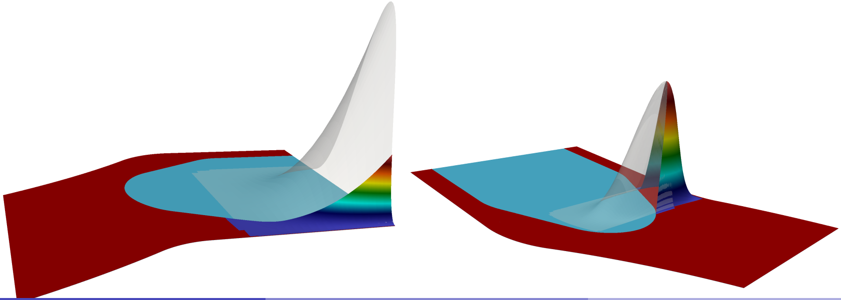

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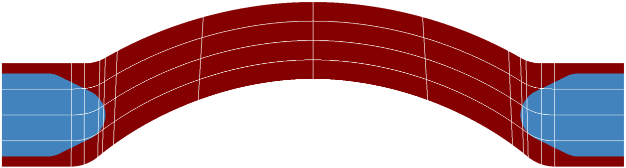

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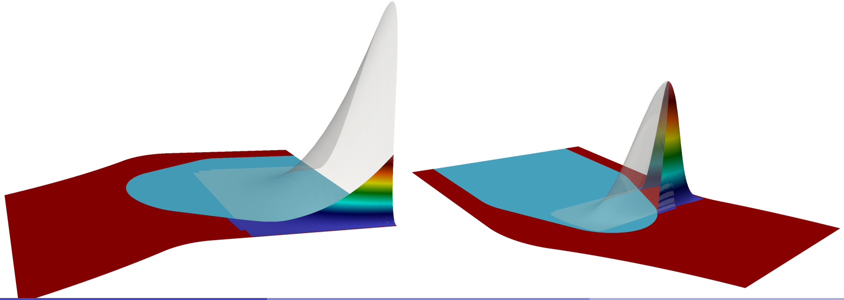
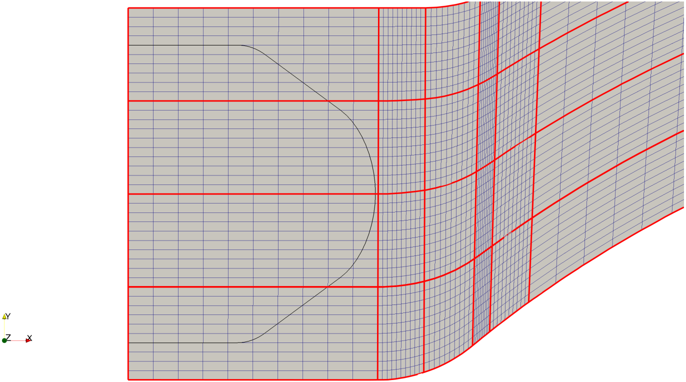
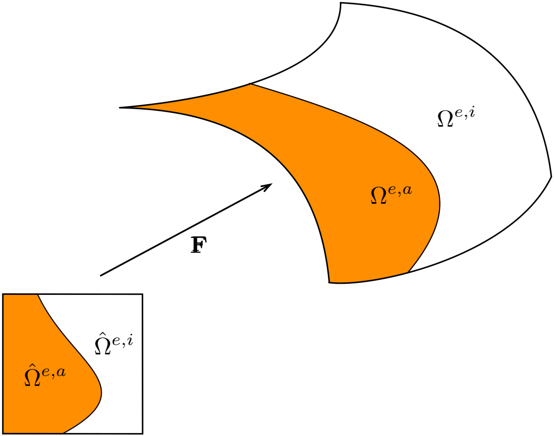


Trimming

- Integrals must be accurately computed for “cut” basis functions: we reparametrize the trimmed elements.

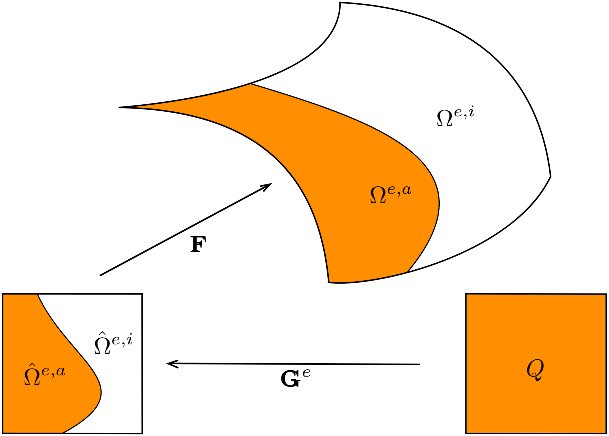


Quadrature on trimmed elements



A trimmed element $\Omega^e = \mathbf{F}(\widehat{\Omega}^e)$ is union of an *active* part $\Omega^{e,a}$ and an *inactive* part $\Omega^{e,i}$.

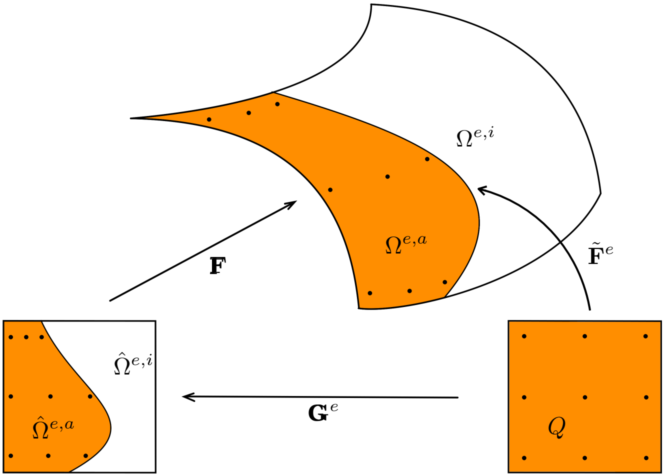
Quadrature on trimmed elements



We can write

$$\Omega^{e,a} = \mathbf{F}(\widehat{\Omega}^{e,a}) = \mathbf{F}(\mathbf{G}^e(Q)) = \widetilde{\mathbf{F}}^e(Q).$$

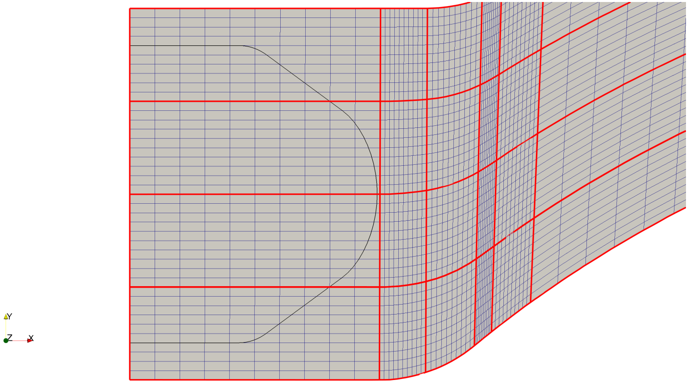
Quadrature on trimmed elements



$$\int_{\Omega^{e,a}} f(\mathbf{x})\, d\mathbf{x} = \int_Q \left(f \circ \widetilde{\mathbf{F}}^e\right)(\zeta) \det\left(D\widetilde{\mathbf{F}}^e\right)(\zeta)\, d\zeta \approx \sum_{q=1}^N \omega_q \left(f \circ \widetilde{\mathbf{F}}^e\right)(\zeta_q) \det\left(D\widetilde{\mathbf{F}}^e\right)(\zeta_q)$$

Trimming

- Integrals must be accurately computed for “cut” basis functions: we reparametrize the trimmed elements.
- the linear system needs a special rescaling to improve conditioning



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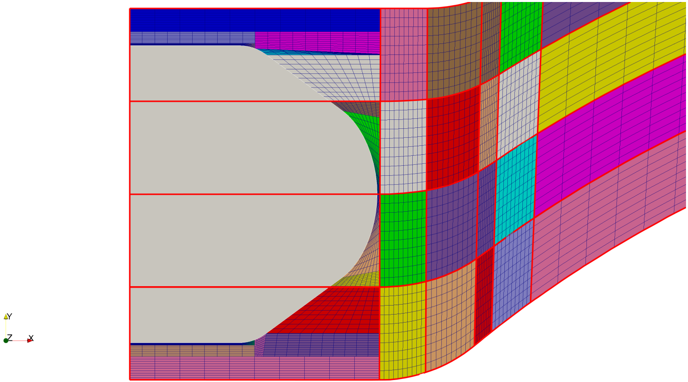

Reparametrization mesh by IRIT, G. Elber, Technion.

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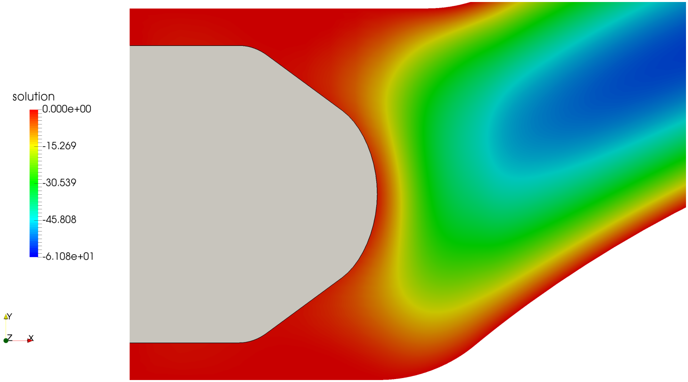

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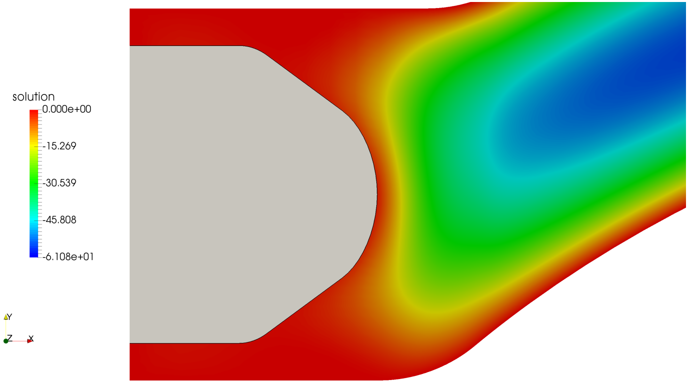

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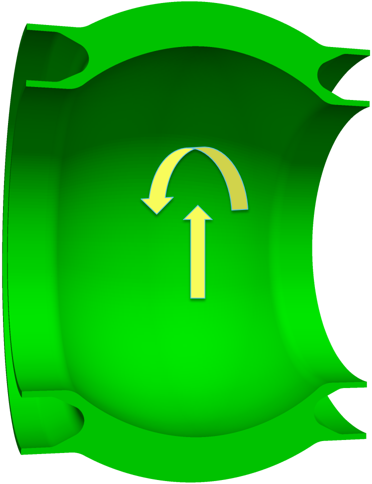
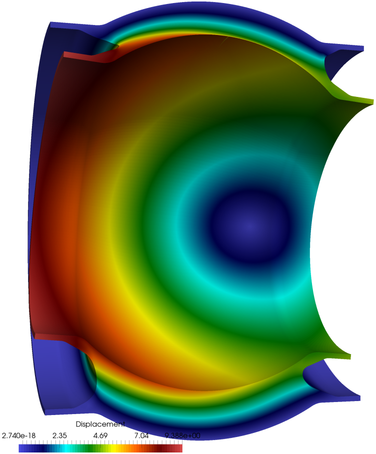
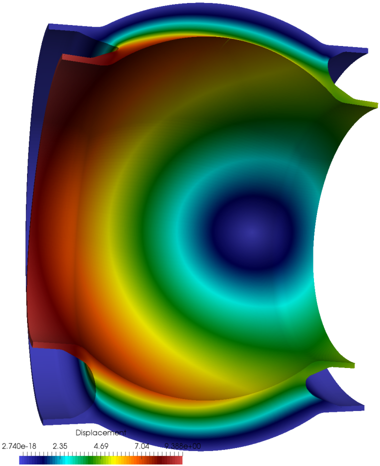
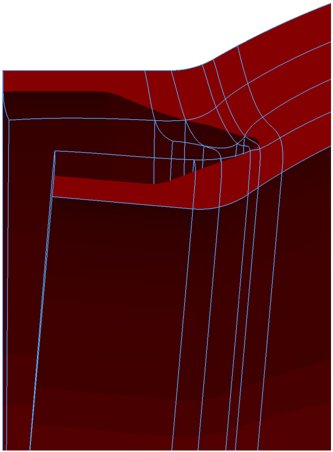


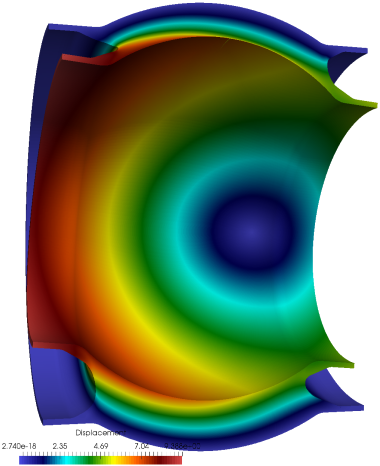


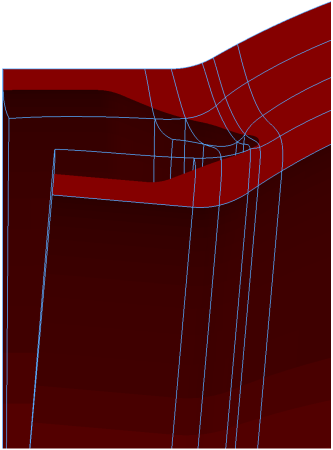
Trimming





Trimming





Handling efficiently trimmed 3D volumes is the next big challenge...

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