CIME-EMS Summer School in Applied Mathematics

Splines and PDEs: Recent Advances
from Approximation Theory to Structured Numerical Linear Algebra

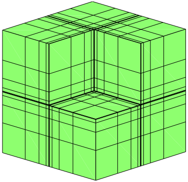
July 3 - July 7, 2017 - Cetraro

Structure preserving and stability.

Giancarlo Sangalli

...with results from many colleagues: A. Buffa, J. Evans, R. Vázquez,...

Finite Element Exterior Calculus



$$\mathbf{u} = -\mathbb{K}\nabla p \quad \mathrm{div}\, \mathbf{u} = \mathbf{g}$$

+ boundary conditions

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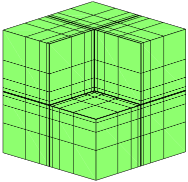

$$\begin{aligned} \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) - \nu \Delta \mathbf{u} + \nabla p &= \mathbf{f} \\ \operatorname{div} \mathbf{u} &= 0 \end{aligned}$$

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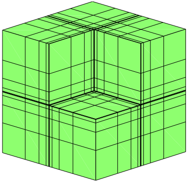

$$\mathbf{curl}\,\mathbf{H} = i\,\omega\,\mathbf{D} + \mathbf{J} \qquad \mathbf{curl}\,\mathbf{E} = -i\,\omega\,\mathbf{B}$$

$$\mathbf{B} = \mu\mathbf{H} \qquad \mathbf{D} = \varepsilon\mathbf{E}$$

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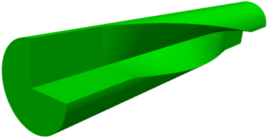

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- Problems which reveal a geometric structure that needs to be preserved at the discrete level to obtain spurious free discretizations

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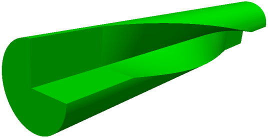

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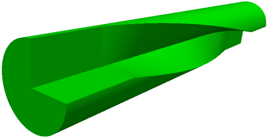
- Math literature: *Finite Element Exterior Calculus (FEEC)* ... started around 2000 and has involved many scientists

  [Arnold, Boffi, Bossavit, Buffa, Costabel, Christiansen, Demkovicz, Dauge, Falk, Hiptmair, Winther ....]

Finite Element Exterior Calculus



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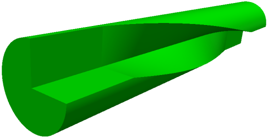

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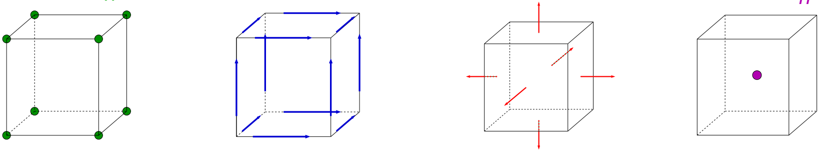
- Edge and Face elements, Mimetic Finite Differences, Finite Volumes methods, Virtual elements...
- Conjugate results from differential geometry, functional analysis and numerical analysis.

Finite element "compatible" vector fields

Λ_h^0 $\quad$ Λ_h^1 $\quad$ Λ_h^2 $\quad$ Λ_h^3

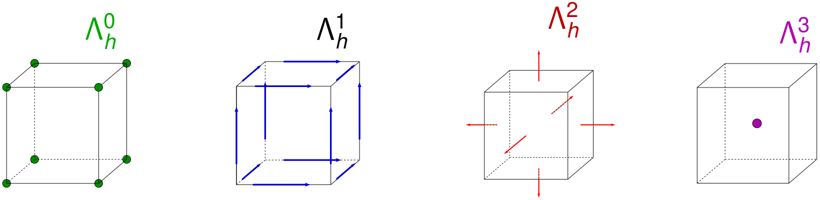


$$\mathbb{Q}^{1,1,1} \qquad \begin{bmatrix} \mathbb{Q}^{0,1,1} \\ \mathbb{Q}^{1,0,1} \\ \mathbb{Q}^{1,1,0} \end{bmatrix} \qquad \begin{bmatrix} \mathbb{Q}^{1,0,0} \\ \mathbb{Q}^{0,1,0} \\ \mathbb{Q}^{0,0,1} \end{bmatrix} \qquad \mathbb{Q}^{0,0,0}$$

These are the Whitney forms:

$$\mathbb{R} \longrightarrow \Lambda_h^0 \xrightarrow{\nabla} \Lambda_h^1 \xrightarrow{\textbf{curl}} \Lambda_h^2 \xrightarrow{\text{div}} \Lambda_h^3 \longrightarrow 0$$

Finite element “compatible” vector fields



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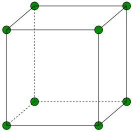
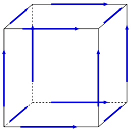
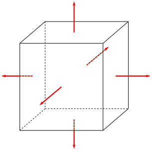
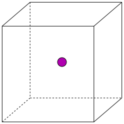
Isogeometric “compatible” vector fields

Λ_h^0 Λ_h^1 Λ_h^2 Λ_h^3

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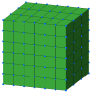
Degree elevation and **knot insertion** on the unit cube $\widehat{\Omega}$:



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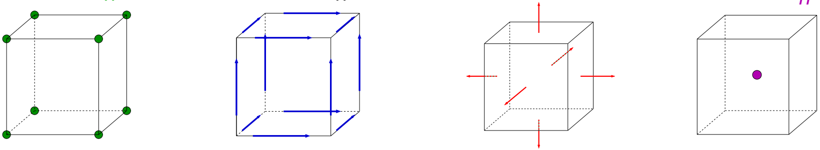

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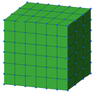

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$$S_h^0 = S_{p,p,p}$$

Isogeometric “compatible” vector fields

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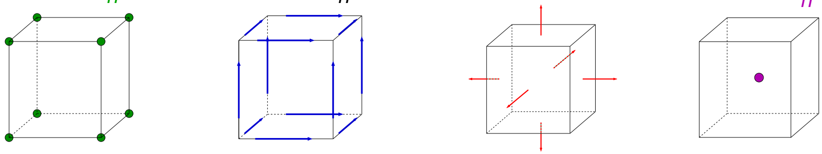

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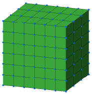

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$$S_h^1 = (S_{p-1,p,p}, S_{p,p-1,p}, S_{p,p,p-1})$$

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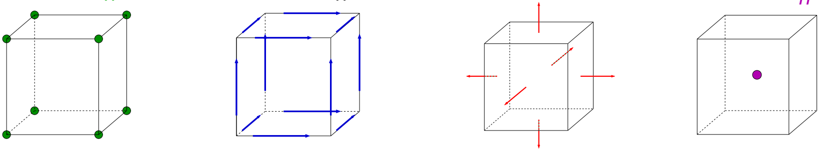
$\Lambda_h^0 \qquad \Lambda_h^1 \qquad \Lambda_h^2 \qquad \Lambda_h^3$



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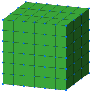

$$\mathbb{R} \longrightarrow S_h^0 \xrightarrow{\nabla} S_h^1 \xrightarrow{\textbf{curl}} S_h^2 \xrightarrow{\text{div}} S_h^3 \longrightarrow 0$$

$$S_h^2 = (S_{p,p-1,p-1}, S_{p-1,p,p-1}, S_{p-1,p-1,p})$$

Isogeometric "compatible" vector fields

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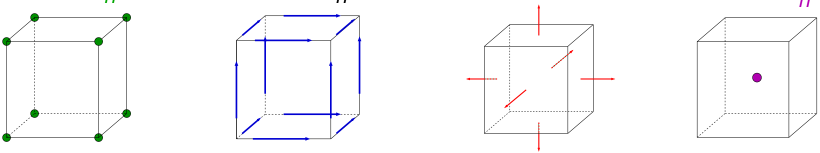

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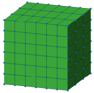

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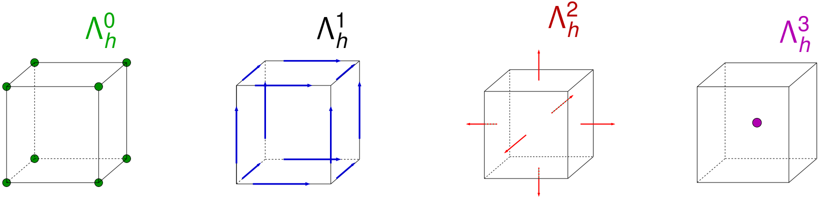
Isogeometric “compatible” vector fields



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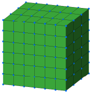

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The diagram is exact: $Im(\nabla) = Ker(\,\textbf{curl}\,)$, $Im(\,\textbf{curl}\,) = Ker(\,\mathrm{div}\,)$

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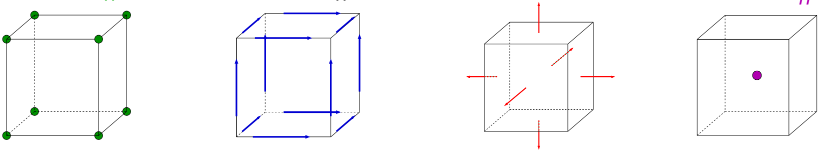

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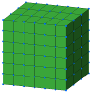

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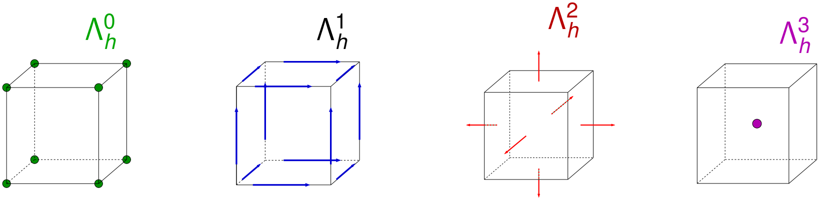
With splines you have smooth and compatible vector spaces!

Isogeometric “compatible” vector fields



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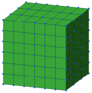

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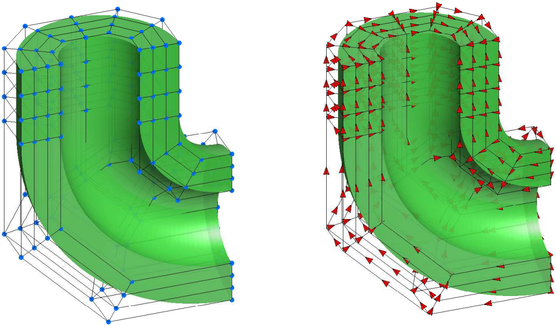
Can be extended to AS Tsplines [Buffa, Sangalli, Vazquez, JCP, 2010]

Representing d.o.f.s on mapped geometries



Left: dof location (blue dots) of isogeometric zero-forms.
Right: dof location (red arrows) of isogeometric one-forms

application to the Stokes Navier-Stokes equation

[Buffa, de Falco, Sangalli, 2010] [Evans, Hughes, 2010-2013]

- **Stokes**: $-\Delta \mathbf{u} + \nabla p = \mathbf{f}, \ \mathrm{div}\,(\mathbf{u}) = 0$

  seek for a $\mathbf{u}_h \in \Lambda_h^2$ such that $\mathrm{div}\,(\mathbf{u}_h) = 0$!

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  Extended to steady/unsteady Navier-Stokes by [Evans, Hughes, 2010-2013]
  - Structure preserving discretization of Navier-Stokes equations
  - Extremely stable w.r.t the Reynold number!

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$$
\begin{aligned}
\nabla \cdot (\mathbf{u} \otimes \mathbf{u}) - \nu \Delta \mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega = (0,1)^2 \\
\mathrm{div}\,\mathbf{u} &= 0 && \text{in } \Omega \\
\mathbf{u} &= 0 && \text{in } \Gamma = \partial\Omega.
\end{aligned}
$$

with weakly imposed non-leaking boundary conditions

application to the Stokes Navier-Stokes equation

[Buffa, de Falco, Sangalli, 2010] [Evans, Hughes, 2010-2013]

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Extended to steady/unsteady Navier-Stokes by [Evans, Hughes, 2010-2013]

- Structure preserving discretization of Navier-Stokes equations
- Extremely stable w.r.t the Reynold number!

Polynomial degree $k' = 2,\ h = 1/16$

Re	0	1	10	100	1000	10000
$\|\mathbf{u} - \mathbf{u}_h\|_h$	5.68e-4	5.68e-4	5.68e-4	5.68e-4	5.68e-4	5.68e-4
$\lvert\mathbf{u} - \mathbf{u}_h\rvert_{\mathbf{H}^1(\Omega)}$	5.68e-4	5.68e-4	5.68e-4	5.68e-4	5.68e-4	5.68e-4
$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{L}^2(\Omega)}$	5.03e-6	5.03e-6	5.03e-6	5.03e-6	5.03e-6	5.03e-6
$\|p - p_h\|_{L^2(\Omega)}$	1.17e-5	1.17e-5	6.50e-6	6.42e-4	6.42e-6	6.42e-6

Courtesy of J. Evans

application to the Stokes Navier-Stokes equation

[Buffa, de Falco, Sangalli, 2010] [Evans, Hughes, 2010-2013]

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- Structure preserving discretization of Navier-Stokes equations
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$\mathbf{Q}_2/Q_1$ velocity/pressure pair, $h = 1/16$

Re	0	1	10	100	1000	10000
$\lvert \mathbf{u} - \mathbf{u}_h \rvert_{\mathbf{H}^1(\Omega)}$	6.78e-4	6.78e-4	7.11e-4	2.26e-3	2.16e-2	2.16e-1
$\lVert \mathbf{u} - \mathbf{u}_h \rVert_{\mathbf{L}^2(\Omega)}$	6.54e-6	6.54e-6	6.79e-6	1.97e-5	1.86e-4	2.35e-3
$\lVert p - p_h \rVert_{L^2(\Omega)}$	1.96e-4	1.96e-4	1.96e-4	1.96e-4	1.96e-4	1.96e-4

Courtesy of J. Evans

application to the Stokes: spectrum analysis

[Evans, Hughes, 2010-2013]

Spectrum Analysis

Consider the two-dimensional periodic Stokes eigenproblem:

$$-\nabla \cdot \left(2\nu\nabla^{s}\mathbf{u}\right) + \nabla p = \lambda\,\mathbf{u} \quad \text{in } \Omega$$

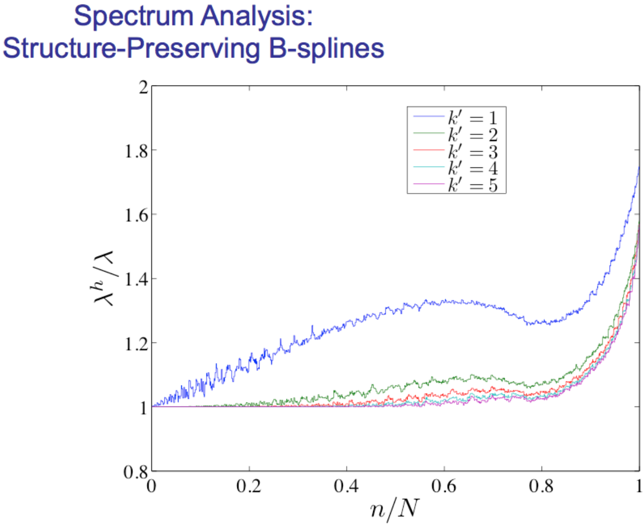
$$\nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Omega$$

w/ Periodic BCs

We compare the discrete spectrum for a specified discretization with the exact spectrum. This analysis sheds light on a given discretization's *diffusive characteristics* and *resolution properties*.

application to the Stokes: spectrum analysis

[Evans, Hughes, 2010-2013]



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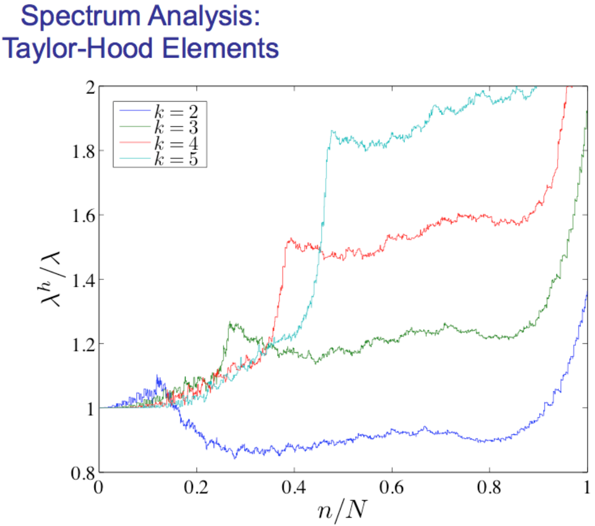
[Evans, Hughes, 2010-2013]



application to the Reissner-Mindlin plate bending

- $\Omega \subset \mathbf{R}^2$ = midsurface of the plate, t = the tickness
- $w =$ deflection, $\phi =$ rotation of the normal fibers
  $f =$ applied scaled normal load.

Exact formulation:
Find $\phi \in H_0^1(\Omega)^2, w \in H_0^1(\Omega)$ s.t. for all $\boldsymbol{\eta} \in H_0^1(\Omega)^2, v \in H_0^1(\Omega)$

$$(\mathbb{C}\varepsilon(\phi), \varepsilon(\boldsymbol{\eta})) + t^{-2}(\phi - \nabla w, \boldsymbol{\eta} - \nabla v) = (f, v)$$

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When $t \to 0$, R-M $\to$ Kirchoff which means: $\phi = \nabla w$.

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To prevent locking

When discretizing, usually the constraint $\phi = \nabla w$ needs to be weakened trough a reduced integration/mixed formulationz

application to the Reissner-Mindlin plate bending

[Beirao da Veiga, Buffa, Lovadina, Martinelli, Sangalli 2011]

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The 2D compatible element is (reasoning in a similar way)...

$$S_{p,p} \xrightarrow{\nabla} S_{p-1,p} \times S_{p,p-1}$$

$$\boldsymbol{\Phi}_h = S_{p-1,p} \times S_{p,p-1} \cap \text{ B.C. } , \; W_h = S_{p,p} \cap \text{ B.C.}$$

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No stabilization, no reduced integration, **locking free**

Convergence analysis [Beirao da Veiga, Buffa, Lovadina, Martinelli, Sangalli, 2011]

Let $\gamma = t^{-2}(\phi - \nabla w)$ and $\gamma_h = t^{-2}(\phi_h - \nabla w_h)$ (shear stresses) then, for regular solutions (and q.u. meshes):

$$\|\phi - \phi_h\|_{H^1} + h^{-1}\|w - w_h\|_{H^1} + (t+h)\|\gamma - \gamma_h\|_{L^2} \leq Ch^{p-1}$$

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Solution with boundary layer at curved sides (simply supported BCs), $t = 10^{-2}$, $p = 3$