Approximation with Ambient B-Splines and Intrinsic PDEs on Manifolds

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Approximation problem

How to approximate a function which is defined on the surface of some object, say . . .

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Approximation problem

- Given: manifold $\omega \subset \mathbb{R}^d$
  - without boundary, compact, smooth
  - codimension 1
  - arbitrary topology
- Given: function $f : \omega \to \mathbb{R}$
  - smooth
  - Sobolev class $W_p^n(\omega)$
- Sought: approximation $s : \omega \to \mathbb{R}$
  - accurate, $\|f - s\| = O(h^n)$
  - smooth, C^k
  - finite-dimensional space
  - simple concept, easy implementation
  - fast evaluation

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Approaches

- piecewise linear
  - flexible, standard in Computer Graphics
  - C^0, low approximation order
- intrinsic functions
  - explicitly known only for elementary geometry
  - otherwise comlicated
- chart-based methods
  - blending artifacts
- piecewise parametrization (subdivision, G-splines)
  - non-trivial quadrature
  - limited smoothness
- radial basis functions in ambient space
  - yes, but ...

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- **alternative:** ambient B-spline method (ABM)

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Basic idea

Define function space on ω by restricting functions in ambient space $\mathbb{R}^d$ to ω. In particular, if S is a spline of order n on $\mathbb{R}^d$, then

$$s := S_{|\omega}$$

is a smooth function on ω.

Benefits:

- standard splines, independent of ω
- higher order smoothness
- adaptive refinement

Challenges:

- stability
- approximation order

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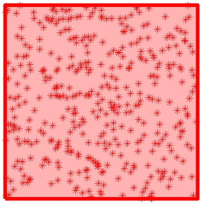
Scattered data approximation

- data sites $x_1, \ldots, x_N \in \mathbb{R}^n$
- corresponding function values $f_1, \ldots, f_N \in \mathbb{R}$
- sought: approximation f such that $f(x_i) \approx f_i$
- try spline: $f(x) = \sum_{k=1}^{M} b_k(x) c_k$
- (overdetermined) linear system

$$\begin{bmatrix} & \vdots & \\ \cdots & b_k(y_i) & \cdots \\ & \vdots & \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_M \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix}$$



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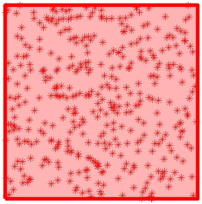
$$BC = F$$



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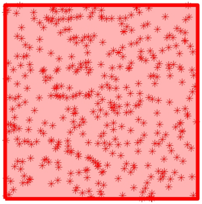
- solve normal equation

$$(B^T B)C = A^T F$$



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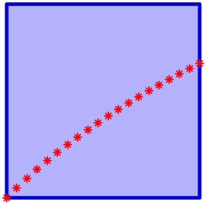
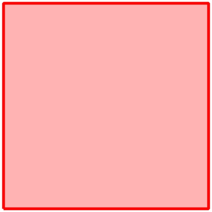


Lack of stability

Condition number of Gramian matrix of TP Bernstein basis on $[0,1]^2$:

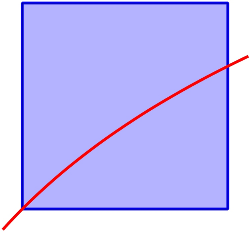
n	2	3	4
cond	8e0	1e2	1e3



Lack of stability

Condition number of Gramian matrix of TP Bernstein basis restricted to curve ω, e.g., graph of $\ln(1+x), 0 \leq x \leq 1$:

n	2	3	4
cond	1e06	1e20	3e32



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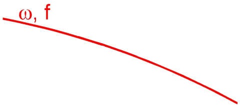
Extension

In order to use B-splines in a neighborhood

$$\Omega \supset \omega$$

of ω, we need to extend the given function,

$$f : \omega \to \mathbb{R}$$



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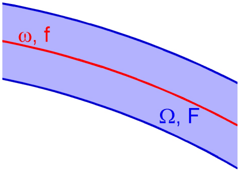
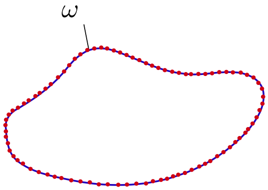
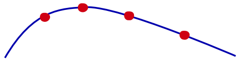
$$f : \omega \to \mathbb{R} \quad \Rightarrow \quad F : \Omega \to \mathbb{R}$$



Ambient B-Spline approximation method

- Given (scattered) data on manifold ω.





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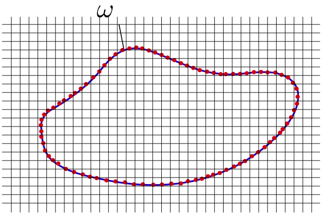
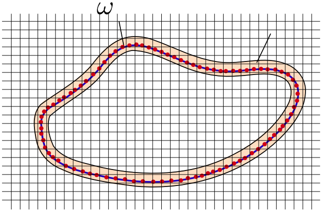
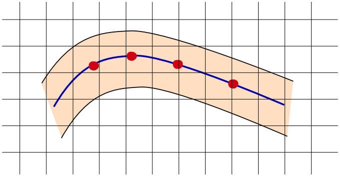


Ambient B-Spline approximation method

- Given (scattered) data on manifold ω.
- Define sufficiently thin tube $\Omega \supset \omega$.





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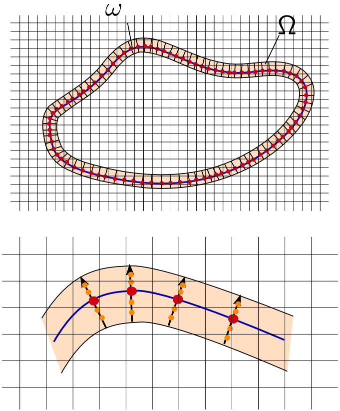
Ambient B-Spline approximation method

- Given (scattered) data on manifold ω.
- Define sufficiently thin tube $\Omega \supset \omega$.
- Extend data to Ω.

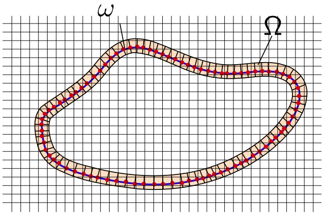


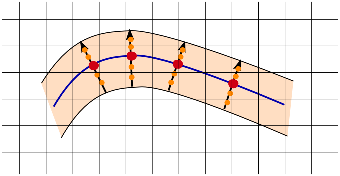
Ambient B-Spline approximation method

manifold ω

f

ambient space $\mathbb{R}^d$





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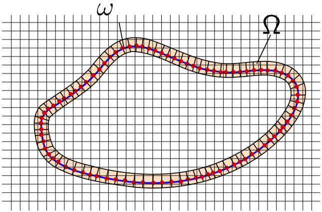
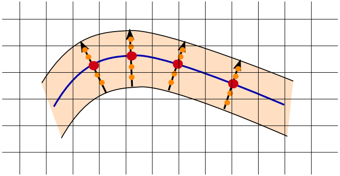
Ambient B-Spline approximation method

manifold ω

ambient space $\mathbb{R}^d$

$f \xrightarrow{\text{extension}} F = Ef$





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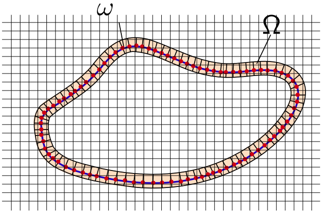
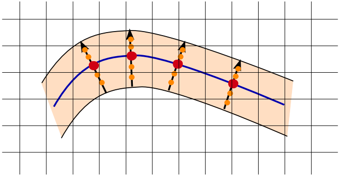
Ambient B-Spline approximation method

manifold ω

ambient space $\mathbb{R}^d$

$$
\begin{array}{ccc}
f & \xrightarrow{\text{extension}} & F = Ef \\
 & & \big\downarrow \text{approx.} \\
 & & S = PF
\end{array}
$$



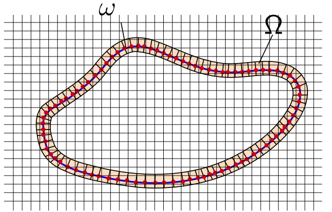


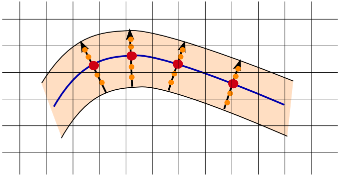
Ambient B-Spline approximation method

manifold ω

ambient space $\mathbb{R}^d$

$$\begin{tikzcd} f \arrow[r, "\text{extension}"] & F = Ef \arrow[d, "\text{approx.}"] \\ s = RS & S = PF \arrow[l, "\text{restriction}"] \end{tikzcd}$$





Ambient B-Spline approximation method

manifold ω | ambient space $\mathbb{R}^d$

$$
\begin{tikzcd}
f \arrow[r, "\text{extension}"] \arrow[d, "\text{ambient method}"'] & F = Ef \arrow[d, "\text{approx.}"] \\
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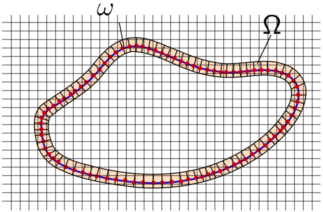

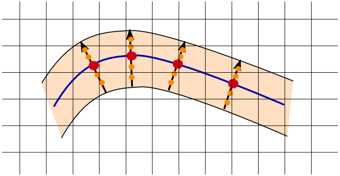

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manifold ω

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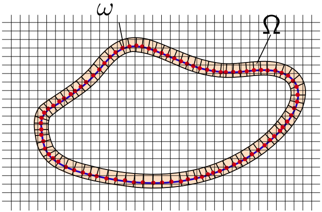

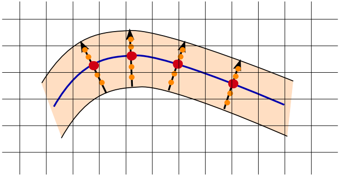

The ambient method: $\quad s = RPEf$

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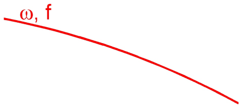
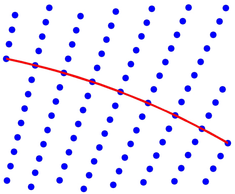
Extension

Given $f : \omega \to \mathbb{R}$, it is not difficult to construct an extension $F : \Omega \to \mathbb{R}$, provided that Ω is small enough:

- constant in normal direction

$$F(x + tn) = f(x), \quad x \in \omega$$





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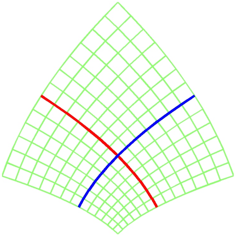
- orthogonal flow, if $\omega = \varphi^{-1}(0)$ is given as a levelset

$$F(\psi(x,t)) = f(x), \quad \partial_t \psi = \frac{\nabla\varphi}{|\nabla\varphi|}, \quad \psi(x,0) = \psi(x)$$



Ambient B-Spline approximation method

Properties:

- based on standard tensor product B-splines

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- higher dimension, but comparable number of control points
- approximation order?

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Approximation order

Theorem (Odathuparambil, R. '15)

Let E_ψ be the extension operator based on some transversal flow ψ. For $f \in W_p^n(\omega)$, the approximation error $\Delta = RPE_\psi f - f$ is bounded by

$$\|\Delta\|_{W_p^m(\omega)} \leq \mathrm{c}\, h^{n-m} \|f\|_{W_p^n(\omega)}, \quad m < n,$$

where c depends on ψ.

Proof is based on:

- approximation properties of P
- Friedrichs' inequality
- Markov inequality
- Faà di Bruno formula

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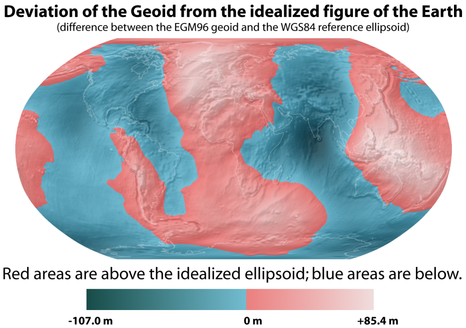
Example: The geoid

The geoid is the equipotential surface of gravitational field corresponding to the mean-ocean surface.

Model currently used **EGM2008**:

- spherical harmonics up to degree 2190 and order 2159,
- more than 4 million coefficients.



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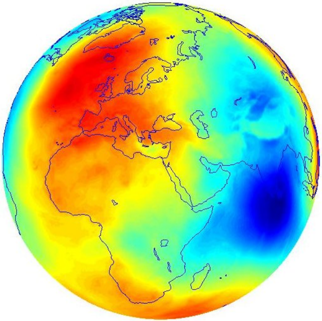
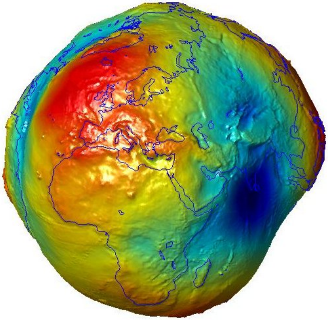
Example: The geoid

Using ambient B-spline approximation method:

- drastically improves evaluation time,
- reduces number of coefficients (hierarchical B-splines).

Local B-Spline method (order 3, $1e6$ coefficients):





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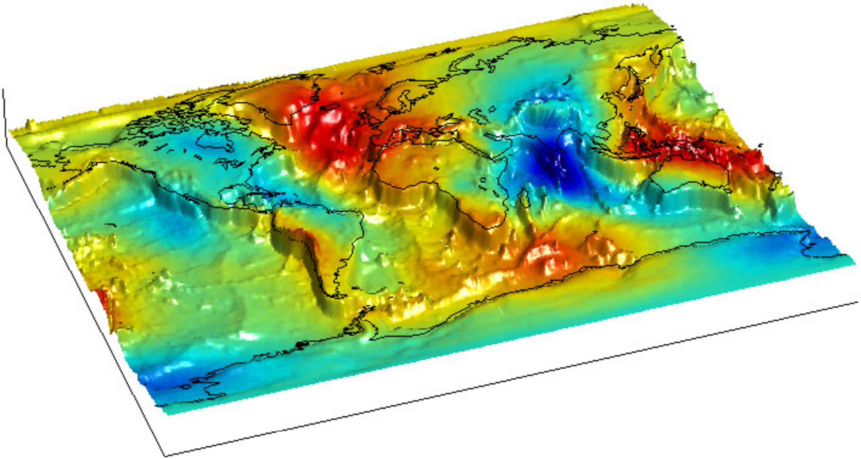

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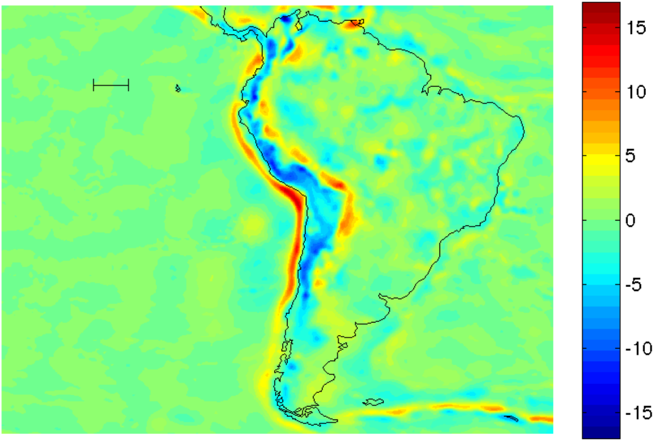

approximation error for $h = \frac{1}{10} R_{\mathrm{earth}}$

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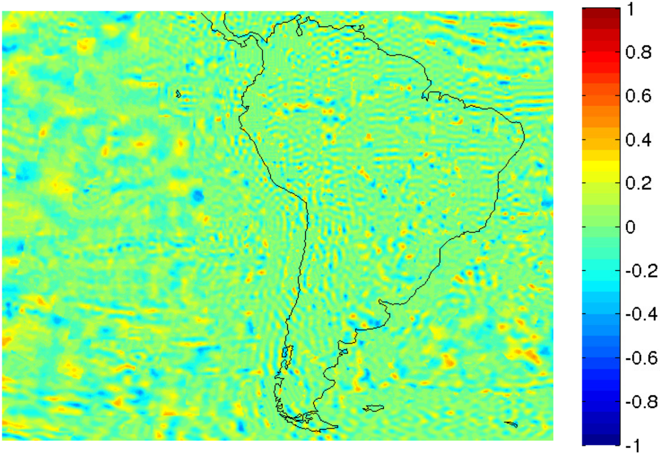
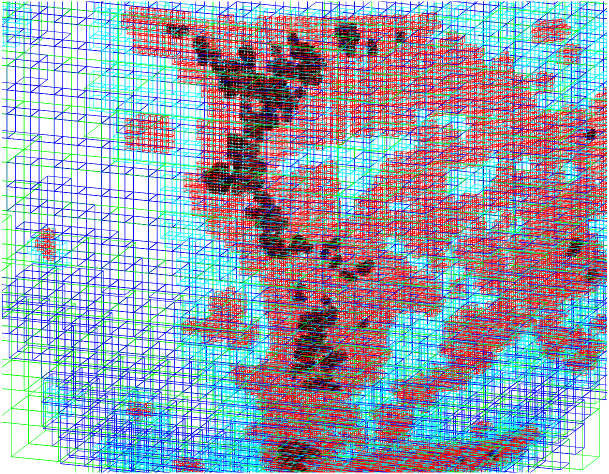
Example: The geoid, adaptive refinement



error for $h_{\min} = \frac{1}{160} R_{\text{earth}}$ and $\sim$ 750.000 B-splines

Example: The geoid, adaptive refinement



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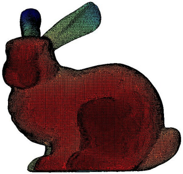
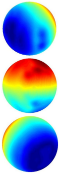
Surface reconstruction



reparametrize
over sphere

approximate
coordinate
functions



smooth
representation

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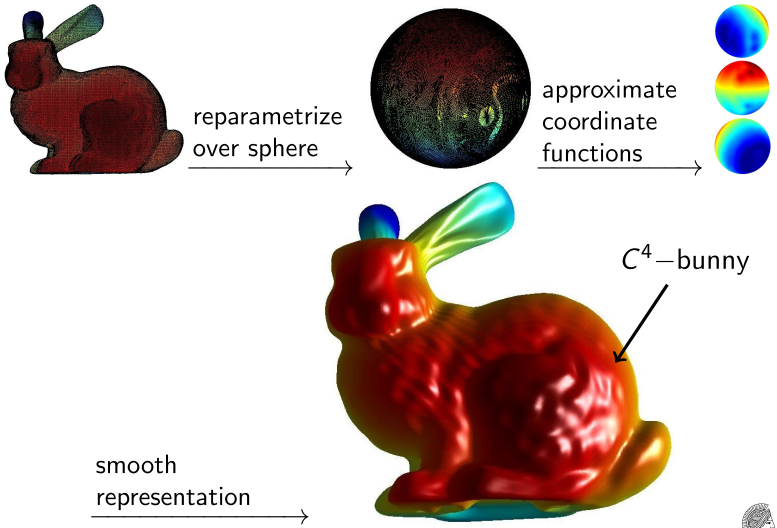
Surface reconstruction



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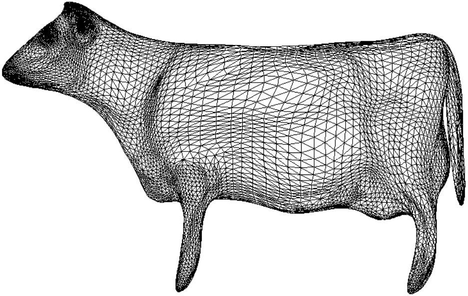
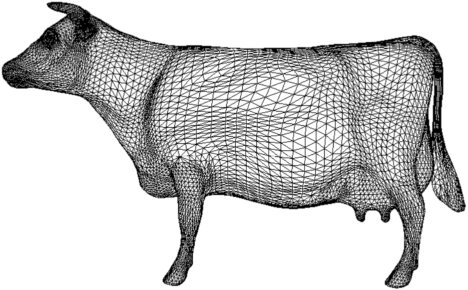


Approximation with $h = 0.2$ and ~ 2000 B-Splines.

Surface reconstruction



Adaptive approximation with $h_{\min} = 0.02$ and ~ 6000 B-Splines.

Surface reconstruction

Benefits:

- simple construction
- arbitrary smoothness
- adaptive refinement
- no extraordinary vertices

Challenges:

- How to find a *good* parametrization?
- How to build an interactive modeling tool?
- How to model sharp creases?

Intrinsic PDEs on manifolds

Intrinsic model equations:

- elliptic

$$\Delta_\omega u + cu = f, \quad c < 0$$

- parabolic

$$u_t = -\Delta_\omega u$$

Applications:

- Computer Graphics (parametrization, segmentation)
- Fluid Dynamics
- Biology/Medicine
- Meteorology
- ...

Intrinsic PDEs on manifolds

- Piecewise linear FE approximation (Wardetzki '07).
- Embedding methods for parabolic PDEs (Bertalmio et. al. '01, Ruuth and Merriman '08). The Laplace-Beltrami operator is computed by

$$\Delta_\omega u = \Delta E_n u$$

  instead of

$$\Delta_\omega u = \frac{\mathrm{div}\left(\sqrt{\det G}\, G^{-1} \nabla u\right)}{\sqrt{\det G}}.$$

- Embedding method for elliptic PDEs (Dziuk and Elliott '13). Problem: Loss of elipticity.

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- Embedding method for elliptic PDEs (Dziuk and Elliott '13). Problem: Loss of elipticity.
- New: Ambient B-spline approximation of extended elliptic PDE.

Intrinsic PDEs on manifolds

Caution: Let u be a solution of the intrinsic PDE

$$\Delta_\omega u + cu = f.$$

Consider the extensions $U := E_n u$ and $F := E_n f$ in normal direction.

Then

$$\Delta U + cU = F \quad \text{on} \quad \omega$$

but

$$\Delta U + cU \neq F \quad \text{on} \quad \Omega.$$

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New approach

Can we define an elliptic operator L such that

$$LEu = E\Delta_\omega u\,?$$

Then, we would have

$$\begin{aligned}
\Delta_\omega u + cu &= f \\
E(\Delta_\omega u + cu) &= Ef \\
LEu + cEu &= Ef \\
LU + cU &= F
\end{aligned}$$

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New approach

Theorem (Odathuparambil, R. '14)

Let d be the signed distance function of ω. Define

- *the matrix*

$$Q := (\mathrm{Id} - dH)^{-1}, \quad H := \nabla^2 d.$$

- *the differential operator*

$$LU := \Delta_Q U := \sum_{i,j} Q_{i,j} (\nabla Q \nabla U)_{i,j}.$$

Then L is uniformly elliptic in a vicinity of ω and satisfies

$$LE_n u = E_n \Delta_\omega u$$

In particular,

$$\Delta_\omega u + cu = f \quad \Rightarrow \quad LU + cU = F, \quad \nabla U \cdot \nabla d = 0.$$

New approach

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$$L E_n u = E_n \Delta_\omega u$$

In particular,

$$\Delta_\omega u + cu = f \quad \Leftrightarrow \quad LU + cU = F, \quad \nabla U \cdot \nabla d = 0.$$

New approach for general levelsets

- $\omega = \varphi^{-1}(0)$ is given as a levelset.
- General second order differential operator on ω

$$L^0 := A^0 * \nabla^2 u + B^0 * \nabla u := \sum_{i,j} A^0_{i,j} \partial_{i,j} u + \sum_i B^0_i \partial_i u$$

- **Sought:** Extension

$$L := A * \nabla^2 U + B * \nabla U$$

to ambient space along the orthogonal flow ψ such that

$$LU = LE_\psi u = E_\psi L^0 u.$$

- **Challenge:** Find a formula for the functions $A = A(X), B = B(X), X \in \Omega$.

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New approach for general levelsets

Theorem (Odathuparambil, R. 15)

Consider the system of ODEs

$$\tilde{A}' = |\nabla\varphi|^{-1}(\tilde{A}H + H\tilde{A}), \quad H := \nabla^2\varphi$$
$$\tilde{B}' = |\nabla\varphi|^{-1}(H\tilde{B} + \tilde{A} * \partial H)$$

with initial conditions $\tilde{A}(0) := A^0, \tilde{B}(0) := B^0$ *and define*

$$A(\psi(x,t)) := \tilde{A}(t), \quad B(\psi(x,t)) := \tilde{B}(t).$$

Then the operator L*, as defined above, is uniformly elliptic in a vicinity of* ω *if so is* L^0*, and satisfies*

$$LU = E_\psi L^0 u$$

In particular,

$$L^0 u = f \quad \Leftrightarrow \quad LU = F, \quad \nabla U \cdot \nabla\varphi = 0.$$

New approach for general levelsets

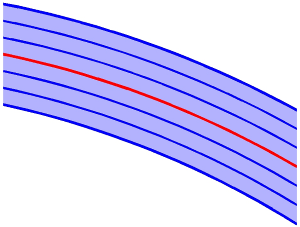
- If the boundary $\partial\Omega$ is given by levelsets, the problem

$$LU = F, \quad \nabla U \cdot \nabla \varphi = 0$$

is equivalent to an elliptic PDE with Neumann boundary conditions,

$$LU = F, \quad \nabla U \cdot \nabla \varphi = 0 \text{ on } \partial\Omega.$$

- Meshing required.

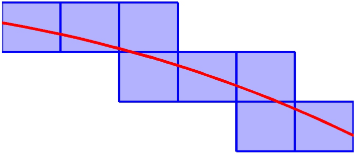


New approach for general levelsets

- If the boundary of Ω is *not* given by levelsets, the problem

$$LU = F, \quad \nabla U \cdot \nabla \varphi = 0$$

  is still well posed. In particular, Ω can be defined as a union of boxes covering ω.
- No meshing required!



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Current work

- Implementation and practical tests
- Ambient smoothing splines (L. Maier)
- Manifolds with boundary
- Error estimates
- ...

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Thanks for your attention!