Lecture 1: The B-spline

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Splines and PDE's: Recent advances from Approximation Theory to structured Numerical Linear Algebra

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Knot sequence

A **knot sequence** $\boldsymbol{\xi}$ is a nondecreasing sequence of real numbers,

$$\boldsymbol{\xi} := \{\xi_i\}_{i=1}^{m} = \{\xi_1 \leq \xi_2 \leq \cdots \leq \xi_m\}, \quad m \in \mathbb{N}.$$

The elements ξ_i are called **knots**.

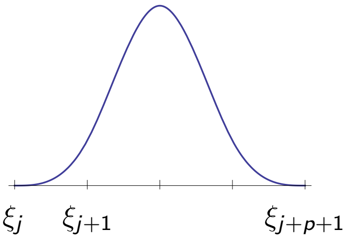
Recursive definition of jth B-splines of degree p on $\boldsymbol{\xi}$

$$B_{j,0,\boldsymbol{\xi}}(x) := \chi_{[\xi_j,\xi_{j+1})}(x) = \begin{cases} 1, & \text{if } x \in [\xi_j, \xi_{j+1}), \\ 0, & \text{otherwise.} \end{cases}$$

For $p \geq 1$

$$B_{j,p,\boldsymbol{\xi}}(x) := \frac{x - \xi_j}{\xi_{j+p} - \xi_j} B_{j,p-1,\boldsymbol{\xi}}(x) + \frac{\xi_{j+p+1} - x}{\xi_{j+p+1} - \xi_{j+1}} B_{j+1,p-1,\boldsymbol{\xi}}(x), \quad x \in \mathbb{R}$$

term with zero denominator is defined to be zero
(the "$\frac{0}{0} := 0$" convention)



Alternative notation $B[\xi_j, \xi_{j+1}, \ldots, \xi_{j+p+1}](x)$

Right continuity

Note that

$$B_{j,0,\boldsymbol{\xi}}(\xi_j) = 1, \quad B_{j,0,\boldsymbol{\xi}}(\xi_{j+1}) = 0.$$

It follows that any B-spline is **right continuous**, i.e., the value at a point x is obtained by taking limits from the right:

$$B_{j,p,\boldsymbol{\xi}}(\xi) = B_{j,p,\boldsymbol{\xi}}(\xi_+),$$

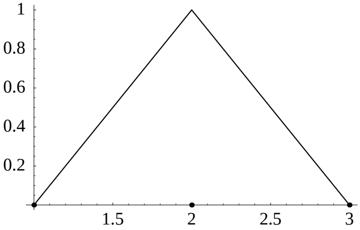
where

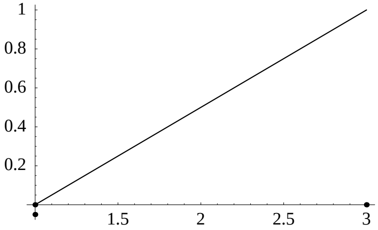
$$x_+ := \lim_{\substack{t \to x \\ t > x}} t, \quad x_- := \lim_{\substack{t \to x \\ t < x}} t, \quad x \in \mathbb{R}.$$

Multiplicity of a knot

We say that a knot has **multiplicity** μ if it occurs exactly μ times in the knot sequence. A knot is called **simple**, **double**, **triple**, ... if its multiplicity is equal to $1, 2, 3, \ldots$, and a **multiple knot** if $\mu > 1$.

jth B-spline of degree $p = 1$ on $\boldsymbol{\xi}$

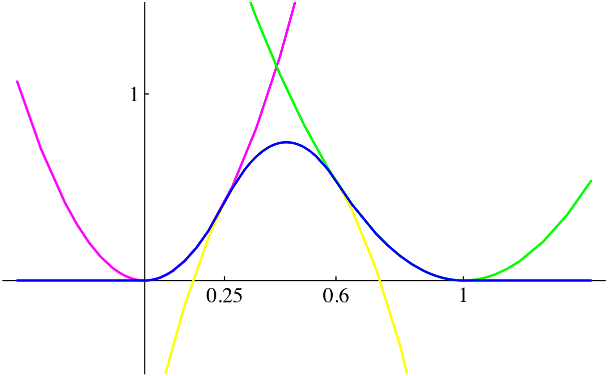




$B[1,2,3](x) := \frac{x-1}{2-1}\chi_{[1,2)}(x) + \frac{3-x}{3-2}\chi_{[2,3)}(x)$, $B[1,3,3](x) := \frac{x-1}{2}\chi_{[1,3)}(x) + \frac{0}{0}$

The linear B-spline is continuous at a simple knot and discontinuous at a double knot.

B-spline of degree 2



$B[0, \frac{1}{4}, \frac{3}{5}, 1](x)$

The quadratic B-spline consists of pieces of parabolas tied together smoothly.

B-splines, properties

Simple to prove properties (induction on p)

- **Local support.** $B_{j,p,\boldsymbol{\xi}}(x) = 0, \quad x \notin [\xi_j, \xi_{j+p+1})$.
- **Piecewise structure.** $B^{\{m\}}_{j,p,\boldsymbol{\xi}} \in \mathbb{P}_p, \quad m = j, \ldots, j+p$.
- **Nonnegativity.** $B_{j,p,\boldsymbol{\xi}}(x) \geq 0, \quad x \in \mathbb{R}$,
- **Positivity.** $B_{j,p,\boldsymbol{\xi}}(x) > 0, \quad x \in (\xi_j, \xi_{j+p+1})$.
- **Translation and scaling invariance.**

$$B_{j,p,\alpha\boldsymbol{\xi}+\beta}(\alpha x + \beta) = B_{j,p,\boldsymbol{\xi}}(x), \quad \alpha, \beta \in \mathbb{R}, \quad \alpha \neq 0,$$

where $\alpha\boldsymbol{\xi} + \beta := (\alpha\xi_j + \beta, \ldots, \alpha\xi_{j+p+1} + \beta)$.

The following identity simplifies many dealings with B-splines.

Theorem (Local Marsden identity)

Suppose

$$\xi_{m-p} \leq \cdots \leq \xi_m < \xi_{m+1} \leq \cdots \leq \xi_{m+p+1}.$$

Then

$$(y-x)^p = \sum_{i=m-p}^{m} \psi_{i,p,\boldsymbol{\xi}}(y) B_{i,p,\boldsymbol{\xi}}(x), \quad x \in [\xi_m, \xi_{m+1}), \quad y \in \mathbb{R}, \tag{1}$$

where $\psi_{j,0,\boldsymbol{\xi}} := 1$ *and*

$$\psi_{j,p,\boldsymbol{\xi}}(y) := (y-\xi_{j+1}) \cdots (y-\xi_{j+p}), \quad y \in \mathbb{R}, \quad p \in \mathbb{N}. \tag{2}$$

We say that $\psi_{j,p,\boldsymbol{\xi}}$ *is the polynomial* **dual** *to* $B_{j,p,\boldsymbol{\xi}}$

Proof

We need to prove

$$(y-x)^p = \sum_{i=m-p}^{m} (y-\xi_{i+1})\cdots(y-\xi_{i+p}) B_{i,p,\boldsymbol{\xi}}(x)$$

- $p = 0$: $1 = 1$
- $p = 1$

$$\begin{aligned}
&(y-\xi_m)B_{m-1,1,\boldsymbol{\xi}}(x) + (y-\xi_{m+1})B_{m,1,\boldsymbol{\xi}}(x) \\
&= (y-\xi_m)\frac{\xi_{m+1}-x}{\xi_{m+1}-\xi_m} + (y-\xi_{m+1})\frac{x-\xi_m}{\xi_{m+1}-\xi_m} \\
&= (y-x)
\end{aligned}$$

  Follows by **linear interpolation** of $\ell_y(x) := y - x$
- $p > 1$ use recurrence relations for B_j and ψ_j

Local properties

$$(y-x)^p = \sum_{i=m-p}^{m} (y-\xi_{j+1})\cdots(y-\xi_{j+p})B_{i,p,\boldsymbol{\xi}}(x)$$

- **Local partition of unity.** Differentiating p times wrt y and cancelling $p!$ gives

$$1 = \sum_{i=m-p}^{m} B_{i,p,\boldsymbol{\xi}}(x), \quad x \in [\xi_m, \xi_{m+1}). \tag{3}$$

- **Local linear independence.** The two sets $\{B_{i,p,\boldsymbol{\xi}}\}_{i=m-p}^{m}$ and $\{\psi_{i,p,\boldsymbol{\xi}}\}_{i=m-p}^{m}$ form both a basis for the polynomial space $\mathbb{P}_p$ on $[\xi_m, \xi_{m+1})$.

Derivative

The derivative of a B-spline can be expressed by means of a simple difference formula.

Theorem (Differentiation formula)

We have

$$D_+ B_{j,p,\boldsymbol{\xi}}(x) = p\left(\frac{B_{j,p-1,\boldsymbol{\xi}}(x)}{\xi_{j+p}-\xi_j} - \frac{B_{j+1,p-1,\boldsymbol{\xi}}(x)}{\xi_{j+p+1}-\xi_{j+1}}\right), \quad p \geq 1, \tag{4}$$

where fractions with zero denominator have value zero.

Right and left derivatives of a function f are defined by

$$D_+ f(x) := \lim_{\substack{h\to 0\\ h>0}} \frac{f(x+h)-f(x)}{h}, \quad D_- f(x) := \lim_{\substack{h\to 0\\ h<0}} \frac{f(x+h)-f(x)}{h}, \tag{5}$$

provided that the limits exist at the point $x \in \mathbb{R}$.

Proof Differentiation formula

Suppose $x \in [\xi_m, \xi_{m+1})$ for some m with $j \leq m \leq j + p$. Differentiating both sides of Marsden with respect to x gives

$$- p(y - x)^{p-1} = \sum_{i=m-p}^{m} \psi_{i,p}(y) D B_{i,p,\boldsymbol{\xi}}(x). \tag{6}$$

On the other hand, using Marsden for degree $p - 1$

$$\begin{aligned}
-p(y-x)^{p-1} &= -p \sum_{i=m-p+1}^{m} \psi_{i,p-1}(y) B_{i,p-1,\boldsymbol{\xi}}(x) \\
&= p \sum_{i=m-p+1}^{m} \left(\frac{\psi_{i,p}(y)}{\xi_{i+p} - \xi_i} - \frac{\psi_{i-1,p}(y)}{\xi_{i+p} - \xi_i} \right) B_{i,p-1,\boldsymbol{\xi}}(x) \\
&= \sum_{i=m-p}^{m} \psi_{i,p}(y) p \left(\frac{B_{i,p-1,\boldsymbol{\xi}}(x)}{\xi_{i+p} - \xi_i} - \frac{B_{i+1,p-1,\boldsymbol{\xi}}(x)}{\xi_{i+p+1} - \xi_{i+1}} \right).
\end{aligned}$$

By comparing this with (6) and using the linear independence of the dual polynomials, it follows that (4) holds for $i = m - p, \ldots, m$. Since $m - p \leq j \leq m$, (4) holds for $i = j$. $\square$

Continuity

Theorem (Smoothness property)
If ξ is a knot of $B_{j,p,\boldsymbol{\xi}}$ of multiplicity $\mu \leq p+1$ then

$$B_{j,p,\boldsymbol{\xi}} \in C^{p-\mu}(\xi),$$

i.e., its derivatives of order $0, 1, \ldots, p-\mu$ are continuous at ξ.

(The derivative of order $p-\mu+1$ has a nonzero jump at ξ.)

"continuity +multiplicity = degree"

Integral of a B-spline

Theorem (Integration formula)

We have

$$\gamma_{j,p,\boldsymbol{\xi}} := \int_{\xi_j}^{\xi_{j+p+1}} B_{j,p,\boldsymbol{\xi}}(x)\,\mathrm{d}x = \frac{\xi_{j+p+1} - \xi_j}{p+1}. \tag{7}$$

Proof integration formula

We define $p+1$ extra knots at each end such that

$$\boldsymbol{\xi} := \{\xi_{j-p-1} = \cdots = \xi_{j-1} < \xi_j \leq \cdots \leq \xi_{j+p+1} < \xi_{j+p+2} = \cdots = \xi_{j+2p+2}\}.$$

On this knot sequence we consider $p+1$ B-splines $B_{i,p+1,\boldsymbol{\xi}}$, $i = j-p-1, \ldots, j-1$ of degree $p+1$. These B-splines are continuous on $\mathbb{R}$. Therefore, we get for $i = j-p-1, \ldots, j-1$,

$$0 = B_{i,p+1,\boldsymbol{\xi}}(\xi_{i+p+2}) - B_{i,p+1,\boldsymbol{\xi}}(\xi_i) = \int_{\xi_i}^{\xi_{i+p+2}} D_+ B_{i,p+1,\boldsymbol{\xi}}(x)\,\mathrm{d}x = E_i - E_{i+1},$$

where by the local support and the differentiation formula (4),

$$E_i := \frac{p+1}{\xi_{i+p+1} - \xi_i} \int_{\xi_i}^{\xi_{i+p+1}} B_{i,p,\boldsymbol{\xi}}(x)\,\mathrm{d}x, \quad i = j-p-1, \ldots, j.$$

This means that $E_j = E_{j-1} = \cdots = E_{j-p-1}$. Moreover, since $\xi_{j-p-1} = \cdots = \xi_{j-1}$, we obtain that

$$E_{j-p-1} = \frac{p+1}{\xi_j - \xi_{j-p-1}} \int_{\xi_{j-p-1}}^{\xi_j} \frac{(\xi_j - x)^p}{(\xi_j - \xi_{j-p-1})^p}\,\mathrm{d}x = 1,$$

and the integration formula (7) follows. $\square$

What does the **B** in B-Splines stand for?

- Beautiful?
- Bell shaped?
- Basic

It started early

Laplace, 1749-1827,

Popoviciu,1906-1975,

Chakalov, 1886-1963

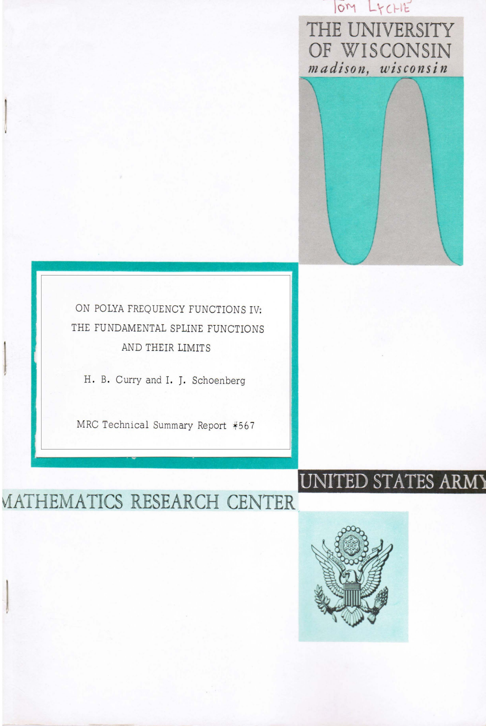
- Laplace 1820: Uniform B-splines
- Popoviciu 1930's B-spline recurrence relation, Marsden identity, knot insertion
- Chakalov1930's: B-splines and divided differences, positivity
- de Boor, Pinkus, J. Approximation Theory, 124(2003),115–123.
  Bojanov, Hakopian, Sahakian 93, Bojanov 96.

Some mathematical works after 1945

- Schoenberg, 1946, B-splines on uniform knots.
- Curry, Schoenberg 1966, General B-spline theory
- Book by Ahlberg, Nilson, Walsh, 1967. Unfortunately they ignored B-splines.
- Marsden's identity, 1970
- Cox, de Boor, 1972, recurrence relation
- Boehm 1980 and Cohen, Lyche, Riesenfeld, 1980, knot insertion.

The Curry Schoenberg paper, 1966

" The present paper was written in 1945 and completed by 1947, but for no good reason has so far not been published".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ON POLYA FREQUENCY FUNCTIONS IV:
THE FUNDAMENTAL SPLINE FUNCTIONS
AND THEIR LIMITS

H. B. Curry and I. J. Schoenberg

MRC Technical Summary Report #567

UNITED STATES ARMY
MATHEMATICS RESEARCH CENTER

Early industrial applications, France

 

Paul de Faget de Casteljau (1930-), Pierre Bézier

Industrial application, variational approach

- Even Mehlum
- Autocon, SI, Norway 1960's aimed at ship building industry
- Autokon had 70% of the world market for ship design systems
- a form of nonlinear splines (Cornu spirals)