Lecture 3: Approximation by Quasi-interpolants

Tom Lyche

Centre of Mathematics for Applications,
Department of Mathematics,
University of Oslo

Splines and PDE's: Recent advances from Approximation Theory to structured Numerical Linear Algebra

CIME-EMS Summer School in Applied Mathematics
3 July - 7 July, 2017

Contents

The space $C^{-1}(I)$

- Let I be a finite interval (open, half open, closed) and $f : I \to \mathbb{R}$. If
  - f is bounded
  - f is continuous except at a finite number of points, where the value is obtained by taking the limit either from the left or the right,

  then f is said to be **piecewise continuous** on I. We denote the space of these functions by $C^{-1}(I)$.
- For any function $f \in C^{-1}([a,b])$ we write $f : [a^+, b^-] \to \mathbb{R}$ to mean that $f(a) := f(a+)$ and $f(b) := f(b-)$. With the notation $f \in C^r([a^+, b^-])$ we mean that f has continuous derivatives up to order r on (a,b), and their limits $\lim_{\substack{x\to a\\ x>a}} D^j f(x)$, $\lim_{\substack{x\to b\\ x<b}} D^j f(x)$, $j = 0, \ldots, r$ exist and are bounded.

The L_q spaces

For $1 \le q \le \infty$ and $I := [a, b]$ the L_q-**spaces** are defined by

$$L_q(I) := \{f : I \to \mathbb{R},\ f \text{ is measurable on } I \text{ and } \|f\|_{L_q(I)} < \infty\}, \tag{1}$$

where the L_q-**norms** are given by

$$\|f\|_{L_\infty(I)} := \operatorname*{ess\,sup}_{x\in I} |f(x)| \tag{2}$$

$$\|f\|_{L_q(I)} := \Big(\int_a^b |f(x)|^q \,\mathrm{d}x \Big)^{1/q}, \quad 1 \le q < \infty. \tag{3}$$

Note that

- If $f \in C^{-1}(I)$ then $\|f\|_{L_\infty(I)} = \sup_{x\in I} |f(x)|$
- if I is closed and $f \in C(I)$ then $\|f\|_{L_\infty(I)} = \max_{x\in I} |f(x)|$.

Hölder inequalities

The **Hölder inequalities** for integrals and sums are given by

$$\int_a^b |f(x)g(x)|\,\mathrm{d}x \le \|f\|_{L_q(I)} \|g\|_{L_{q'}(I)},$$

$$\sum_{j=1}^{n} |x_j\, y_j| \le \|\boldsymbol{x}\|_q\, \|\boldsymbol{y}\|_{q'},$$

where q, q' are integers so that

$$\frac{1}{q} + \frac{1}{q'} = 1, \quad 1 \le q \le \infty.$$

In particular, $q' = \infty$ if $q = 1$ and $q' = 2$ if $q = 2$.

1D Sobolev spaces

- For I a finite interval, $r \in \mathbb{N}_0$ and $1 \leq q \leq \infty$ the **one-dimensional Sobolev spaces** are given by

$$W_q^r(I) = \left\{ f : I \to \mathbb{R} : f \in C^{r-1}(I),\ D^r f \in L_q(I) \right\}. \qquad (4)$$

- The Sobolev spaces are complete normed spaces with norm

$$\|f\|_{W_q^r(I)}^2 := \sum_{j=0}^{r} \|D^j f\|_{L_q(I)}^2, \qquad (5)$$

called the **Sobolev norm**. It is an inner product norm for $q = 2$.

Taylor approximation

The **Taylor polynomial** of degree p at the point a to a function $f \in W_\infty^{p+1}([a,b])$ is defined by

$$\mathcal{T}_{a,p} f(x) := \sum_{j=0}^{p} \frac{(x-a)^j}{j!} D^j f(a), \tag{6}$$

and its approximation error can be expressed in integral form as

$$f(x) - \mathcal{T}_{a,p} f(x) = \frac{1}{p!} \int_a^b (x-y)_+^p D^{p+1} f(y) \, \mathrm{d}y, \quad x \in [a,b], \tag{7}$$

where $(x-y)_+^p := \max((x-y)^p, 0)$.

Taylor approximation error

By differentiating the integral form of the Taylor approximation error we find for $0 \leq r \leq p$

$$|D^r(f - \mathcal{T}_{a,p}f)(x)| = \frac{1}{(p-r)!} \int_a^b (x-y)_+^{p-r} D^{p+1} f(y)\, \mathrm{d}y.$$

Using the Hölder inequality and taking L_q norms we obtain

Theorem

Let $f \in W_\infty^{p+1}([a,b])$ with $1 \leq q \leq \infty$, Then, for any $x \in [a,b]$ and $0 \leq r \leq p$,

$$\|D^r(f - \mathcal{T}_{a,p}f)\|_{L_q([a,b])} \leq \frac{(b-a)^{p+1-r}}{(p-r)!} \|D^{p+1} f\|_{L_q([a,b])}. \tag{8}$$

What is a quasi-interpolant?

Quasi-interpolants

In general, a spline approximating a function f can be written in terms of B-splines as

$$\mathcal{Q}f(x) := \sum_{j=1}^{n} \lambda_j f \, B_{j,p,\boldsymbol{\xi}}(x), \quad x \in [a,b] := [a,b], \tag{9}$$

for suitable coefficients $\lambda_j f$. The spline will be referred to as a **quasi-interpolant** to f whenever it provides a **reasonable** approximation to f.

Remarks

- Both interpolation and least squares are examples of quasi-interpolation methods. They are global methods since we have to solve an n by n system of linear equations to find the coefficients $\lambda_j f$. It follows that the value of the spline (9) at a point depends on all the data.
- Here we focus on **local and linear methods**, i.e., methods where each λ_j is a linear functional only depending on the values of f "near" the support of $B_{j,p,\boldsymbol{\xi}}$. This implies that the value of the spline approximation $\mathcal{Q}f$ at a point depends only on the data in a local neighborhood of the point.

Notation and preliminaries

We assume throughout that $\boldsymbol{\xi}$ is a $(p+1)$-open knot sequence.

- $I_j := [\xi_j, \xi_{j+p+1}]$, the support of $B_{j,p,\boldsymbol{\xi}}$
- If $x \in [\xi_m, \xi_{m+1})$ then by the local support property

$$\mathcal{Q}f(x) = \sum_{j=1}^{n} \lambda_j f \, B_{j,p,\boldsymbol{\xi}}(x) = \sum_{j=m-p}^{m} \lambda_j f \, B_{j,p,\boldsymbol{\xi}}(x), \tag{10}$$

  therefore, by nonnegativity and partition of unity

$$|\mathcal{Q}f(x)| \leq \max_{m-p \leq j \leq m} |\lambda_j f|. \tag{11}$$

- $$J_m := \bigcup_{j=m-p}^{m} I_j = [\xi_{m-p}, \xi_{m+p+1}], \quad p+1 \leq m \leq n$$

Length of knot intervals

- $h_{j,p,\boldsymbol{\xi}} := \max_{j \leq k \leq j+p} \xi_{k+1} - \xi_k$ is the largest length of a knot interval in $I_j = [\xi_j, \xi_{j+p+1}]$
- $h_{m,\boldsymbol{\xi}}$ is the largest length of a knot interval in $J_m = [\xi_{m-p}, \xi_{m+p+1}]$
- $h_{\boldsymbol{\xi}} := \max_{p+1 \leq j \leq n} \xi_{j+1} - \xi_j$ the largest length of a knot interval in $[a, b]$

A class of bounded local linear quasi-interpolants on $[a, b] := [\xi_{p+1}, \xi_{n+1}]$

1. **locality**: each $\lambda_j : C^{-1}([a, b]) \to \mathbb{R}$ is a linear functional with support on I_j, i.e., $\lambda_j f = 0$ for any $f \in C^{-1}([a, b])$ which vanishes on I_j.
2. **Reproduces $\mathbb{P}_\ell$ for some ℓ with** $0 \leq \ell \leq p$:

   $$\mathcal{Q}g(x) = g(x) \text{ for all } x \in [a, b] \text{ and all } g \text{ in } \mathbb{P}_\ell,$$

3. **Bounded linear functionals**: There is a constant $C_\mathcal{Q}$ such that for $j = 1, \ldots, n$ and all $f \in C^{-1}(I_j)$ we have

   $$|\lambda_j f| \leq C_\mathcal{Q} h_{j,p,\boldsymbol{\xi}}^{-1/q} \|f\|_{L_q(I_j)} \tag{12}$$

   for some q, $1 \leq q \leq \infty$.

Local and global approximation

Theorem

Let

- $\mathcal{Q}$ *be a bounded local linear quasi-interpolant,*
- $f \in W_q^{\ell+1}(J_m)$ *some* m *with* $\xi_m < \xi_{m+1}$.

Then,

$$\|f - \mathcal{Q}f\|_{L_q([\xi_m,\xi_{m+1}])} \leq \frac{(2p+1)^{\ell+1}}{\ell!}(1 + C_{\mathcal{Q}})h_{m,\boldsymbol{\xi}}^{\ell+1}\|D^{\ell+1}f\|_{L_q(J_m)}. \tag{13}$$

Moreover, if $f \in W_q^{\ell+1}([a,b])$ *then*

$$\|f - \mathcal{Q}f\|_{L_q([a,b])} \leq \frac{(2p+1)^{\ell+1+1/q}}{\ell!}(1 + C_{\mathcal{Q}})h_{\boldsymbol{\xi}}^{\ell+1}\|D^{\ell+1}f\|_{L_q([a,b])}, \tag{14}$$

Proof Local approximation

Suppose $x \in [\xi_m, \xi_{m+1})$. Then by (11) and (12)

$$|\mathcal{Q}f(x)| \leq C_{\mathcal{Q}} \max_{m-p \leq j \leq m} h_{j,p,\boldsymbol{\xi}}^{-1/q} \|f\|_{L_q(I_j)} \leq C_{\mathcal{Q}} (\xi_{m+1} - \xi_m)^{-1/q} \|f\|_{L_q(J_m)}.$$

Taking L_q-norm we find

$$\|\mathcal{Q}f\|_{L_q([\xi_m, \xi_{m+1}])} \leq C_{\mathcal{Q}} \|f\|_{L_q(J_m)}. \tag{15}$$

Since $\mathcal{Q}$ reproduces any polynomial $g \in \mathbb{P}_\ell$ and $\mathcal{Q}$ is linear we have

$$f - \mathcal{Q}f = f - g + \mathcal{Q}g - \mathcal{Q}f = (f - g) + \mathcal{Q}(f - g).$$

Taking norms, using the triangle inequality and (15)

$$\begin{aligned} \|f - \mathcal{Q}f\|_{L_q([\xi_m, \xi_{m+1}])} &\leq \|f - g\|_{L_q([\xi_m, \xi_{m+1}])} + \|\mathcal{Q}(f - g)\|_{L_q([\xi_m, \xi_{m+1}])} \\ &\leq (1 + C_{\mathcal{Q}}) \|f - g\|_{L_q(J_m)}. \end{aligned}$$

Let us now choose $g := \mathcal{T}_{\xi_{m-p},\ell} f$, where $\mathcal{T}_{\xi_{m-p},\ell} f$ is the Taylor polynomial of degree ℓ using $a = \xi_{m-p}$. By the error term for Taylor approximation

$$\|f - \mathcal{Q}f\|_{L_q([\xi_m, \xi_{m+1}])} \leq (1 + C_{\mathcal{Q}}) \frac{(\xi_{m+p+1} - \xi_{m-p})^{l+1}}{l!} \|D^{\ell+1} f\|_{L_q(J_m)}.$$

Since $\xi_{m+p+1} - \xi_{m-p} \leq (2p + 1) h_{m,\boldsymbol{\xi}}$ we obtain the local bound.
Summing the qth power of local estimates we obtain the global one. $\square$

Example; the Schoenberg operator

$$\mathcal{V}_{p,\boldsymbol{\xi}} f(x) := \sum_{j=1}^{n} f\Big(\frac{\xi_{j+1} + \cdots + \xi_{j+p}}{p}\Big) B_{j,p,\boldsymbol{\xi}}(x), \tag{16}$$

This is a local linear quasi-interpolant with $\ell = 1$. It is bounded in the L_∞-norm with $C_{\mathcal{Q}} = 1$. The local and global approximation Theorem implies for any $f \in W_\infty^2([a,b])$,

$$\|f - \mathcal{V}_{p,\boldsymbol{\xi}} f\|_{L_\infty([a,b])} \le 2(2p+1)^2 h_{\boldsymbol{\xi}}^2 \|D^2 f\|_{L_\infty([a,b])}. \tag{17}$$

The Schoenberg operator preserves positivity, monotonicity and convexity.

Spline reproduction

The next proposition gives a sufficient condition for a quasi-interpolant to reproduce the whole spline space.

Proposition

Suppose

- $\mathcal{Q}f := \sum_{j=1}^{n} \lambda_j f \, B_{j,p,\boldsymbol{\xi}}$ *is a linear quasi-interpolant*
- $\mathcal{Q}$ *reproduces* $\mathbb{P}_{p,\boldsymbol{\xi}}$
- *each linear functional* λ_j *is supported on one knot interval*

Then

$$\mathcal{Q}s(x) = s(x), \quad s \in \mathbb{S}_{p,\boldsymbol{\xi}}, \quad x \in [a, b), \tag{18}$$

In other words $\mathcal{Q}$ *is a projector onto the spline space* $\mathbb{S}_{p,\boldsymbol{\xi}}$.

Proof Spline reproduction

Since $\mathcal{Q}B_{i,p,\boldsymbol{\xi}} = \sum_{j=1}^{n} (\lambda_j B_{i,p,\boldsymbol{\xi}})\, B_{j,p,\boldsymbol{\xi}}$ it suffices to show that $\lambda_j(B_{i,p,\boldsymbol{\xi}}) = \delta_{i,j}$, $i, j = 1, \ldots, n$.
Let j with $1 \le j \le n$ be fixed and consider an interval $[\xi_m, \xi_{m+1})$

- If $i \notin \{m-p, \ldots, m\}$ the local support property implies $\lambda_j(B_{i,p,\boldsymbol{\xi}}) = 0$.
- Suppose $i \in \{m-p, \ldots, m\}$.
  Since $B_{i,p,\boldsymbol{\xi}} \in \mathbb{P}_p$ on one knot interval, we have

$$B_{i,p,\boldsymbol{\xi}}(x) = \mathcal{Q}(B_{i,p,\boldsymbol{\xi}})(x) = \sum_{k=m-p}^{m} \lambda_k(B_{i,p,\boldsymbol{\xi}}) B_{k,p,\boldsymbol{\xi}}(x), \quad x \in [\xi_m, \xi_{m+1}).$$

This implies $\lambda_k(B_{i,p,\boldsymbol{\xi}}) = \delta_{k,i} \quad k = m-p, \ldots, m$.
In particular, it holds for $k = i$.

$\square$

Example; A bounded quadratic spline projector

Let $p = 2$ and let $\boldsymbol{\xi}$ be a 3-open knot sequence. We consider the operator $\mathcal{Q}_{2,\boldsymbol{\xi}} f(x) := \sum_{j=1}^{n} \lambda_j f\, B_{j,2,\boldsymbol{\xi}}(x)$, where

$$\lambda_j f := -\frac{1}{2} f(\xi_{j+1}) + 2f\left(\frac{\xi_{j+2} + \xi_{j+1})}{2}\right) - \frac{1}{2} f(\xi_{j+2}).$$

Clearly $|\lambda_j f| \leq 3\|f\|_{L_\infty([\xi_{j+1}, \xi_{j+2}])}$ for any $f \in C^{-1}([a, b])$, and since

$$\mathcal{Q}_{2,\boldsymbol{\xi}} 1 = \sum_{j=1}^{n} B_{j,2,\boldsymbol{\xi}}(x) = 1,$$

$$\mathcal{Q}_{2,\boldsymbol{\xi}} x = \sum_{j=1}^{n} \frac{\xi_{j+1} + \xi_{j+2}}{2} B_{j,2,\boldsymbol{\xi}}(x) = x,$$

$$\mathcal{Q}_{2,\boldsymbol{\xi}} x^2 = \sum_{j=1}^{n} \xi_{j+1}\xi_{j+2} B_{j,2,\boldsymbol{\xi}}(x) = x^2,$$

it follows that $\mathcal{Q}_{2,\boldsymbol{\xi}}$ reproduces $\mathbb{P}_2$. It is a projector onto the spline space $\mathbb{S}_{2,\boldsymbol{\xi}}$ since λ_j has support on one knot interval. For any $f \in W^3_\infty([a, b])$, the L_∞ error is $O(h^3_{\boldsymbol{\xi}})$.

Quadratic splines are well conditioned

$$s = \sum_{j=1}^{n} c_j B_{j,2,\boldsymbol{\xi}} \in \mathbb{S}_{p,\boldsymbol{\xi}} \implies c_j = \mathcal{Q}_{2,\boldsymbol{\xi}} s.$$

This shows that each c_j can be at most 3 times as large as $\|s\|_{L_\infty([a,b])}$ independently of $\boldsymbol{\xi}$.

Degree of reproduction

The following proposition can be used to find the degree ℓ of polynomials reproduced by a linear quasi-interpolant.

Proposition

Let

$$\{\varphi_{j,0}, \ldots, \varphi_{j,\ell}\}, \quad j = 1, \ldots, n, \quad 0 \leq \ell \leq p \tag{19}$$

be n sets of basis functions for polynomials in $\mathbb{P}_\ell$ and let

$$\varphi_{j,r} = \sum_{m=1}^{n} c_{j,r,m} B_{m,p,\boldsymbol{\xi}} \tag{20}$$

be their B-spline representations. The linear quasi-interpolant (9) *reproduces $\mathbb{P}_\ell$ provided the corresponding linear functionals satisfy*

$$\lambda_j(\varphi_{j,r}) = c_{j,r,j}, \quad j = 1, \ldots, n, \quad r = 0, \ldots, \ell. \tag{21}$$

Proof Degree of reproduction

Any $g \in \mathbb{P}_\ell$ can be written both in terms of the B-splines and the φ's, say

$$g = \sum_{m=1}^{n} b_m B_{m,p,\boldsymbol{\xi}} = \sum_{r=0}^{\ell} b_{j,r} \varphi_{j,r}, \quad j = 1, \ldots, n. \tag{22}$$

By (20) and (22) for $j = 1, \ldots, n$,

$$g = \sum_{r=0}^{\ell} b_{j,r} \left(\sum_{m=1}^{n} c_{j,r,m} B_{m,p,\boldsymbol{\xi}} \right) = \sum_{m=1}^{n} \left(\sum_{r=0}^{\ell} b_{j,r} c_{j,r,m} \right) B_{m,p,\boldsymbol{\xi}} = \sum_{m=1}^{n} b_m B_{m,p,\boldsymbol{\xi}}.$$

By linear independence of the B-splines and choosing $i = m$ we obtain

$$b_m = \sum_{r=0}^{\ell} b_{m,r} c_{m,r,m}. \tag{23}$$

Similarly, for $\mathcal{Q}g$ using (22) with $j = m$,

$$\mathcal{Q}g := \sum_{m=1}^{n} \lambda_m(g) B_{m,p,\boldsymbol{\xi}} = \sum_{m=1}^{n} \lambda_m \left(\sum_{r=0}^{\ell} b_{m,r} \varphi_{m,r} \right) B_{m,p,\boldsymbol{\xi}}.$$

From the linearity of λ_m and (21), (23) and finally (22) again we obtain

$$\mathcal{Q}g = \sum_{m=1}^{n} \sum_{r=0}^{\ell} b_{m,r} \lambda_m(\varphi_{m,r}) B_{m,p,\boldsymbol{\xi}} = \sum_{m=1}^{n} \sum_{r=0}^{\ell} b_{m,r} c_{m,r,m} B_{m,p,\boldsymbol{\xi}} = \sum_{m=1}^{n} b_m B_{m,p,\boldsymbol{\xi}} = g.$$

□