Lecture 2: Splines

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Splines and PDE's: Recent advances from Approximation Theory to structured Numerical Linear Algebra

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Contents

The word Spline

Figure: A physical spline with ducks.

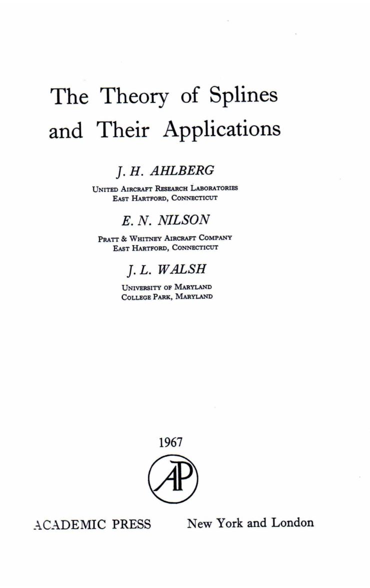
- Physical spline: a flexible beam with minimal potential energy.
- Kept in place using "ducks".
- Nonlinear spline: minimal integrated square curvature
- Holladay 1957. Rediscover minimal norm property of cubic splines. (known to Euler).

What is a spline?

- **Variational approach..** It is not a spline unless it minimizes something (for a long time the French view)
- **Constructive approach1.** A spline is a linear combination of B-splines (Carl de Boor).
- **Constructive approach2.** Any piecewise polynomial is a spline (Larry Schumaker).
- **Constructive approach3.** Any piecewise analytic function is a spline (Even Mehlum).
- **Constructive approach4.** Any piecewise sufficiently smooth function is a spline.

I will only talk about piecewise polynomials.

The variational approach


The Theory of Splines and Their Applications

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E. N. NILSON
PRATT & WHITNEY AIRCRAFT COMPANY
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J. L. WALSH
UNIVERSITY OF MARYLAND
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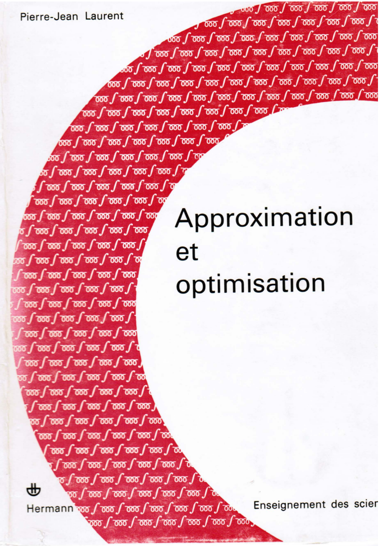
1967

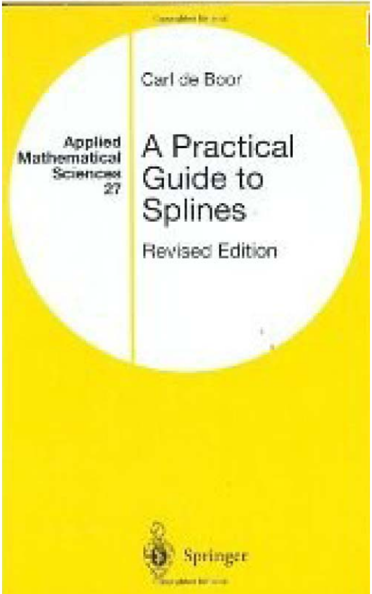
ACADEMIC PRESS New York and London

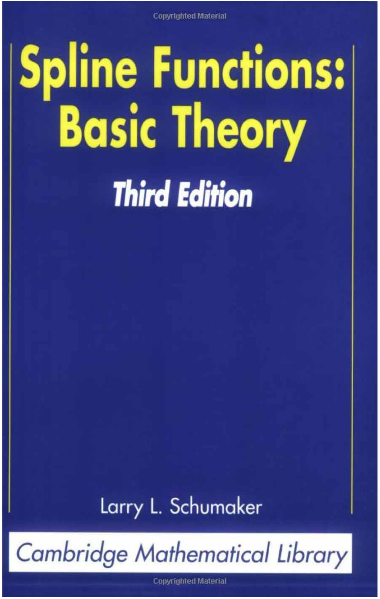


Ahlberg Nilson, Walsh, 1967

P-J Laurent, 1972

The constructive approach





Linear combinations of B-splines

Knot sequence

Suppose for integers $n > p \geq 0$ that a knot sequence

$$\boldsymbol{\xi} := \{\xi_i\}_{i=1}^{n+p+1} = \{\xi_1 \leq \xi_2 \leq \cdots \leq \xi_{n+p+1}\}, \quad n \in \mathbb{N}, \quad p \in \mathbb{N}_0,$$

is given. This knot sequence allows us to define a set of n B-splines of degree p, namely

$$\{B_{1,p,\boldsymbol{\xi}}, \ldots, B_{n,p,\boldsymbol{\xi}}\}. \tag{1}$$

The spline space

We consider the space

$$\mathbb{S}_{p,\boldsymbol{\xi}} := \left\{ s : [\xi_{p+1}, \xi_{n+1}] \to \mathbb{R} : s = \sum_{j=1}^{n} c_j B_{j,p,\boldsymbol{\xi}},\ c_j \in \mathbb{R} \right\}. \tag{2}$$

This is the space of **splines** over the **basic interval**

$$[a, b] := [\xi_{p+1}, \xi_{n+1}].$$

Open knot sequence

A knot sequence $\boldsymbol{\xi} = (\xi_j)_{j=1}^{n+p+1}$ is called $(p+1)$-**open** if

- $\xi_{j+p+1} > \xi_j$, $j = 1, \ldots, n$
- $\xi_1 = \cdots = \xi_{p+1} < \xi_{p+2} \leq \cdots \leq \xi_n < \xi_{n+1} = \cdots = \xi_{n+p+1}$.

Unless said otherwise $\boldsymbol{\xi}$ is an $(p+1)$-open knot sequences. This is no restriction and we have

$$\int_a^b B_{j,p,\boldsymbol{\xi}}(x)dx = \frac{\xi_{j+p+1} - \xi_j}{p+1}, \quad j = 1, \ldots, n.$$

The right endpoint

- We consider B-splines on a closed interval

$$[a, b] := [\xi_{p+1}, \xi_{n+1}].$$

In order to avoid the asymmetry at the right endpoint we define the B-splines to be **left continuous** at the right endpoint, i.e., its value at $b = \xi_{n+1}$ is obtained by taking limits from the left:

$$B_{j,p,\boldsymbol{\xi}}(b) := \lim_{\substack{x \to b \\ x < b}} B_{j,p,\boldsymbol{\xi}}(x), \quad j = 1, \ldots, n. \tag{3}$$

Properties

From the properties of B-splines, we immediately conclude the following properties of the spline $s = \sum_{j=1}^{n} c_j B_{j,p,\boldsymbol{\xi}}$.

- **Smoothness.**

  "smoothness + multiplicity= degree"

- **Local support.**

$$\sum_{j=1}^{n} c_j B_{j,p,\boldsymbol{\xi}}(x) = \sum_{j=m-p}^{m} c_j B_{j,p,\boldsymbol{\xi}}(x), \quad x \in [\xi_m, \xi_{m+1}), \quad p+1 \le m \le n. \tag{4}$$

- **Linear independence.** The spline space $\mathbb{S}_{p,\boldsymbol{\xi}}$ is a vector space of dimension n.

- **differentiation.**

$$s \in \mathbb{S}_{p,\boldsymbol{\xi}} \implies D_+ s \in D_+ \mathbb{S}_{p,\boldsymbol{\xi}} := \mathbb{S}_{p-1,\boldsymbol{\xi}_1}, \quad \boldsymbol{\xi}_1 = (\xi_j)_{j=2}^{n+p}$$

Two evaluation algorithms

(can easily be extended to compute derivatives)

Computing nonzero cubic B-splines, $B_{j,3,\boldsymbol{\tau}}$

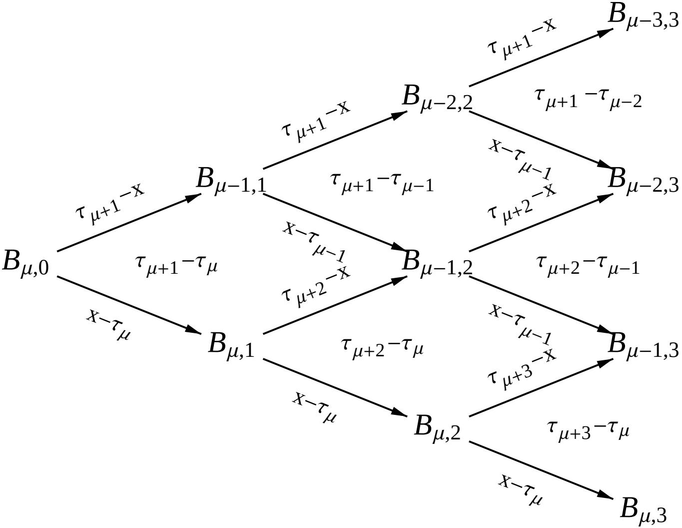


Figure: A triangular algorithm for computation of all the nonzero cubic B-splines at $x \in [\tau_\mu, \tau_{\mu+1})$.

de Boor algorithm. Compute $s(x) = \sum_{j=1}^{n} c_{j,0} B_{j,3,\boldsymbol{\tau}}(x)$

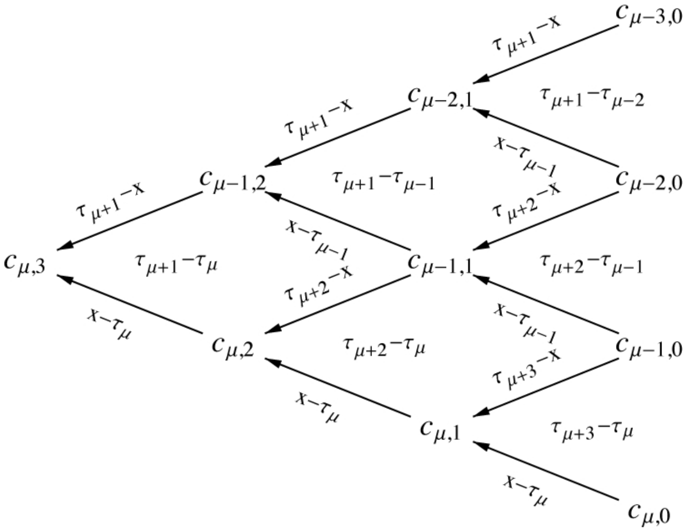


Figure: A triangular algorithm for computing the value of a cubic spline with B-spline coefficients $\boldsymbol{c}$ at $x \in [\tau_\mu, \tau_{\mu+1})$.

Piecewise polynomials

Let Δ be a sequence of distinct real numbers,

$$\Delta := \{\eta_0 < \eta_1 < \cdots < \eta_{\ell+1}\}.$$

The elements in Δ are called **break points**. Moreover, let $\boldsymbol{r} := (r_1, \ldots, r_\ell)$ be a vector of integers such that $-1 \leq r_i \leq p$ for $i = 1, \ldots, \ell$. The space $\mathbb{S}_p^{\boldsymbol{r}}(\Delta)$ of piecewise polynomials of degree p with smoothness $\boldsymbol{r}$ over the partition Δ is defined by

$$\begin{aligned} \mathbb{S}_p^{\boldsymbol{r}}(\Delta) := \big\{ s : [\eta_0, \eta_{\ell+1}] \to \mathbb{R} : \; & s \in \mathbb{P}_p([\eta_i, \eta_{i+1})), \; i = 0, \ldots, \ell, \\ & s \in C^{r_i}(\eta_i), \; i = 1, \ldots, \ell \big\}. \end{aligned} \tag{5}$$

This space is denoted by $\mathbb{S}_p^r(\Delta)$ when $r = r_1 = \cdots = r_\ell$.

Piecewise polynomials and B-splines

The next theorem shows that the set of B-splines in (**??**) defined over a specific knot sequence $\boldsymbol{\xi}$ forms a basis for $\mathbb{S}_p^{\boldsymbol{r}}(\Delta)$.

Theorem (Curry–Schoenberg)

The piecewise polynomial space $\mathbb{S}_p^{\boldsymbol{r}}(\Delta)$ is characterized in terms of B-splines by

$$\mathbb{S}_p^{\boldsymbol{r}}(\Delta) = \mathbb{S}_{p,\boldsymbol{\xi}},$$

where the knot sequence $\boldsymbol{\xi} := \{\xi_i\}_{i=1}^{n+p+1}$ with $n := \dim(\mathbb{S}_{p,\boldsymbol{\xi}})$ is constructed such that

$$\xi_1 \leq \cdots \leq \xi_{p+1} := \eta_0, \quad \eta_{\ell+1} =: \xi_{n+1} \leq \cdots \leq \xi_{n+p+1},$$

and

$$\xi_{p+2}, \ldots, \xi_n := \overbrace{\eta_1, \ldots, \eta_1}^{p-r_1}, \ldots, \overbrace{\eta_\ell, \ldots, \eta_\ell}^{p-r_\ell}.$$

Marsden's identity

The local Marsden identity implies

Theorem

We have

$$(y-x)^p = \sum_{j=1}^{n} \psi_{j,p,\boldsymbol{\xi}}(y) B_{j,p,\boldsymbol{\xi}}(x), \quad x \in [a,b], \quad y \in \mathbb{R}, \tag{6}$$

where $\psi_{j,p,\boldsymbol{\xi}}(y) := (y-\xi_{j+1}) \cdots (y-\xi_{j+p})$ *is the polynomial of degree* p *that is dual to* $B_{j,p,\boldsymbol{\xi}}$.

Shifted monomials as B-splines

Differentiating $p-k$ times with respect to y in

$$(y-x)^p = \sum_{j=1}^{n} \psi_{j,p,\boldsymbol{\xi}}(y) B_{j,p,\boldsymbol{\xi}}(x)$$

results in

Corollary

For $k = 0, 1, \ldots, p$ we have

$$\frac{(y-x)^k}{k!} = \sum_{j=1}^{n} \left(\frac{1}{p!} D^{p-k} \psi_{j,p,\boldsymbol{\xi}}(y) \right) B_{j,p,\boldsymbol{\xi}}(x), \quad x \in [\xi_{p+1}, \xi_{n+1}], \quad y \in \mathbb{R}. \tag{7}$$

Greville points

$$(y-\xi_{j+1})\cdots(y-\xi_{j+p}) = y^p - (\xi_{j+1}+\cdots+\xi_{j+p})y^{p-1}+\cdots$$

Taking $k=1$ in

$$\frac{(y-x)^k}{k!} = \sum_{j=1}^{n}\left(\frac{1}{p!}D^{p-k}\psi_{j,p,\boldsymbol{\xi}}(y)\right)B_{j,p,\boldsymbol{\xi}}(x),$$

setting $y=0$ and simplifying

$$x = \sum_{j=1}^{n}\xi^*_{j,p,\boldsymbol{\xi}}B_{j,p,\boldsymbol{\xi}}(x), \quad x\in[\xi_{p+1},\xi_{n+1}], \tag{8}$$

where

$$\xi^*_{j,p,\boldsymbol{\xi}} := \xi^{*,1}_{j,p,\boldsymbol{\xi}} = \frac{\xi_{j+1}+\cdots+\xi_{j+p}}{p}. \tag{9}$$

The number $\xi^*_{j,p,\boldsymbol{\xi}}$ is called a **Greville point**. It is also known as a **knot average** or a **node**.

Cubic powers

For $p = 3$ we obtain

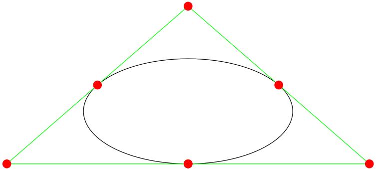
$$
\begin{aligned}
1 &= \sum_{j=1}^{n} B_{j,3,\boldsymbol{\xi}}, \\
x &= \sum_{j=1}^{n} \frac{\xi_{j+1} + \xi_{j+2} + \xi_{j+3}}{3} \, B_{j,3,\boldsymbol{\xi}}, \\
x^2 &= \sum_{j=1}^{n} \frac{\xi_{j+1}\xi_{j+2} + \xi_{j+1}\xi_{j+3} + \xi_{j+2}\xi_{j+3}}{3} \, B_{j,3,\boldsymbol{\xi}}, \\
x^3 &= \sum_{j=1}^{n} \xi_{j+1}\xi_{j+2}\xi_{j+3} \, B_{j,3,\boldsymbol{\xi}}.
\end{aligned}
$$

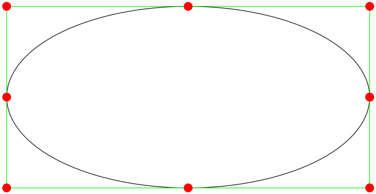
NURBS

NURBS

- **NURBS basis function.**
  $R_{i,p,\boldsymbol{\xi},\boldsymbol{w}}(x) := \frac{w_i B_{i,p,\boldsymbol{\xi}}(x)}{\sum_{j=1}^n w_j B_{j,p,\boldsymbol{\xi}}(x)},\ w_j > 0.$
- **NURBS basis function properties.**
  Positivity, Partition of unity, Compact support, Smoothness related to knot multiplicity
- **NURBS curve.** $\sum_{i=1}^n \boldsymbol{c}_i R_{i,p,\boldsymbol{\xi}}(t), \quad \boldsymbol{c}_i \in \mathbb{R}^d,\ d \geq 2.$
- **projective transformation of a B-spline curve in $\mathbb{R}^{d+1}$.**
  $\left(\sum_{i=1}^n \boldsymbol{c}_i w_i B_{i,p,\boldsymbol{\xi}}(t),\ \sum_{i=1}^n w_i B_{i,p,\boldsymbol{\xi}}(t) \right)$

NURBS ellipses, $(p = 2)$





3 120 degree segments,
$\boldsymbol{\xi} := (0, 0, 0, 1, 1, 2, 2, 3, 3, 3)$
$\boldsymbol{w} := (1, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, 1)$

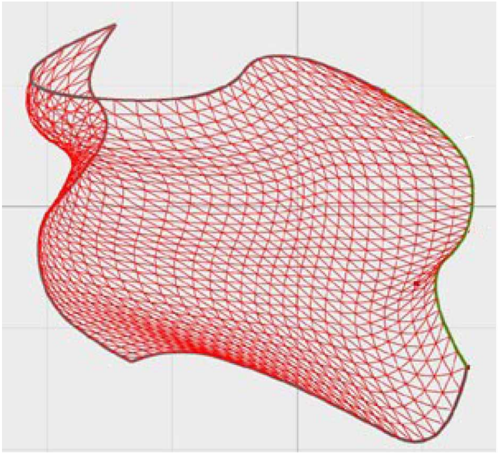
4 quarter ellipses
$\boldsymbol{\xi} := (0, 0, 0, 1, 1, 2, 2, 3, 3, 4, 4, 4)$
$\boldsymbol{w} := \{1, \frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}, 1\}$

Surfaces

- Coons patches
- Tensor-product splines
- T-splines, hierarchical splines, LR splines, $\cdots$
- Multivariate B-splines
  - Simplex splines
  - Box splines
- Splines on triangulations and simplices

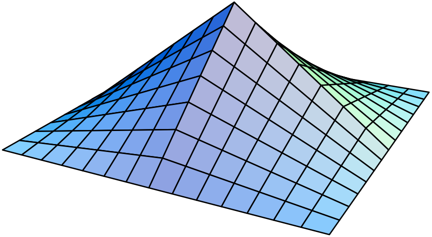
Coons Patch

Hermite interpolation to boundary curves.



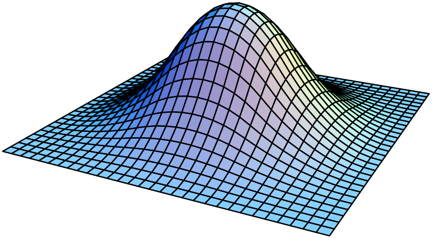
Tensor-product splines

- **degrees** $\boldsymbol{p} := (p_1, \dots p_d)$.
- **knot sequences** $\Xi := (\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_d)$.
- **multi index** $\boldsymbol{j} := (j_1, \dots, j_d)$.
- **multi variable** $\boldsymbol{x} := (x_1, \dots, x_d)$.
- **Tensor product B-spline**
  $B_{\boldsymbol{j},\boldsymbol{p},\Xi}(\boldsymbol{x}) := B_{j_1,p_1,\boldsymbol{\xi}_1}(x_1) \cdots B_{j_d,p_d,\boldsymbol{\xi}_d}(x_d), \quad \boldsymbol{x} \in \mathbb{R}^d$.

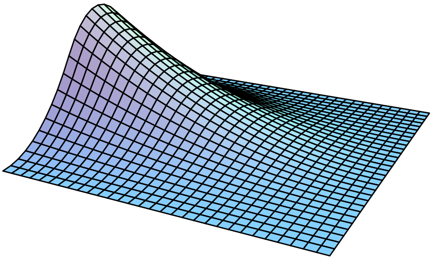


$B[0,1,2;0,1,2]$, $(d = 2,\ p_1 = p_2 = 1)$

Quadratic tensor-product B-splines

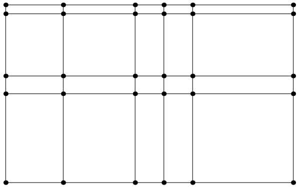


$B[0,1,2,3;0,1,2,3]$,

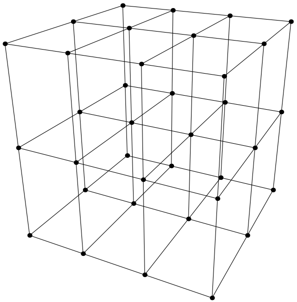


$B[0,0,0,3;0,1,2,3]$,

Tensor-product grids



$d = 2,$



$d = 3$

Tensor-product Splines

The tensor product of d spline spaces $\mathbb{S}_1, \ldots, \mathbb{S}_d$ of dimensions $\boldsymbol{n} := (n_1, \ldots, n_d)$ is defined to be the family of all functions of the form

$$\begin{aligned} f(\boldsymbol{x}) &:= \sum_{1 \le \boldsymbol{j} \le \boldsymbol{n}} c_{\boldsymbol{j}} B_{\boldsymbol{j},\boldsymbol{p},\Xi}(\boldsymbol{x}) \\ &:= \sum_{j_1=1}^{n_1} \cdots \sum_{j_d=1}^{n_d} c_{j_1,\ldots,j_d} B_{j_1,p_1,\boldsymbol{\xi}_1}(x_1) \cdots B_{j_d,p_d,\boldsymbol{\xi}_d}(x_d), \end{aligned}$$

where the coefficients $(c_{\boldsymbol{j}})$ can be any real numbers. This linear space of functions is denoted $\mathbb{S}_1 \otimes \cdots \otimes \mathbb{S}_d$.
This space has dimension $n_1 \cdots n_d$.

Things not treated

- **Uniqueness of spline interpolation.**
- **Knot insertion.** (h-refinement)
- **Degree raising.** (p-refinement)
- **Combined degree raising and knot insertion.** (k-refinement)
- **Geometric properties.**
- $\cdots$**.**