Lecture 4: Spline space distance

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Splines and PDE's: Recent advances from Approximation Theory to structured Numerical Linear Algebra

CIME-EMS Summer School in Applied Mathematics
3 July - 7 July, 2017

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Distance to a spline space

For $f \in L_q([a,b])$ we define

$$\mathrm{dist}_q(f, \mathbb{S}_{p,\boldsymbol{\xi}}) := \inf_{s \in \mathbb{S}_{p,\boldsymbol{\xi}}} \|f - s\|_{L_q([a,b])}, \quad 1 \le q \le \infty. \qquad (1)$$

We are also interested in estimates for the distance between derivatives of f and derivative spline spaces.

- For a given $f \in W_q^r([a,b])$ with $1 \le q \le \infty$ and $0 \le r \le p$, we define

$$\mathrm{dist}_q(D^r f, D^r \mathbb{S}_{p,\boldsymbol{\xi}}) := \inf_{s \in \mathbb{S}_{p,\boldsymbol{\xi}}} \|D^r(f - s)\|_{L_q([a,b])}, \qquad (2)$$

  where $D = D_+$.

Main theorem

We will derive the following upper bound for $\mathrm{dist}_q(D^r f, D^r \mathbb{S}_{p,\boldsymbol{\xi}})$.

Theorem

For any $0 \leq r \leq \ell \leq p$ *and* $f \in W_q^{\ell+1}([a,b])$ *with* $1 \leq q \leq \infty$ *we have*

$$\mathrm{dist}_q(D^r f, D^r \mathbb{S}_{p,\boldsymbol{\xi}}) \leq K (h_{\boldsymbol{\xi}})^{\ell+1-r} \| D^{\ell+1} f \|_{L_q([a,b])},$$

where $h_{\boldsymbol{\xi}} := \max_{p+1 \leq j \leq n} (\xi_{j+1} - \xi_j)$ *and* K *is a constant depending only on* p*.*

For $\ell = p$ the upper bound behaves like $(h_{\boldsymbol{\xi}})^{p+1-r}$ for sufficiently smooth f.

Main approximation theorem

The distance result will be shown by explicitly constructing a suitable spline quasi-interpolant s_p which achieves this order of approximation for all r.

Theorem

Let $f \in W_q^{\ell+1}([a,b])$ with $1 \leq q \leq \infty$ and $0 \leq \ell \leq p$. Then, there exists $s_p \in \mathbb{S}_{p,\boldsymbol{\xi}}$ such that

$$\|D^r(f - s_p)\|_{L_q([a,b])} \leq K h_{\boldsymbol{\xi}}^{\ell+1-r} \|D^{\ell+1} f\|_{L_q([a,b])}, \quad 0 \leq r \leq \ell, \tag{3}$$

where $h_{\boldsymbol{\xi}} := \max_{p+1 \leq j \leq n}(\xi_{j+1} - \xi_j)$ and K is a constant depending only on p.

The Quasi-interpolant

Let $[\xi_{m_j}, \xi_{m_j+1}]$ be a knot interval of largest length in $[\xi_j, \xi_{j+p+1}]$ for any $j = 1, \dots, n$ and $h_{j,p,\boldsymbol{\xi}} := \xi_{m_j+1} - \xi_{m_j} > 0$. Define

$$\mathcal{Q}_{p,\boldsymbol{\xi}} f(x) := \sum_{j=1}^{n} \mathcal{L}_{j,p,\boldsymbol{\xi}} f \, B_{j,p,\boldsymbol{\xi}}(x), \tag{4}$$

where

$$\mathcal{L}_{j,p,\boldsymbol{\xi}} f := \frac{1}{h_{j,p,\boldsymbol{\xi}}} \int_{\xi_{m_j}}^{\xi_{m_j+1}} \Big(\sum_{i=0}^{p} \alpha_{j,i} \varphi_{j,i}(x) \Big) f(x) \, \mathrm{d}x, \quad \varphi_{j,i}(x) := \Big(\frac{x - \xi_{m_j}}{h_{j,p,\boldsymbol{\xi}}} \Big)^i \tag{5}$$

and the coefficients $\alpha_{j,i}$, $i = 0, \dots, p$ are such that

$$\mathcal{L}_{j,p,\boldsymbol{\xi}} \varphi_{j,i} = c_{j,i,j}, \quad i = 0, \dots, p, \tag{6}$$

where

$$\varphi_{j,i}(x) = \sum_{k=m_j-p}^{m_j} c_{j,i,k} B_{k,p,\boldsymbol{\xi}}(x), \quad x \in [\xi_{m_j}, \xi_{m_j+1}), \quad i = 0, \dots, p. \tag{7}$$

$\mathcal{Q}_{p,\boldsymbol{\xi}}$ is a projector

Lemma

The above spline approximation is well defined and reproduces polynomials, i.e., for any polynomial $g \in \mathbb{P}_p$ *we have*

$$\mathcal{Q}_{p,\boldsymbol{\xi}} g(x) = g(x), \quad x \in [a,b]. \tag{8}$$

Moreover, it is a projector onto the spline space $\mathbb{S}_{p,\boldsymbol{\xi}}$*, i.e., for any spline* $s \in \mathbb{S}_{p,\boldsymbol{\xi}}$ *we have*

$$\mathcal{Q}_{p,\boldsymbol{\xi}} s(x) = s(x), \quad x \in [a,b], \tag{9}$$

and, in particular,

$$s(x) = \sum_{j=1}^{n} (\mathcal{L}_{j,p,\boldsymbol{\xi}} s) B_{j,p,\boldsymbol{\xi}}(x), \quad x \in [a,b]. \tag{10}$$

Proof $\mathcal{Q}_{p,\boldsymbol{\xi}}$ is a projector

By the degree of reproduction and spline projection propositions it follows that $\mathcal{Q}_{p,\boldsymbol{\xi}}$ is a projection onto the spline space provided it is well defined, i.e., we can find $\alpha_{j,i}$ such that (6) holds . For this we find

$$\mathcal{L}_{j,p,\boldsymbol{\xi}}\varphi_{j,r} = \frac{1}{h_{j,p,\boldsymbol{\xi}}}\int_{\xi_{m_j}}^{\xi_{m_j+1}} \Big(\sum_{i=0}^{p} \alpha_{j,i}\varphi_{j,i}(x)\Big)\varphi_{j,r}(x)\,\mathrm{d}x = \sum_{i=0}^{p} \alpha_{j,i} H_{i+1,r+1},$$

where

$$H_{i+1,r+1} = \frac{1}{h_{j,p,\boldsymbol{\xi}}}\int_{\xi_{m_j}}^{\xi_{m_j+1}} \varphi_{j,i+r}(x)\,\mathrm{d}x = \frac{1}{i+r+1}, \quad i,r = 0,\ldots,p.$$

It follows that the coefficients $\alpha_{j,i}$ are given by the solution of the linear system

$$H\boldsymbol{\alpha}_j = \boldsymbol{c}_j, \tag{11}$$

where $\boldsymbol{\alpha}_j := (\alpha_{j,0},\ldots,\alpha_{j,p})^T$, $\boldsymbol{c}_j := (c_{j,0,j},\ldots,c_{j,p,j})^T$, and $H \in \mathbb{R}^{(p+1)\times(p+1)}$ is the well known Hilbert matrix which is nonsingular. $\square$

The linear functionals are bounded

Lemma

For $p \geq 0$ and $1 \leq q \leq \infty$ we have for any $f \in L_q([\xi_{m_j}, \xi_{m_j+1}])$,

$$|\mathcal{L}_{j,p,\boldsymbol{\xi}} f| \leq C_{\mathcal{Q}} h_{j,p,\boldsymbol{\xi}}^{-1/q} \|f\|_{L_q([\xi_{m_j}, \xi_{m_j+1}])}, \quad j = 1, \ldots, n, \tag{12}$$

where $C_{\mathcal{Q}}$ is a constant depending only on p.

Proof that the linear functionals are bounded

Since $0 \leq \frac{x-\xi_{m_j}}{h_{j,p,\boldsymbol{\xi}}} \leq 1$ for $x \in [\xi_{m_j}, \xi_{m_j+1}]$, we get

$$
\begin{aligned}
|\mathcal{L}_{j,p,\boldsymbol{\xi}} f| &= \Big|\frac{1}{h_{j,p,\boldsymbol{\xi}}} \int_{\xi_{m_j}}^{\xi_{m_j+1}} \Big(\sum_{i=0}^{p} \alpha_{j,i}\varphi_{j,i}(x)\Big) f(x)\,\mathrm{d}x\Big| \\
&\leq (p+1) h_{j,p,\boldsymbol{\xi}}^{-1} \|\boldsymbol{\alpha}_j\|_\infty \, \|f\|_{L_1([\xi_{m_j},\xi_{m_j+1}])} \\
&\leq (p+1) h_{j,p,\boldsymbol{\xi}}^{-1} \|H^{-1}\|_\infty \|\boldsymbol{c}_j\|_\infty \|f\|_{L_1([\xi_{m_j},\xi_{m_j+1}])}.
\end{aligned}
$$

We have $\boldsymbol{c}_j := (c_{j,0,j}, \ldots, c_{j,p,j})^T$ and

$$
|c_{j,i,j}| = \frac{i!}{p!}\left|\frac{D^{p-i}\psi_{j,p,\boldsymbol{\xi}}(\xi_{m_j})}{h_{j,p,\boldsymbol{\xi}}^i}\right| \leq \left(\frac{\xi_{j+p+1}-\xi_j}{h_{j,p,\boldsymbol{\xi}}}\right)^i \leq (p+1)^i, \quad i = 0, \ldots, p.
$$

This gives

$$
|\mathcal{L}_{j,p,\boldsymbol{\xi}} f| \leq C_{\mathcal{Q}} h_{j,p,\boldsymbol{\xi}}^{-1} \|f\|_{L_1([\xi_{m_j},\xi_{m_j+1}])}, \quad C_{\mathcal{Q}} := \|H^{-1}\|_\infty (p+1)^{p+1},
$$

where $C_{\mathcal{Q}}$ only depends on p. By the Hölder inequality for integrals we arrive at (12).

$\square$

Bounding derivatives of B-splines

From partition of unity and nonnegativity:

$$B_{j,p,\boldsymbol{\xi}}(x) \leq 1, \quad x \in \mathbb{R}.$$

If $x \in [\xi_m, \xi_{m+1})$ with $j \leq m \leq j+p$ then it follows from the differentiation formula

$$D_+ B_{j,p,\boldsymbol{\xi}}(x) = p\left(\frac{B_{j,p-1,\boldsymbol{\xi}}(x)}{\xi_{j+p}-\xi_j} - \frac{B_{j+1,p-1,\boldsymbol{\xi}}(x)}{\xi_{j+p+1}-\xi_{j+1}}\right)$$

that

$$|D_+ B_{j,p,\boldsymbol{\xi}}(x)| \leq \frac{2p}{\xi_{m+1}-\xi_m}.$$

More generally using the differentiation formula for B-splines we can prove

Proposition

The r-th derivative of the B-spline $B_{j,p,\boldsymbol{\xi}}$ for $0 \le r \le p$ can be bounded as follows. For any $x \in [\xi_m, \xi_{m+1})$ with $j \le m \le j+p$ we have

$$|D^r B_{j,p,\boldsymbol{\xi}}(x)| \le 2^r \frac{p!}{(p-r)!} \prod_{k=p-r+1}^{p} \frac{1}{\Delta_{m,k}}, \tag{13}$$

where

$$\Delta_{m,k} := \min_{m-k+1 \le i \le m} \xi_{i+k} - \xi_i, \quad k = 1, \dots, p. \tag{14}$$

For $r = 0, 1$ the right-hand side becomes the upper bound we had on the previous slide

- $r = 0$*: right-hand side* $= 1$
- $r = 1$*: right-hand side* $\frac{2p}{\xi_{m+1} - \xi_m}$.

$\mathcal{Q}_{p,\boldsymbol{\xi}} f$; bound for derivatives

Lemma

For $0 \le r \le p$ and $1 \le q \le \infty$ we have for any $f \in L_q([\xi_{m-p}, \xi_{m+p+1}])$ with $p + 1 \le m \le n$,

$$\|D^r(\mathcal{Q}_{p,\boldsymbol{\xi}} f)\|_{L_q([\xi_m, \xi_{m+1}])} \le C \left(\prod_{k=p-r+1}^{p} \frac{1}{\Delta_{m,k}} \right) \|f\|_{L_q(J_m)},$$

where $J_m := [\xi_{m-p}, \xi_{m+p+1}]$ and

$$\Delta_{m,k} := \min_{m-k+1 \le i \le m} \xi_{i+k} - \xi_i, \quad k = 1, \dots, p,$$

and C is a constant depending only on p.

Proof $\mathcal{Q}_{p,\boldsymbol{\xi}} f$; bound for derivatives

For $x \in [\xi_m, \xi_{m+1})$,

$$
\begin{aligned}
|D^r(\mathcal{Q}_{p,\boldsymbol{\xi}} f)(x)| &= \left| \sum_{j=m-p}^{m} \mathcal{L}_{j,p,\boldsymbol{\xi}}(f) D^r B_{j,p,\boldsymbol{\xi}}(x) \right| \\
&\le \max_{m-p \le j \le m} |D^r B_{j,p,\boldsymbol{\xi}}(x)| \sum_{j=m-p}^{m} |\mathcal{L}_{j,p,\boldsymbol{\xi}}(f)| \\
&\le (p+1) \max_{m-p \le j \le m} |D^r B_{j,p,\boldsymbol{\xi}}(x)| \max_{m-p \le j \le m} (\xi_{m+1} - \xi_m)^{-1/q} \|f\|_{L_q(J_m)}.
\end{aligned}
$$

This follows since $[\xi_m, \xi_{m+1}] \subset [\xi_j, \xi_{j+p+1}]$ and $h_{j,p,\boldsymbol{\xi}}$ is the length of the largest knot interval in $[\xi_j, \xi_{j+p+1}]$ so we have $\xi_{m+1} - \xi_m \le h_{j,p,\boldsymbol{\xi}}$ for $j = m-p, \ldots, m$.
Replacing $|D^r B_{j,p,\boldsymbol{\xi}}(x)|$ by an upper bound and taking the L_q-norm complete the proof. $\square$

The quasi-interpolant $\mathcal{Q}_{p,\boldsymbol{\xi}} f$ can be used to obtain an upper bound for the distance between a given function f and the spline space $\mathbb{S}_{p,\boldsymbol{\xi}}$ for $p \geq 0$, $n > p+1$ and $\boldsymbol{\xi} := \{\xi_j\}_{j=1}^{n+p+1}$. We start by giving a local and global upper bound for (the derivatives of) the difference between f and $\mathcal{Q}_{p,\boldsymbol{\xi}} f$.

Local and global upper bound

Proposition

Suppose $\xi_m < \xi_{m+1}$ for some $p+1 \le m \le n$, and let $f \in W_q^{\ell+1}([\xi_{m-p}, \xi_{m+p+1}])$ with $0 \le \ell \le p$ and $1 \le q \le \infty$. Then, for any $0 \le r \le \ell$,

$$\|D^r(f - \mathcal{Q}_{p,\boldsymbol{\xi}} f)\|_{L_q([\xi_m,\xi_{m+1}])} \le K_m(\xi_{m+p+1} - \xi_{m-p})^{\ell+1-r} \|D^{\ell+1} f\|_{L_q([\xi_{m-p},\xi_{m+p+1}])}.$$

Here,

$$K_m := 1 + C \prod_{k=p-r+1}^{p} \frac{\xi_{m+p+1} - \xi_{m-p}}{\Delta_{m,k}},$$

$$\Delta_{m,k} := \min_{m-k+1 \le i \le m} \xi_{i+k} - \xi_i, \quad k = 1, \ldots, p,$$

and C is a constant depending only on p.

Proof Local upper bound

Since $\mathcal{Q}_{p,\boldsymbol{\xi}}$ reproduces any polynomial in $\mathbb{P}_\ell$

$$\begin{aligned} &\|D^r(f - \mathcal{Q}_{p,\boldsymbol{\xi}} f)\|_{L_q([\xi_m,\xi_{m+1}])} \\ &\quad \leq \|D^r(f-g)\|_{L_q([\xi_m,\xi_{m+1}])} + \|D^r \mathcal{Q}_{p,\boldsymbol{\xi}}(f-g)\|_{L_q([\xi_m,\xi_{m+1}])}, \end{aligned}$$

for any $g \in \mathbb{P}_\ell$. Let us now set $g := \mathcal{T}_{\xi_m,\ell} f$, where $\mathcal{T}_{\xi_m,\ell} f$ is the Taylor polynomial of degree ℓ. Then

$$\|D^r(f-g)\|_{L_q([\xi_m,\xi_{m+1}])} \leq (\xi_{m+1} - \xi_m)^{\ell+1-r} \|D^{\ell+1} f\|_{L_q([\xi_m,\xi_{m+1}])}.$$

On the other hand, since $f - g \in C(J_m)$, it follows from Lemma 6 that

$$\|D^r \mathcal{Q}_{p,\boldsymbol{\xi}}(f-g)\|_{L_q([\xi_m,\xi_{m+1}])} \leq C \left(\prod_{k=p-r+1}^{p} \frac{1}{\Delta_{m,k}} \right) \|f-g\|_{L_q([\xi_{m-p},\xi_{m+p+1}])},$$

where C is a constant depending only on p. Combining the above three inequalities gives the result. $\square$

Global upper bound

Proposition

Let $f \in W_q^{\ell+1}([a,b])$ with $0 \le \ell \le p$ and $1 \le q \le \infty$. If $\mathcal{Q}_{p,\boldsymbol{\xi}} f$ is defined as in (4) then, for any $0 \le r \le \ell$,

$$\|D^r(f - \mathcal{Q}_{p,\boldsymbol{\xi}} f)\|_{L_q([a,b])} \le K h_{\boldsymbol{\xi}}^{\ell+1-r} \|D^{\ell+1} f\|_{L_q([a,b])}, \qquad (15)$$

where $h_{\boldsymbol{\xi}} := \max_{p+1 \le j \le n}(\xi_{j+1} - \xi_j)$, and

$$K := (2p+1)^{\ell+2-r} \left[1 + C \max_{p+1 \le m \le n} \prod_{k=p-r+1}^{p} \frac{\xi_{m+p+1} - \xi_{m-p}}{\Delta_{m,k}} \right],$$

$$\Delta_{m,k} := \min_{m-k+1 \le i \le m} h_{i,k}, \quad h_{i,k} := \xi_{i+k} - \xi_i, \quad k = 1, \ldots, p,$$

and C is a constant depending only on p.

Proof Global upper bound

For $q = \infty$ the result follows immediately from Proposition 7 by taking into account that $\boldsymbol{\xi}$ can be assumed to be a $(p+1)$-open knot sequence. We now assume $1 \leq q < \infty$. Since

$$\max_{p+1 \leq m \leq n} (\xi_{m+p+1} - \xi_{m-p}) \leq (2p+1) h_{\boldsymbol{\xi}},$$

the result follows from the local error bound in Proposition 7. $\square$

Local mesh ration

We know that the ratio $\frac{\xi_{m+p+1}-\xi_{m-p}}{\Delta_{m,k}}$ is well defined because $\Delta_{m,k} > 0$. For a uniform knot sequence

$$\frac{\xi_{m+p+1} - \xi_{m-p}}{\Delta_{m,k}} = \frac{2p+1}{k}.$$

For a general knot sequence it is related to the **local mesh ratio**, i.e., the ratio between the lengths of the largest and smallest knot intervals in a neighborhood of ξ_m.

Knot thinning

The expression K in the upper bound in Theorem 8 depends on the position of the knots for $r > 0$. However, for any knot sequence $\boldsymbol{\xi}$, it is possible to construct a coarser knot sequence $\boldsymbol{\xi}^\sharp$ such that the corresponding K only depends on p. This can be obtained by a clever thinning process. The idea of thinning out a knot sequence to get a quasi-uniform sequence is credited to Sharma and Meir,1966. Since $\boldsymbol{\xi}^\sharp$ is a subsequence of $\boldsymbol{\xi}$, we have that $\mathbb{S}_{p,\boldsymbol{\xi}^\sharp}$ is a subspace of $\mathbb{S}_{p,\boldsymbol{\xi}}$. In particular, for any $f \in L_\infty([a,b])$ the spline approximation

$$s_p := \mathcal{Q}_{p,\boldsymbol{\xi}^\sharp} f$$

as defined in (4) belongs to the spline space $\mathbb{S}_{p,\boldsymbol{\xi}}$. This spline quasi-interpolant leads to the following important result.

Main approximation theorem

Theorem

Let $f \in W_q^{\ell+1}([a,b])$ *with* $1 \le q \le \infty$ *and* $0 \le \ell \le p$*. Then, there exists* $s_p \in \mathbb{S}_{p,\boldsymbol{\xi}}$ *such that*

$$\|D^r(f - s_p)\|_{L_q([a,b])} \le K h_{\boldsymbol{\xi}}^{\ell+1-r} \|D^{\ell+1} f\|_{L_q([a,b])}, \quad 0 \le r \le \ell, \tag{16}$$

where $h_{\boldsymbol{\xi}} := \max_{p+1 \le j \le n}(\xi_{j+1} - \xi_j)$ *and*

$$K \le (2p+1)^{\ell+2-r}\left[1 + C \cdot 3^r (2p+1)^r\right].$$

K is a constant depending only on r, p, since C is the constant in the global upper bound proposition depending only on p.

The constant depends on the degree

Theorem 9 implies Theorem 1. The constant K in Theorem 9 is independent of the mesh, but grows exponentially with p. However, this dependency on p can be removed in some cases.