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Adaptive Multiscale Methods for the Numerical Treatment of Systems of PDEs

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Sketch of Contents

- Elliptic and parabolic partial differential equations (PDEs) in weak form; regularity of solutions
- Control problems constrained by elliptic and parabolic PDEs
- Numerical approximations of solutions on uniform and non-uniform/adaptive grids
- Concepts of multiscale methods and adaptivity; convergence proofs and complexity estimates
- Realization of these concepts by B-spline-wavelets
- Fast solvers: multilevel preconditioning; implementation issues

Literature: see References in notes_kunoth.pdf

Part IV: Adaptive Wavelet Methods for Control Problems Constrained by Elliptic PDEs

In this part: adaptive (a-posteriori) discretizations

Loop: SOLVE $\longrightarrow$ ESTIMATE $\longrightarrow$ REFINE $\longrightarrow$ SOLVE ... until target acccuracy reached

- Introduction: uniform and adaptive approximations
- Tracking type control problem constrained by elliptic PDE in wavelet coordinates
- Inexact gradient methods: convergence
- Complexity estimates

Literature:

[CDD1] A. Cohen, W. Dahmen, and R. DeVore, Adaptive wavelet methods for elliptic operator equations—Convergence rates, Math. Comp., 70 (2001), pp. 27–75.

[CDD2] A. Cohen, W. Dahmen, and R. DeVore, Adaptive wavelet methods *II* — Beyond the elliptic case, Found. Comput. Math., 2 (2002), pp. 203–245.

[DK] W. Dahmen, A. Kunoth, Adaptive wavelet methods for linear-quadratic elliptic control problems: Convergence rates, SIAM J. Control Optim., 43 (5) (2005), pp. 1640–1675.

[De] R. DeVore, Nonlinear approximation, Acta Numerica, 7 (1998), 51–150.

Introduction: Uniform versus Adaptive Approximations

Recall: A-priori estimates for finite elements (or B-splines/generators in multiresolution analysis)
Quality measure: Approximation in $L_2(\Omega)$ norm $\|y - y_h\|_{L_2(\Omega)} \leq \varepsilon$

A–priori error estimates: $\Omega \subset \mathbb{R}^d$ $\qquad \dim V_h = N \sim h^{-d}$ $\qquad$ uniform grid

$$\|y - y_h\|_{L_2(\Omega)} \lesssim h^s \|y\|_{H^s(\Omega)} \qquad y_h \in V_h \qquad 0 \leq s \leq p+1$$

$$\Longleftrightarrow \qquad \|y - y_N\|_{L_2(\Omega)} \lesssim N^{-s/d} \|y\|_{H^s(\Omega)}$$

N degrees of freedom $\longleftrightarrow$ maximal achievable accuracy $\mathcal{O}(N^{-(p+1)/d})$

Approximation rate determined by

(i) (piecewiese polynomials of degree $p \rightsquigarrow$) approximation order $p+1$ of V_h

(ii) space dimension d

(iii) amount of smoothness of y in L_2 measured in H^s norm

For approximation in $H^1(\Omega)$ norm (energy norm for elliptic PDE) with one order less:

$$\|y - y_N\|_{H^1(\Omega)} \lesssim N^{-(s-1)/d} \|y\|_{H^s(\Omega)}$$

Problem: one needs to know s to prescribe accuracy $\varepsilon = N^{-(s-1)/d}$

In addition: if $y \notin H^s(\Omega)$ for some $s > 1$ (or only valid for $s > 1$ with small s),
estimates cannot be used (or are not useful)

Idea in this case: use non-uniform grid to approximate y and achieve same rate

$\rightsquigarrow$ adaptive (or nonlinear) approximation of y

Example: Uniform Versus Adaptive Approximations [De]

Consider problem to approximate function on $\Omega = (0,1)$ by piecewise constants on grid

$0 = x_0 < x_1 < \cdots < x_N = 1 \qquad \Omega_j := [x_j, x_{j+1}) \qquad$ and approximation in $L_\infty(\Omega)$ norm

Case 1: f Lipschitz-continuous on $[0,1]$

approximation $f_N(x) := f(x_{n-1})$ for all $x \in [x_{n-1}, x_n)$

$$\Longrightarrow \quad |f(x) - f_N(x)| = |f(x) - f(x_{n-1})| = \left| \int_{x_{n-1}}^{x} f'(t)\,dt \right| \leq h_n \|f'\|_{L_\infty(x_{n-1},x_n)}$$

with grid spacing $h_n := |x_n - x_{n-1}|$

$$\Longrightarrow \quad \|f - f_N\|_{L_\infty(\Omega)} \leq \frac{1}{N} \|f'\|_{L_\infty(\Omega)} \text{ for } h_n = \frac{1}{N} \text{ (uniform grid)}$$

Case 2: assume $\|f'\|_{L_1(\Omega)} = 1$

define non-decreasing function $\phi(x) := \int_0^x |f'(t)|\,dt \quad \Rightarrow \quad \phi(0) = 0$ and $\phi(1) = 1$

consider now partition $0 = x_0 < x_1 < \cdots < x_N = 1$ such that

$$\int_{x_{n-1}}^{x_n} |f'(t)|\,dt = \phi(x_n) - \phi(x_{n-1}) = \frac{1}{N}$$

$\Longrightarrow \quad$ for $\quad x \in [x_{n-1}, x_n]$:

$$|f(x) - f(x_{n-1})| = \left| \int_{x_{n-1}}^{x} f'(t)\,dt \right| \leq \int_{x_{n-1}}^{x} |f'(t)|\,dt \leq \int_{x_{n-1}}^{x_n} |f'(t)|\,dt = \frac{1}{N}$$

$$\Longrightarrow \quad \|f - f_N\|_{L_\infty(\Omega)} \leq \frac{1}{N} \|f'\|_{L_1(\Omega)} \qquad \text{(grid adapted to } f\text{)}$$

Example: Uniform Versus Adaptive Approximations [De]

Result:

$$f \in W^1(L_\infty(\Omega)) \quad \text{(case 1)}: \qquad x_j = \frac{j}{N} \quad |\Omega_j| = \frac{1}{N}$$

$$\Longrightarrow \inf_{S \text{ piecewise constant}} \|f - S\|_{L_\infty(\Omega)} \leq N^{-1} |f|_{W^1(L_\infty(\Omega))}$$

$$f \in W^1(L_1(\Omega)) \quad \text{(case 2)}: \quad \text{choose } \Omega_j \text{ such that } \int_{\Omega_j} |f'(t)|dt = N^{-1}\|f'\|_{L_1(\Omega)}$$

$$\Longrightarrow \inf_{S \text{ piecewise constant}} \|f - S\|_{L_\infty(\Omega)} \leq N^{-1} |f|_{W^1(L_1(\Omega))}$$

This means:
convergence rate of same order N^{-1} for rougher function just satisfying $\|f'\|_{L_1(\Omega)} < \infty$ than for function $f' \in L_\infty(\Omega)$ by adapting grid to f

Generalization to solutions of (systems of) PDEs ? Function to be approximated is unknown

Wish list for method adapted to (systems of) PDEs:

- does not need any a-priori information (i.e., on smoothness of solutions)
- realizes theoretically optimal order under minimal smoothness assumptions
- can be uniformly used for different types of problems

In the following: adaptive methods for control problems constrained by elliptic or parabolic PDEs

Control problems constrained by elliptic Neumann problem

Linear–Quadratic Elliptic Control Problems: Neumann Problem with Distributed Control

Given y_*, f, $\omega > 0$

$$
\begin{array}{lrll}
\text{minimize} & J(y,u) & = \frac{1}{2}\|y - y_*\|^2_{H^{1-s}(\Omega)} + \frac{\omega}{2}\|u\|^2_{(H^{1-t}(\Omega))'} & \\
\text{subject to} & -\Delta y + y & = f + u & \text{in } \Omega \subset \mathbb{R}^d \\
 & \frac{\partial y}{\partial n} & = 0 & \text{on } \partial\Omega
\end{array}
$$

$0 \le s \le 1$ smoothness parameter for state y
$0 \le t$ smoothness parameter for control u

$A : H^1(\Omega) \to (H^1(\Omega))'$ weak formulation employing $\langle Av, w\rangle := \int_\Omega (\nabla v \cdot \nabla w + vw)dx$

nontrivial solution for $y_* \not\equiv A^{-1}f$

$$
\begin{array}{lrl}
\text{minimize} & J(y,u) & = \frac{1}{2}\|y - y_*\|^2_{H^{1-s}(\Omega)} + \frac{\omega}{2}\|u\|^2_{(H^{1-t}(\Omega))'} \\
\text{subject to} & Ay & = f + u
\end{array}
\qquad (1)
$$

Control problems in (infinite) wavelet coordinates

Neumann problem (1) with distributed control without Riesz operators

Minimize

$$\mathbf{J}(\mathbf{y},\tilde{\mathbf{u}}) \;=\; \tfrac{1}{2}\,\|\mathbf{D}^{-s}(\mathbf{y}-\mathbf{y}_*)\|^2 + \tfrac{\omega}{2}\|\mathbf{D}^{t}\,\tilde{\mathbf{u}}\|^2 \qquad 0 \le s \le 1, \quad 0 \le t$$

subject to

$$\mathbf{A}\mathbf{y} = \mathbf{f} + \tilde{\mathbf{u}}$$

$$\mathbf{A}:\ell_2 \to \ell_2 \text{ automorphism} \qquad \|\cdot\| := \|\cdot\|_{\ell_2}$$

Necessary (and Sufficient Conditions) for Optimality

$$\mathbf{Lagr}(\mathbf{y},\mathbf{u},\mathbf{p}) \;:=\; \mathbf{J}(\mathbf{y},\mathbf{u}) + \left\langle \mathbf{p},\, \mathbf{A}\mathbf{y} - (\mathbf{f} + \mathbf{D}^{-t}\mathbf{u})\right\rangle \qquad \text{and} \qquad \delta\mathbf{Lagr} = 0 \;\rightsquigarrow$$

$$\boxed{\begin{aligned} \mathbf{A}\mathbf{y} &= \mathbf{f} + \mathbf{D}^{-t}\,\mathbf{u} \\ \mathbf{A}^T\mathbf{p} &= -\,\mathbf{D}^{-s}\mathbf{D}^{-s}(\mathbf{y}-\mathbf{y}_*) \qquad (2) \\ \omega\mathbf{u} &= \mathbf{D}^{-t}\,\mathbf{p} \end{aligned}} \qquad \Longleftrightarrow \qquad \mathbf{Q}\,\mathbf{u} = \mathbf{g} \qquad (3)$$

$$\mathbf{Q}:\ell_2 \to \ell_2 \quad \text{automorphism}$$

where

$$\mathbf{Q} \;:=\; \mathbf{D}^{-t}\mathbf{A}^{-T}\mathbf{D}^{-2s}\mathbf{A}^{-1}\mathbf{D}^{-t} + \omega\mathbf{I} \qquad \text{symmetric positive definite}$$

$$\mathbf{g} \;:=\; \mathbf{D}^{-t}\mathbf{A}^{-T}\mathbf{D}^{-2s}(\mathbf{y}_* - \mathbf{A}^{-1}\mathbf{f})$$

Q cannot be realized by setting up and inverting **A** explicitly !

Condensed form (3) useful for deriving a convergent numerical scheme — but realization done through extended form (2)

Starting point: Convergent iteration for the ∞-dimensional problem

Iterative solution of (4) $\mathbf{Qu} = \mathbf{g}$ — $\mathbf{Q}$ symmetric positive definite

$\rightsquigarrow \quad \mathbf{u}^{n+1} = \mathbf{u}^n + \alpha\,(\mathbf{g} - \mathbf{Qu}^n) \qquad n = 0, 1, 2, \ldots \quad (5) \qquad 0 < \alpha_* \le \alpha \le \alpha^*$

$\rightsquigarrow \quad \|\mathbf{u}^{n+1} - \mathbf{u}\| \le \rho\,\|\mathbf{u}^n - \mathbf{u}\| \qquad (6) \quad \text{where} \quad \rho := \|\mathbf{I} - \alpha\mathbf{Q}\| < 1$

(guaranteed by asumptotically optimal conditioning in wavelet coordinates)

Ideal iteration $\rightsquigarrow$ computable scheme for evaluation of

$$\mathbf{Qu}^n = (\mathbf{D}^{-t}\mathbf{A}^{-T}\mathbf{D}^{-2s}\mathbf{A}^{-1}\mathbf{D}^{-t} + \omega\mathbf{I})\mathbf{u}^n \quad \text{and} \quad \mathbf{g}$$

$\rightsquigarrow$ Techniques from [CDD1,CDD2] developing adaptive methods for one elliptic PDE applied to

$\mathbf{Qu} = \mathbf{g}$

described next . . .

Adaptive Approximate Iterations

RES $[\eta, \mathbf{Q}, \mathbf{g}, \mathbf{v}] \to \mathbf{r}_\eta$ DETERMINES FOR GIVEN $\eta > 0$
A FINITELY SUPPORTED $\mathbf{r}_\eta$ SATISFYING $\|\mathbf{g} - \mathbf{Q}\mathbf{v} - \mathbf{r}_\eta\| \leq \eta$

COARSE $[\eta, \mathbf{w}] \to \mathbf{w}_\eta$ DETERMINES FOR GIVEN $\eta > 0$
A FINITELY SUPPORTED $\mathbf{w}_\eta$ SATISFYING $\|\mathbf{w} - \mathbf{w}_\eta\| \leq \eta$

Realization: sort $\mathbf{w}$ into nonincreasing order
find smallest k such that sum of k largest coefficients exceeds $\|\mathbf{w}\|^2 - \eta^2$

Cost: for $N = \#(\mathrm{supp}\,\mathbf{w})$:
$2N$ and N for level-wise (binned) sorting (instead of $N \log N$ for component-wise sorting)

Main algorithm: $\#$ interior iterations is $K := \min\{\ell : \ \rho^{\ell-1}(\alpha\ell + \rho) \leq \frac{1}{10}\}$
(α relaxation weight $\rho < 1$ contraction number in (5), (6))

SOLVE $[\varepsilon, \mathbf{Q}, \mathbf{g}] \to \mathbf{u}_\varepsilon$ $\quad$ α relaxation weight $\quad$ $\rho < 1$ contraction number

(I) $j = 0 \qquad \mathbf{u}^0 = \mathbf{0} \qquad \varepsilon_0 := \frac{1}{2} c_{\mathbf{A}}^{-1}(c_{\mathbf{A}}^{-1}\|\mathbf{f}\| + \|\mathbf{y}_*\|)$

(II) IF $\varepsilon_j \leq \varepsilon$: STOP AND SET $\mathbf{u}_\varepsilon := \mathbf{u}^j$

OTHERWISE $\mathbf{v}^0 := \mathbf{u}^j$

(II.1) FOR $n = 0, \ldots, K-1$ COMPUTE

RES $[\rho^n \varepsilon_j, \mathbf{Q}, \mathbf{g}, \mathbf{v}^n] \to \mathbf{r}^n$ (7)

$\mathbf{v}^{n+1} := \mathbf{v}^n + \alpha\, \mathbf{r}^n$

(II.2) APPLY COARSE $[\frac{2}{5}\varepsilon_j, \mathbf{v}^K] \to \mathbf{u}^{j+1}$

SET $\varepsilon_{j+1} := \frac{1}{2}\varepsilon_j$

$j + 1 \mapsto j$

GO TO (II)

Convergence: application of

Theorem [CDD1] $\quad$ For any $\varepsilon > 0$

SOLVE $[\varepsilon, \mathbf{Q}, \mathbf{g}] \to \mathbf{u}_\varepsilon$ terminates after finitely many steps and $\|\mathbf{u} - \mathbf{u}_\varepsilon\| \leq \varepsilon$

Proof: Bootstrapping argument for convergence analogous to proof of Proposition for $\mathbf{Q}$ in Part III for condensed equation (3) but more involved
COARSE is employed only for optimal computational complexity (later)

Routines for Realization of RES

APPLY $[\eta, \mathbf{A}, \mathbf{v}] \to \mathbf{w}_\eta$ COMPUTES FOR GIVEN $\eta > 0$
A FINITELY SUPPORTED $\mathbf{w}_\eta$ SATISFYING $\|\mathbf{A}\mathbf{v} - \mathbf{w}_\eta\| \le \eta$

SOLVE $[\eta, \mathbf{A}, \mathbf{f} + \mathbf{u}, \overline{\mathbf{y}}^0, \varepsilon_0] \to \mathbf{y}_\eta$ COMPUTES FOR GIVEN $\eta > 0$, INITIAL GUESS $\overline{\mathbf{y}}^0$ FOR $\mathbf{y}$
WITH ACCURACY ε_0
A FINITELY SUPPORTED $\mathbf{y}_\eta$ SATISFYING $\|\mathbf{y} - \mathbf{y}_\eta\| \le \eta$

employs

RES$_{\text{ELL}}$ $[\eta, \mathbf{A}, \mathbf{f} + \mathbf{u}, \overline{\mathbf{y}}] \to \mathbf{r}_\eta$

(i) APPLY $[\frac{1}{3}\eta, \mathbf{A}, \overline{\mathbf{y}}] \to \mathbf{w}_\eta$

(ii) COARSE$[\frac{1}{3}\eta, \mathbf{f}] \to \mathbf{f}_\eta$

COARSE$[\frac{1}{3}\eta, \mathbf{u}] \to \mathbf{u}_\eta$

(iii) SET $\mathbf{r}_\eta := \mathbf{f}_\eta + \mathbf{u}_\eta - \mathbf{w}_\eta$

SOLVE $[\eta, \mathbf{A}^T, -\mathbf{y} + \mathbf{y}_*, \overline{\mathbf{p}}^0, \varepsilon_0] \to \mathbf{p}_\eta$ COMPUTES FOR GIVEN $\eta > 0$
A FINITELY SUPPORTED $\mathbf{p}_\eta$ SATISFYING $\|\mathbf{p} - \mathbf{p}_\eta\| \le \eta$

Realization of $\textsc{Res}\,[\rho^n\varepsilon_j,\ldots]$ — $(n+1)$th iterate in $(j+1)$th block in (7)

$\eta := \rho^n\varepsilon_j$

$\textsc{Res}\;[\eta, \mathbf{Q}, \mathbf{g}, \mathbf{v}] \to \mathbf{r}_\eta$

(I) $\delta_{\mathbf{u}} := \rho^{n-1}\varepsilon_j(\rho+\alpha n) \qquad \delta_{\mathbf{y}} := c_{\mathbf{A}}^{-1}\,\delta_{\mathbf{u}} + \eta$

(II) $\textsc{Coarse}[4\delta_{\mathbf{y}}, \mathbf{y}^{j+1,n}] \to \mathbf{y}_\eta^{j+1,n+1,0}$

(III) $\textsc{Solve}\,[\frac{1}{2}\,c_{\mathbf{A}}\,\eta, \mathbf{A}, \mathbf{f}, \mathbf{u}^{j+1,n}, \mathbf{y}_\eta^{j+1,n+1,0}] \to \mathbf{y}_\eta \qquad =: \mathbf{y}^{j+1,n+1}$

(IV) $\textsc{Coarse}[\frac{4}{\omega}\,\delta_{\mathbf{u}}, \mathbf{p}^{j+1,n}] \to \mathbf{p}_\eta^{j+1,n+1,0}$

(V) $\textsc{Solve}\,[\frac{1}{2}\eta, \mathbf{A}^T, -\mathbf{y}_\eta + \mathbf{y}_*, \mathbf{p}_\eta^{j+1,n+1,0}] \to \mathbf{p}_\eta \qquad =: \mathbf{p}^{j+1,n+1}$

(VI) SET $\mathbf{r}_\eta := \mathbf{p}_\eta - \omega\mathbf{v}$

Main ideas from [CDD1]:

Ideal benchmark: Best (wavelet) N-term approximation

Show that SOLVE realizes asymptotically the work/accuracy balance

of best wavelet N–term approximation $\quad \|\mathbf{v}-\mathbf{v}_N\| := \min_{\#\operatorname{supp}\mathbf{w}\le N} \|\mathbf{v}-\mathbf{w}\|$

$$\boxed{\text{Target accuracy } \varepsilon \quad (\sim N^{-s}) \quad \longleftrightarrow \quad \text{Work } \varepsilon^{-1/s} \quad (\sim N)}$$

$\rightsquigarrow$ classify 'sparse' sequences in ℓ_2 whose best N–term approximation decays

at certain rate $\quad \mathcal{A}^s := \{\mathbf{v} \in \ell_2 : \|\mathbf{v}-\mathbf{v}_N\| \lesssim N^{-s}\}$

Coarsening Lemma $\quad \mathbf{v} \in \mathcal{A}^s$ and $\mathbf{w}$ finitely supported such that $\|\mathbf{v}-\mathbf{w}\| \le \eta$

$\Longrightarrow$ output $\mathbf{w}_\eta$ of COARSE $[4\eta, \mathbf{w}]$ satisfies

$$\#\operatorname{supp}\mathbf{w}_\eta \lesssim \eta^{-1/s}, \qquad \|\mathbf{v}-\mathbf{w}_\eta\| \lesssim 5\eta, \qquad \|\mathbf{w}_\eta\|_{\mathcal{A}^s} \lesssim \|\mathbf{v}\|_{\mathcal{A}^s}$$

Convergence and Complexity (for a single Elliptic PDE)

(Idealized) iteration (for symmetric $\mathbf{A}$)

$\mathbf{v}^{n+1} = \mathbf{v}^n + (\mathbf{f} - \mathbf{A}\mathbf{v}^n)$ update via $\textsc{Res}\,[\eta, \mathbf{A}, \mathbf{f}, \mathbf{v}] \to \mathbf{r}_\eta$ $\rightsquigarrow$ $\textsc{Solve}\,[\varepsilon, \mathbf{A}, \mathbf{f}] \to \mathbf{v}_\varepsilon$

Theorem [Cohen, Dahmen, DeVore '01/'02]

Vanishing moments (CP) for wavelets $\Longrightarrow$ $\mathbf{A}$ is s^*–compressible

$\Longrightarrow$ for variational problem satisfying (MP) scheme $\textsc{Solve}$ can be designed with properties:

(I) For every target accuracy $\varepsilon > 0$ $\textsc{Solve}$ produces after finitely many steps approximate solution $\mathbf{v}_\varepsilon$ such that $\|\mathbf{v} - \mathbf{v}_\varepsilon\| \leq \varepsilon$

(II) Exact solution $\mathbf{v} \in \mathcal{A}^s$ $\Longrightarrow$ $\operatorname{supp} \mathbf{v}_\varepsilon$, # flops $\sim \varepsilon^{-1/s} \sim N$

Core ingredient of $\textsc{Solve}$: compressible operators (later)

Convergence and Complexity Analysis for Control Problem with Elliptic PDE

Essential ideas: RES for SOLVE $[\ldots, \mathbf{Q}, \ldots]$ reduced to RES for SOLVE $[\ldots, \mathbf{A}, \ldots]$

and KKT system $\longleftrightarrow$ condensed system $\mathbf{Qu} = \mathbf{g}$

'Benchmark' Theorem (for control with elliptic PDE [Dahmen, Kunoth, SICON '05])

For any target accuracy $\varepsilon > 0$ SOLVE $[\varepsilon, \mathbf{Q}, \mathbf{g}] \to \mathbf{u}_\varepsilon$ converges in finitely many steps

$\|\mathbf{u} - \mathbf{u}_\varepsilon\| \leq \varepsilon \quad \|\mathbf{y} - \mathbf{y}_\varepsilon\| \lesssim \varepsilon \quad \|\mathbf{p} - \mathbf{p}_\varepsilon\| \lesssim \varepsilon \quad \mathbf{u}_\varepsilon, \mathbf{y}_\varepsilon, \mathbf{p}_\varepsilon$ finitely supported

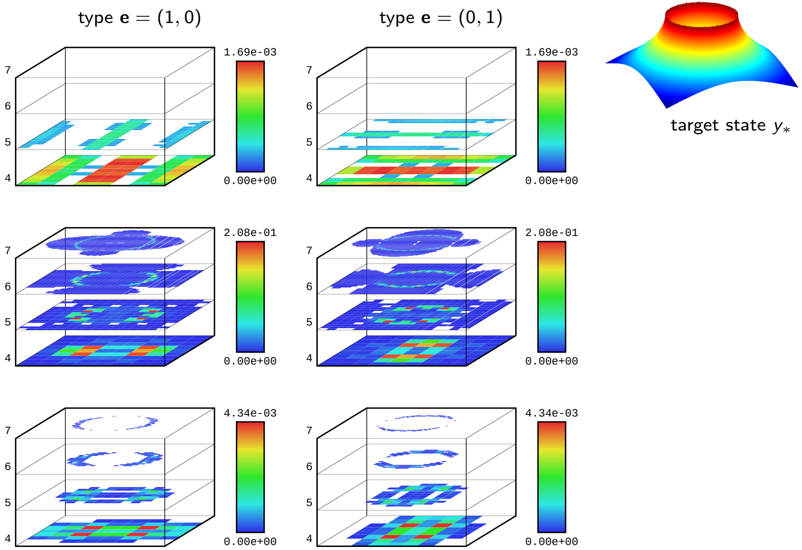
$\mathbf{u}, \mathbf{y}, \mathbf{p} \in \mathcal{A}^s \Longrightarrow$

$$(\#\,\mathrm{supp}\,\mathbf{u}_\varepsilon) + (\#\,\mathrm{supp}\,\mathbf{y}_\varepsilon) + (\#\,\mathrm{supp}\,\mathbf{p}_\varepsilon) \lesssim \varepsilon^{-1/s}\left(\|\mathbf{u}\|_{\mathcal{A}^s}^{1/s} + \|\mathbf{y}\|_{\mathcal{A}^s}^{1/s} + \|\mathbf{p}\|_{\mathcal{A}^s}^{1/s}\right)$$

$$\|\mathbf{u}_\varepsilon\|_{\mathcal{A}^s} + \|\mathbf{y}_\varepsilon\|_{\mathcal{A}^s} + \|\mathbf{p}_\varepsilon\|_{\mathcal{A}^s} \lesssim \|\mathbf{u}\|_{\mathcal{A}^s} + \|\mathbf{y}\|_{\mathcal{A}^s} + \|\mathbf{p}\|_{\mathcal{A}^s}$$

$\#$flops $\sim \varepsilon^{-1/s}$

Numerical Example for Elliptic Control Problem (2D)



[Burstedde '05], [Burstedde, Kunoth '08]

Observation: in wavelet coordinates, each variable obtains its own adaptive "refinement"