CIME–EMS Summer School "Splines and PDEs", 2017

Adaptive Multiscale Methods for the Numerical Treatment of Systems of PDEs

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Sketch of Contents

- Elliptic and parabolic partial differential equations (PDEs) in weak form; regularity of solutions
- Control problems constrained by elliptic and parabolic PDEs
- Numerical approximations of solutions on uniform and non-uniform/adaptive grids
- Concepts of multiscale methods and adaptivity; convergence proofs and complexity estimates
- Realization of these concepts by B-spline-wavelets
- Fast solvers: multilevel preconditioning; implementation issues

Literature: see References in notes_kunoth.pdf

Part V: Adaptive Wavelet Methods for Control Problems Constrained by Parabolic PDEs

Loop: SOLVE $\longrightarrow$ ESTIMATE $\longrightarrow$ REFINE $\longrightarrow$ SOLVE ... until target acccuracy reached

- Tracking type control problem constrained by parabolic PDE in wavelet coordinates
- Full weak space-time formulation of a single parabolic PDE
- Inexact gradient methods: convergence
- Complexity estimates

Literature:

[CDD1] A. Cohen, W. Dahmen, and R. DeVore, Adaptive wavelet methods for elliptic operator equations—Convergence rates, Math. Comp., 70 (2001), pp. 27–75.

[CDD2] A. Cohen, W. Dahmen, and R. DeVore, Adaptive wavelet methods *II* — Beyond the elliptic case, Found. Comput. Math., 2 (2002), pp. 203–245.

[GK] M.D. Gunzburger and A. Kunoth, Space-time adaptive wavelet methods for optimal control problems constrained by parabolic evolution equations, SIAM J. Contr. Optim., 49(3) (2011) pp. 1150–1170.

Constrained minimization problem

$\inf\limits_{(y,u)\in\mathcal{Y}\times\mathcal{U}}$	$\mathcal{J}(y,u)$	$\mathcal{J}:\mathcal{Y}\times\mathcal{U}\to\mathbb{R}$	$\mathcal{Y},\mathcal{U},\mathcal{Q}$	Hilbert spaces
subject to	$\mathcal{K}(y,u)=0$	$\mathcal{K}:\mathcal{Y}\times\mathcal{U}\to\mathcal{Q}'$		control $u\in\mathcal{U}$, state $y\in\mathcal{Y}$

Assumption on $\mathcal{K}$: for given $u\in\mathcal{U}$, there exists unique state $y\in\mathcal{Y}$

Solution approach: compute zeroes of first order Fréchet derivatives of Lagrangian functional

$\mathcal{L}(y,u,p) := \mathcal{J}(y,u) + \langle\mathcal{K}(y,u),p\rangle_{\mathcal{Q}'\times\mathcal{Q}}$ $\quad\mathcal{L}:\mathcal{Y}\times\mathcal{U}\times\mathcal{Q}\to\mathbb{R}$ $\quad$ costate/adjoint $p\in\mathcal{Q}$

$$\rightsquigarrow\quad \delta\mathcal{L}(y,u,p) := \begin{pmatrix}\mathcal{L}_y(y,u,p)\\ \mathcal{L}_u(y,u,p)\\ \mathcal{L}_p(z,u,p)\end{pmatrix} = 0 \quad\Longleftrightarrow\quad \begin{pmatrix}\mathcal{J}_y(y,u)+\langle\mathcal{K}_y(y,u),p\rangle_{\mathcal{Q}'\times\mathcal{Q}}\\ \mathcal{J}_u(y,u)+\langle\mathcal{K}_u(y,u),p\rangle_{\mathcal{Q}'\times\mathcal{Q}}\\ \mathcal{K}(y,u)\end{pmatrix} = 0$$

Special case: $\mathcal{J}$ quadratic in y,u $\quad\mathcal{K}$ linear in y,u

$\Longrightarrow$ necessary conditions for optimality are sufficient

$\rightsquigarrow$ linear (Karush-Kuhn-Tucker (KKT) or saddle point) system

$$\begin{pmatrix}\mathcal{L}_{yy} & \mathcal{L}_{yu} & \mathcal{K}_y^*\\ \mathcal{L}_{uy} & \mathcal{L}_{uu} & \mathcal{K}_u^*\\ \mathcal{K}_y & \mathcal{K}_u & 0\end{pmatrix}\begin{pmatrix}y\\u\\p\end{pmatrix} = g \quad\Longleftrightarrow:\quad \begin{pmatrix}\mathcal{A} & \mathcal{B}^*\\ \mathcal{B} & 0\end{pmatrix}\begin{pmatrix}(y,u)^T\\ p\end{pmatrix} = g \quad\Longleftrightarrow:\quad \mathcal{G}\,q = g$$

$\langle\mathcal{C}^*q,r\rangle := \langle q,\mathcal{C}r\rangle$

$\mathcal{A},\mathcal{B}$ linear, continuous; $\mathcal{A}$ invertible on $\ker\mathcal{B}$; $\operatorname{im}\mathcal{B}=\mathcal{Q}' \Longrightarrow$ $\mathcal{G}$ boundedly invertible

Optimal Control Problem Constrained by a Parabolic PDE with Distributed Control

Given $y_*(t,\cdot)$ $\quad f \quad$ $\omega > 0$ $\quad$ end time $T > 0$ $\quad$ initial condition y_0

$$
\begin{array}{lrcll}
\text{minimize} & \mathcal{J}(y,u) & = & \frac{1}{2}\int_0^T \|y(t,\cdot) - y_*(t,\cdot)\|_Z^2 \, dt + \frac{\omega}{2}\int_0^T \|u(t,\cdot)\|_U^2 \, dt & \\
\text{subject to} & y'(t) + A(t)y(t) & = & f(t) + u(t) \qquad \text{a.e. } t \in (0,T) =: I & \text{(PDE)} \\
 & y(0) & = & y_0 &
\end{array}
$$

$y' := \frac{\partial}{\partial t} y$ $\qquad$ $y = y(t,x)$ state $\qquad$ $u = u(t,x)$ control

$Y = H_0^1(\Omega)$ state space $\quad$ $Z = Y = H_0^1(\Omega)$ observation space $\quad$ $U = Y' = H^{-1}(\Omega)$ control space

$A(t) : Y \to Y'$ $\langle A(t)v(t,\cdot), w(t,\cdot)\rangle := \int_\Omega [\nabla v(t,x) \cdot \nabla w(t,x) + v(t,x)w(t,x)]\, dx$ $\qquad \Omega \subset \mathbb{R}^d$

$A(t)$ 2nd order linear selfadjoint coercive & continuous operator on Y

PDE-constrained control problem $\quad \rightsquigarrow \quad$ requires repeated solution of PDE constraint

$$
\begin{array}{rcl}
y'(t) + A(t)y(t) & = & f(t) + u(t) \\
y(0) & = & y_0
\end{array}
$$

Optimal control problem constrained by parabolic PDE

$\rightsquigarrow$ System of parabolic PDEs coupled globally in time (and space)

$$
\begin{array}{rcll}
y'(t) + A(t)\,y(t) & = & f(t) + u(t) & \text{a.e. } t \in I \\
y(0) & = & y_0 & \\
\omega\,\tilde{R}^{-1} u(t) + p(t) & = & 0 & \text{a.e. } t \in I \\
-p'(t) + A(t)^T p(t) & = & \tilde{R}\,(y_*(t) - y(t)) & \text{a.e. } t \in I \\
p(T) & = & 0 &
\end{array}
$$

Riesz operator $\tilde{R}$ defined by $\langle v, \tilde{R}w\rangle_{Y\times Y'} := (v,w)_Y$ for all $v, w \in Y$

Obstructions for numerical solution:

- convential time discretizations: time-marching methods
  $\rightsquigarrow$ need storage of $y(t_i), u(t_i), p(t_i)$ for all discrete times $0 = t_0, \ldots, T = t_N$
- in each time step: solve elliptic PDE $\rightsquigarrow$ large linear system of equations
  $\rightsquigarrow$ iterative solver $\rightsquigarrow$ need preconditioning in (conjugate) gradient method
- singularities in data/domain: adaptive (FE) mesh(es) for $y(t_i), u(t_i), p(t_i)$ for all t_i
  one mesh for all variables, refinement/coarsening ? [Meidner, Vexler '07], ...
  convergence ? complexity ??

Solution Ansatz here: full weak space-time form of parabolic PDE constraint

Variational Space-Time Form for a Single Parabolic Evolution PDE

[Ladyshenskaya et al 1967], [Wloka '82], [Dautray, Lions '92], [Schwab, Stevenson '09], [Chegini, Stevenson '11], [Stapel '11] ...

$$\text{(PDE)} \qquad \begin{aligned} y'(t) + A(t)\,y(t) &= f(t) \qquad \text{a.e. } t \in I \\ y(0) &= y_0 \end{aligned}$$

solution space: Lebesgue-Bochner space $\mathcal{Y} := (L_2(I) \otimes Y) \cap (H^1(I) \otimes Y') \hookrightarrow \mathcal{C}^0(\overline{I}) \otimes L_2(\Omega)$

with norm $\|w\|_{\mathcal{Y}}^2 := \|w\|_{L_2(I)\otimes Y}^2 + \|w'\|_{H^1(I)\otimes Y'}^2$

test space: $\mathcal{Q} := (L_2(I) \otimes Y) \times L_2(\Omega)$ with norm $\|v\|_{\mathcal{Q}}^2 := \|v_1\|_{L_2(I)\otimes Y}^2 + \|v_2\|_{L_2(\Omega)}^2$

bilinear form $b(\cdot,\cdot) : \mathcal{Y} \times \mathcal{Q} \to \mathbb{R}$

$$b(w,(v_1,v_2)) := \int_I \left[\langle w'(t,\cdot), v_1(t,\cdot)\rangle + \langle A(t)w(t,\cdot), v_1(t,\cdot)\rangle\right]\,dt + \langle w(0,\cdot), v_2\rangle =: \langle Bw, v\rangle$$

right hand side

$$\langle f, v\rangle := \int_I \langle f(t,\cdot), v_1(t,\cdot)\rangle\,dt + \langle y_0, v_2\rangle$$

(PDE) $\rightsquigarrow$ given $f \in \mathcal{Q}'$, find $y \in \mathcal{Y}$: $\boxed{By = f}$

Existence and uniqueness of solution:

Theorem $\|Bw\|_{\mathcal{Q}'} \sim \|w\|_{\mathcal{Y}}$ for all $w \in \mathcal{Q}$ mapping property (MP)

Formulations with 1/2 time derivatives on $\mathbb{R}$: [Fontes "99], [Larsson, Schwab '15]

Reformulation of PDE-Constrained Optimal Control Problem

$$\text{minimize} \quad \mathcal{J}(y,u) = \tfrac{1}{2}\|y - y_*\|^2_{L_2(I)\otimes Y} + \tfrac{\omega}{2}\|u\|^2_{L_2(I)\otimes Y'}$$

$$\text{subject to} \quad B\,y = f + E\,u \quad \text{(PDE)} \qquad B : \mathcal{Y} \to \mathcal{Q}' \text{ satisfies (MP)}$$

$$E := (\mathrm{Id}, 0) : L_2(I) \otimes Y' \to \mathcal{Q}'$$

Necessary and Sufficient Conditions — Karush-Kuhn-Tucker (KKT) system

$$\mathcal{L}(y,u,p) := \mathcal{J}(y,u) + \langle p, By - f - Eu\rangle$$

Riesz operator $\langle v, Rw\rangle_{(L_2(I)\otimes Y)\times(L_2(I)\otimes Y')} := (v,w)_{L_2(I)\otimes Y}$

$$\delta\mathcal{L} = 0 \;\rightsquigarrow\; \boxed{\begin{aligned} B^* p &= R(y_* - y) \\ \omega R^{-1} u &= E^* p \\ B\,y &= f + E\,u \end{aligned}}$$

$$\iff \boxed{\begin{pmatrix} R & 0 & B^* \\ 0 & \omega R^{-1} & -E^* \\ B & -E & 0 \end{pmatrix} \begin{pmatrix} y \\ u \\ p \end{pmatrix} = \begin{pmatrix} Ry_* \\ 0 \\ f \end{pmatrix} \quad \text{(SPP)}}$$

$\rightsquigarrow$ saddle point operator

$$\langle \mathcal{G}q, \tilde{q}\rangle := \left\langle \begin{pmatrix} R & 0 & B^* \\ 0 & \omega R^{-1} & -E^* \\ B & -E & 0 \end{pmatrix} q, \tilde{q} \right\rangle; \quad \mathcal{A} := \mathrm{diag}(R, \omega R^{-1}); \mathcal{B} := (B, -E)$$

symmetric, continuous, boundedly invertible on $\boldsymbol{\mathcal{X}} := \mathcal{Y} \times \mathcal{U} \times \mathcal{Q}$

$\Longrightarrow$ unique solution $\begin{pmatrix} y \\ u \\ p \end{pmatrix} =: q$ of system of PDEs (SPP)

Formulations with 1/2 time derivatives: [Langer, Wolfmayr '13], [Kunoth, Mollet '15, in revision]

Building Blocks: (Biorthogonal Spline–) Wavelets

H Hilbert space on domain $\Omega \subset \mathbb{R}^d$ with $\|\cdot\|_H$ $\qquad$ H' dual space for H with $\langle \cdot, \cdot \rangle$

$\Psi := \{\psi_\lambda : \lambda \in \mathbb{I}\} \subset H$ **Wavelets** $\qquad$ $\mathbb{I}$ (infinite) index set

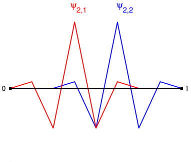
(NE) Ψ Riesz basis for H

$$v \in H: \quad v = \mathbf{v}^T \Psi := \sum_{\lambda \in \mathbb{I}} \langle v, \tilde{\psi}_\lambda \rangle \, \psi_\lambda \qquad \text{such that} \qquad \|v\|_H \sim \|\mathbf{v}\|_{\ell_2(\mathbb{I})}$$

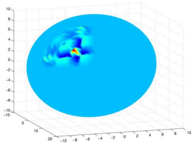
(L) Locality $\qquad$ $\operatorname{diam}(\operatorname{supp} \psi_\lambda) \sim 2^{-|\lambda|}$ $\qquad$ $|\lambda|$ resolution

ψ_λ centered around $2^{-|\lambda|} k$

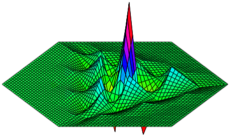
(CP) Vanishing moments $\langle v, \psi_\lambda \rangle \lesssim 2^{-|\lambda|(\frac{d}{2}+\tilde{m})} \|v^{(\tilde{m})}\|_{L_\infty(\operatorname{supp} \psi_\lambda)}$ $\qquad$ for some $\tilde{m}$



[Dahmen, Kunoth, Urban '99]



[Dahmen, Schneider '99], [Kunoth, Sahner '06]



[Harbrecht, Schneider '00]

Paradigm of Adaptive Wavelet Method for One Stationary PDE

[Cohen, Dahmen, DeVore '01/'02]

(i) Well–posed variational problem: given $f \in \mathcal{Q}'$, $B : \mathcal{Y} \to \mathcal{Q}'$, find $y \in \mathcal{Y}$ such that $\boxed{By = f}$

$$\boxed{\text{(MP)} \qquad \|Bw\|_{\mathcal{Q}'} \sim \|w\|_{\mathcal{Y}} \qquad \text{for all } w \in \mathcal{Y} \qquad \text{mapping property}}$$

(ii) $\Psi^{\mathcal{Y}}, \Psi^{\mathcal{Q}}$ wavelet bases for $\mathcal{Y}$, $\mathcal{Q}$:

$$\boxed{\text{(NE)} \qquad \|\mathbf{w}^T \Psi^{\mathcal{Y}}\|_{\mathcal{Y}} \;\sim\; \|\mathbf{w}\|_{\ell_2} \qquad \text{for all } \mathbf{w} = (w_\lambda)_{\lambda \in \mathbb{I}} \in \ell_2}$$

$$\mathbf{Bw} := (\langle \psi_\lambda^{\mathcal{Y}}, Bw \rangle)_{\lambda \in \mathbb{I}} \qquad \mathbf{f} := (\langle \psi_\lambda^{\mathcal{Y}}, f \rangle)_{\lambda \in \mathbb{I}}$$

$\rightsquigarrow$ **Theorem** $\quad By = f \iff \mathbf{By} = \mathbf{f}$ well-posed in ℓ_2 $\qquad (\mathbf{B} : \ell_2 \to \ell_2)$

$$\text{(MP)} + \text{(NE)} \iff \qquad \|\mathbf{Bw}\|_{\ell_2} \sim \|\mathbf{w}\|_{\ell_2} \quad \text{for all } \mathbf{w} \in \ell_2$$

(iii) Practical solution schemes for $\mathbf{By} = \mathbf{f}$:

(A) Perturbed Richardson iteration (for symmetric $\mathbf{B}$):

(A.1) $\mathbf{y}^{n+1} = \mathbf{y}^n + (\mathbf{f} - \mathbf{B}\mathbf{y}^n) \quad n = 0, 1, 2, \ldots \qquad \|\mathbf{y}^{n+1} - \mathbf{y}\|_{\ell_2} \le \rho \, \|\mathbf{y}^n - \mathbf{y}\|_{\ell_2} \quad \rho < 1$

(A.2) Approximate realization: adaptive evaluation of $\mathbf{B}\mathbf{y}^n$ in SOLVE $[\varepsilon, \mathbf{B}, \mathbf{f}] \to \mathbf{y}_\varepsilon$

(A.3) Coarsening (thresholding) of the iterands (for complexity)

(B) Adaptive wavelet Galerkin method and bulk chasing strategy

Extension to a Single Parabolic Evolution PDE

[Schwab, Stevenson '09]

(i) Variational space-time form of (PDE)

$$\begin{aligned} y'(t) + A(t)\,y(t) &= f(t) \qquad \text{a.e. } t \in I \\ y(0) &= y_0 \end{aligned}$$

solution space: Lebesgue-Bochner space $\mathcal{Y} := (L_2(I) \otimes Y) \cap (H^1(I) \otimes Y')$

with norm $\|w\|_{\mathcal{Y}}^2 := \|w\|_{L_2(I)\otimes Y}^2 + \|w'\|_{H^1(I)\otimes Y'}^2$

test space $\mathcal{Q} := L_2(I; Y) \times L_2(\Omega)$ with norm $\|v\|_{\mathcal{Q}}^2 := \|v_1\|_{L_2(I)\otimes Y}^2 + \|v_2\|_{L_2(\Omega)}^2$

bilinear form $b(\cdot,\cdot) : \mathcal{Y} \times \mathcal{Q} \to \mathbb{R}$

$$b(y,(v_1,v_2)) :=$$
$$\int_I \left[\langle y'(t,\cdot), v_1(t,\cdot)\rangle + \langle A(t)y(t,\cdot), v_1(t,\cdot)\rangle\right] dt + \langle y(0,\cdot), v_2\rangle =: \langle By, v\rangle$$

right hand side

$$\langle f, v\rangle := \int_I \langle f(t,\cdot), v_1(t,\cdot)\rangle \, dt + \langle y_0, v_2\rangle$$

(PDE) $\rightsquigarrow$ given $f \in \mathcal{Q}'$, find $y \in \mathcal{Y}$: $\boxed{By = f}$

Theorem	(MP)	$\|Bw\|_{\mathcal{Q}'} \sim \|w\|_{\mathcal{Y}}$	for all $w \in \mathcal{Y}$	mapping property

(ii) $\Psi^{\mathcal{Y}}, \Psi^{\mathcal{Q}}$ wavelet bases for $\mathcal{Y}, \mathcal{Q}$ $\rightsquigarrow$ $\mathbf{B}\mathbf{y} := (\langle \psi_\lambda^{\mathcal{Q}}, By\rangle)_{\lambda\in\mathbb{I}}$ $\quad \mathbf{f} := (\langle \psi_\lambda^{\mathcal{Q}}, f\rangle)_{\lambda\in\mathbb{I}}$

Theorem $By = f \iff \mathbf{By} = \mathbf{f}$ $\quad \mathbf{B} : \ell_2 \to \ell_2$ and $\mathbf{By} = \mathbf{f}$ well-posed in ℓ_2

(MP) + (NE) $\Longrightarrow$ $\|\mathbf{Bv}\|_{\ell_2} \sim \|\mathbf{v}\|_{\ell_2}, \quad \mathbf{v} \in \ell_2$ $\qquad$ $\mathbf{B}$ unsymmetric

Application to PDE-Constrained Optimal Control Problem

Control problem in wavelet coordinates

minimize $\quad \mathbf{J}(\mathbf{y},\mathbf{u}) = \frac{1}{2}\|\mathbf{R}^{1/2}(\mathbf{y}-\mathbf{y}_*)\|^2 + \frac{\omega}{2}\|\mathbf{R}^{-1/2}\mathbf{u}\|^2$

subject to $\quad \mathbf{B}\mathbf{y} = \mathbf{f} + \mathbf{u} \qquad \mathbf{B} : \ell_2 \to \ell_2$ automorphism $\qquad \|\cdot\| := \|\cdot\|_{\ell_2}$

Necessary and Sufficient Conditions — Karush-Kuhn-Tucker (KKT) system

$$\mathbf{L}(\mathbf{y},\mathbf{u},\mathbf{p}) := \mathbf{J}(\mathbf{y},\mathbf{u}) + \langle \mathbf{p},\, \mathbf{B}\mathbf{y} - (\mathbf{f}+\mathbf{u})\rangle$$

$$\delta\mathbf{L} = 0 \rightsquigarrow \boxed{\begin{aligned} \mathbf{B}\mathbf{y} &= \mathbf{f}+\mathbf{u} \\ \omega\mathbf{R}^{-1}\mathbf{u} &= \mathbf{p} \\ \mathbf{B}^*\mathbf{p} &= \mathbf{R}(\mathbf{y}_* - \mathbf{y}) \end{aligned}} \iff \boxed{\mathbf{Q}\,\mathbf{u} = \mathbf{g}}$$

$$\iff \boxed{\begin{pmatrix} \mathbf{R} & \mathbf{0} & \mathbf{B}^* \\ \mathbf{0} & \omega\mathbf{R}^{-1} & -\mathbf{E} \\ \mathbf{B} & -\mathbf{E} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{R}\mathbf{y}_* \\ \mathbf{0} \\ \mathbf{f} \end{pmatrix} \quad \text{(SPP)}} \qquad \mathbf{Q} : \ell_2 \to \ell_2 \ \text{automorphism}$$

where
$$\begin{aligned} \mathbf{Q} &:= \mathbf{B}^{-*}\mathbf{R}\mathbf{B}^{-1} + \omega\mathbf{R}^{-1} \\ \mathbf{g} &:= \mathbf{B}^{-*}(\mathbf{R}\mathbf{y}_* - \mathbf{R}\mathbf{B}^{-1}\mathbf{f}) \end{aligned}$$

Complexity Analysis

Based on benchmark:
decay rate s for (wavelet-)best N term approximation $\mathcal{A}^s := \{\mathbf{v} \in \ell_2 : \|\mathbf{v} - \mathbf{v}_N\| \lesssim N^{-s}\}$

Work/accuracy balance of best N term approximation:

$$\boxed{\text{Target accuracy } \varepsilon \ (\sim N^{-s}) \quad \longleftrightarrow \quad \text{Work } \varepsilon^{-1/s} \ (\sim N)}$$

Convergence and Complexity

For solution routine (A): (Idealized) iteration (for symmetric $\mathbf{B}$)
$\mathbf{v}^{n+1} = \mathbf{v}^n + (\mathbf{f} - \mathbf{B}\mathbf{v}^n)$ update via $\textsc{Res}\,[\eta, \mathbf{B}, \mathbf{f}, \mathbf{v}] \to \mathbf{r}_\eta$ $\rightsquigarrow$ $\textsc{Solve}\,[\varepsilon, \mathbf{B}, \mathbf{f}] \to \mathbf{v}_\varepsilon$

Theorem [Cohen, Dahmen, DeVore '01/'02]

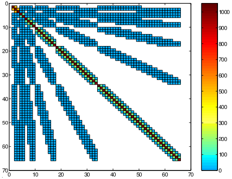
Vanishing moments (CP) for wavelets $\Longrightarrow$ $\mathbf{B}$ is s^*–compressible

$\Longrightarrow$ for variational problem satisfying (MP) scheme $\textsc{Solve}$ can be designed with properties:

(I) For every target accuracy $\varepsilon > 0$ $\textsc{Solve}$ produces after finitely many steps approximate solution $\mathbf{v}_\varepsilon$ such that $\|\mathbf{v} - \mathbf{v}_\varepsilon\| \leq \varepsilon$

(II) Exact solution $\mathbf{v} \in \mathcal{A}^s \Longrightarrow \operatorname{supp} \mathbf{v}_\varepsilon$, $\#$ flops $\sim \varepsilon^{-1/s} \sim N$

Core Ingredient of SOLVE : Compressible Operators



(CP) $\rightsquigarrow$ **B** is s^*–compressible:
for every $s \in (0, s^*)$ there exists $\mathbf{B}_j$ with
$\leq \alpha_j 2^j$ nonzero entries per row and column s.th. for $j \in \mathbb{N}_0$

$\|\mathbf{B} - \mathbf{B}_j\| \leq \alpha_j 2^{-sj}$; $\sum_{j \in \mathbb{N}_0} \alpha_j < \infty$ (**B** 'close to' sparse matrix)

Application of (Non)Linear Operators in Wavelet Bases

Theory: [Dahmen, Schneider, Xu '00], [Cohen, Dahmen, DeVore '03] ...
$d = 2$, isotropic tensor-product wavelets: [Vorloeper '10] general d: [Stapel '11], [Mollet, Pabel '12], [Pabel '15]

Input: finitely supported vector $\mathbf{v} = (v_\mu)_{\mu \in \Lambda}$ $\Lambda \subset \mathbb{I}$ finite

Output: approximation of **Bv** with infinite-dimensional operator $\mathbf{B} : \ell_2(\mathbb{I}) \to \ell_2(\mathbb{I})$

$B : \mathcal{Y} \to \mathcal{Q}' \rightsquigarrow$ expand $Bv \in \mathcal{Q}'$ in dual wavelet basis for $\mathcal{Q}'$ and v in primal wavelet basis for $\mathcal{Y}$
$\rightsquigarrow$

$$Bv = (\mathbf{Bv})^T \tilde{\Psi} = \sum_{\lambda \in \mathbb{I}} \langle Bv, \psi_\lambda \rangle \, \tilde{\psi}_\lambda = \sum_{\lambda \in \mathbb{I}} \langle B(\sum_{\mu \in \Lambda} v_\mu \psi_\mu, \psi_\lambda) \rangle \, \tilde{\psi}_\lambda = \sum_{\lambda \in \mathbb{I}} \sum_{\mu \in \Lambda} v_\mu \langle B\psi_\mu, \psi_\lambda \rangle \, \tilde{\psi}_\lambda$$

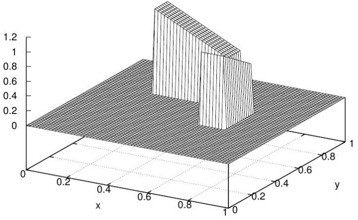
$\rightsquigarrow$ compute $\langle B\psi_\mu, \psi_\lambda \rangle$ for given $\mu \in \Lambda$ (finite) and all $\lambda \in \mathbb{I}$

Compressibility of B: $|\langle B\psi_\mu, \psi_\lambda \rangle| \leq C_{\|\mathbf{v}\|} \sup_{\mu:\ S_\lambda \cap S_\mu \neq \emptyset} 2^{-\gamma(|\lambda| - |\mu|)} \, |v_\mu|$ $\gamma > \frac{d}{2} + 1$
follows from wavelet property (CP)

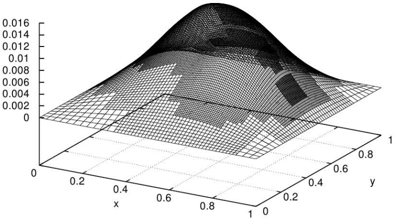
Essential data structure (for nonlinear operators): tree-type index sets
input $\mathbf{v} \rightsquigarrow$ prediction of tree index set based on supp $\mathbf{v}$ and properties of **B**
$\rightsquigarrow$ computation of $(\mathbf{Bv})_\lambda$ after transformation to piecewise polynomials
$\rightsquigarrow$ application of **B** in optimal linear complexity

[Mollet, Pabel '12], [Pabel '15]

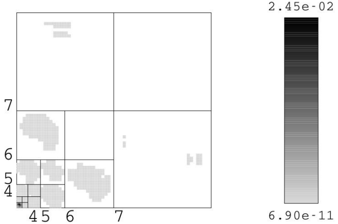
PDE with nonlinear term
$$\begin{aligned} -\Delta y + y^3 &= f && \text{in } \Omega := (0,1)^2 \\ y &= 0 && \text{on } \partial\Omega \end{aligned}$$



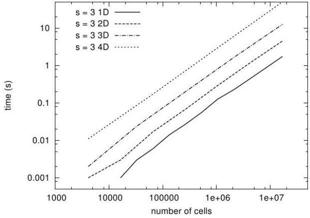
right hand side f



solution y (with Richardson scheme and residual error bound 10^{-3})



distribution of 7177 active wavelet coefficients



Runtime (seconds) for evaluating y^3 for $d \leq 4$

Convergence and Complexity Analysis for Control Problem with Parabolic PDE Constraints

Essential ideas: $\textsc{Res}$ for $\textsc{Solve}\,[\ldots, \mathbf{Q}, \ldots]$ reduced to $\textsc{Res}$ for $\textsc{Solve}\,[\ldots, \mathbf{B}, \ldots]$

applied to normal equations

and KKT system $\longleftrightarrow$ condensed system $\mathbf{Q}\mathbf{u} = \mathbf{g}$

'Benchmark' Theorem control problem with parabolic PDE: [Gunzburger, Kunoth, SICON '11]

For any target accuracy $\varepsilon > 0$ $\textsc{Solve}\,[\varepsilon, \mathbf{Q}, \mathbf{g}] \to \mathbf{u}_\varepsilon$ converges in finitely many steps

$$\|\mathbf{u} - \mathbf{u}_\varepsilon\| \le \varepsilon \quad \|\mathbf{y} - \mathbf{y}_\varepsilon\| \lesssim \varepsilon \quad \|\mathbf{p} - \mathbf{p}_\varepsilon\| \lesssim \varepsilon \quad \mathbf{u}_\varepsilon, \mathbf{y}_\varepsilon, \mathbf{p}_\varepsilon \text{ finitely supported}$$

$\mathbf{u}, \mathbf{y}, \mathbf{p} \in \mathcal{A}^s \Longrightarrow$

$$(\#\,\mathrm{supp}\,\mathbf{u}_\varepsilon) + (\#\,\mathrm{supp}\,\mathbf{y}_\varepsilon) + (\#\,\mathrm{supp}\,\mathbf{p}_\varepsilon) \lesssim \varepsilon^{-1/s}\left(\|\mathbf{u}\|_{\mathcal{A}^s}^{1/s} + \|\mathbf{y}\|_{\mathcal{A}^s}^{1/s} + \|\mathbf{p}\|_{\mathcal{A}^s}^{1/s}\right)$$

$$\|\mathbf{u}_\varepsilon\|_{\mathcal{A}^s} + \|\mathbf{y}_\varepsilon\|_{\mathcal{A}^s} + \|\mathbf{p}_\varepsilon\|_{\mathcal{A}^s} \lesssim \|\mathbf{u}\|_{\mathcal{A}^s} + \|\mathbf{y}\|_{\mathcal{A}^s} + \|\mathbf{p}\|_{\mathcal{A}^s}$$

$\#$flops $\sim \varepsilon^{-1/s}$

Numerical Example for One Parabolic PDE

[Chegini, Stevenson '11], [Stapel '11]

Compute $y = y(t,x)$ such that

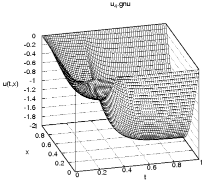
$$
\begin{array}{rlll}
y_t(t,x) - y_{xx}(t,x) & = g(t) \otimes (-\pi^2)\sin(\pi x) & & \text{in } \; I \times \Omega := (0,1)^2 \\
y(t,0) & = \; y(t,1) \; = \; 0 & & \text{for } t \geq 0 \\
y(0,x) & = 0 & & \text{for } x \in (0,1)
\end{array}
$$

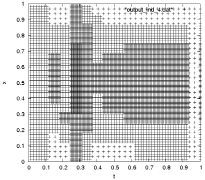
and $g(t) := \begin{cases} 1 & t \in [0, \frac{1}{3}) \\ 2 & t \in [\frac{1}{3}, 1] \end{cases}$

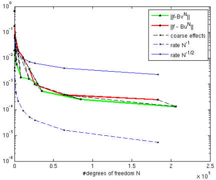
Problem formulation and implementation:

- Modified problem with zero initial conditions $\rightsquigarrow$
  solution space $\mathcal{Y} = (L_2(I) \otimes H^1(\Omega)) \cap (H^1_{(0}(I) \otimes H^{-1}(\Omega))$ and test space $\mathcal{Q} = L_2(I) \otimes Y$
- Inhomogeneous initial data: homogenization of initial conditions $\rightsquigarrow$ modification of r.h.s.
- Implementation based on AWM Toolbox by [Vorloeper '10]
  biorthogonal isotropic wavelets of order $m = 2, \tilde{m} = 4$
- Iterative solution by GMRES

Plot of Solution, Refined Grid and Residual Error Reduction







8526 degrees of freedom Expected rate in H^1 (isotropic wavelets): $1/2$ red: after coarsening

PDE-Constrained Control Problems: Summary and Extensions

- Control problem constrained by parabolic PDE
  Full weak space-time formulation of evolution PDE
  $\rightsquigarrow$ saddle point system of PDEs coupled globally in time and space
- For smooth solutions: multilevel/wavelet preconditioners + nested iteration
  $\rightsquigarrow$ numerical solution scheme with optimal complexity
- For non-smooth solutions:
  proofs of convergence and optimal complexity based on adaptive wavelets
- Extension to control problems with elliptic or parabolic PDE with stochastic coefficients
  [Kunoth, Schwab '13], [Kunoth, Schwab '16]
- Inequality constraints on control and/or state

Beyond Wavelets

- Optimal preconditioning: multilevel and multigrid methods (for normal equations);
  fast iterative solvers on (non)uniform grids
- (A posteriori) error estimates for PDE constrained control problems: one grid [Liu et al ... et al ...]
- Convergence theory of adaptive (finite element) method for control problem
  with linear elliptic PDE constraints; one mesh for all variables [Kohls, Siebert, Rösch '14]
- Complexity estimates ? Optimal complexity ? Application of PDE operator ?
- Convergence theory of adaptive (FE/DG) methods for control problems
  constrained by linear evolutionary PDE ?