# Adaptive Multiscale Methods for the Numerical Treatment of Systems of PDEs

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Sketch of Contents

- Elliptic and parabolic partial differential equations (PDEs) in weak form; regularity of solutions
- Control problems constrained by elliptic and parabolic PDEs
- Numerical approximations of solutions on uniform and non-uniform/adaptive grids
- Concepts of multiscale methods and adaptivity; convergence proofs and complexity estimates
- Realization of these concepts by B-spline-wavelets
- Fast solvers: multilevel preconditioning; implementation issues

Literature: see References in notes_kunoth.pdf

## Part III: Control Problems Constrained by Elliptic PDEs

Optimization problems with PDEs:

- Tracking type control problems constrained by PDE, e.g. flow control
- Topology optimization
- Shape optimization
- Variables: state, control, adjoint (or co-)state
- Additional inequality constraints on control and/or state

In the following main subjects:

- Tracking type control problem constrained by elliptic (or parabolic) PDE
  $\rightsquigarrow$ system of coupled PDEs
- Variables: state, control, adjoint (or co-)state
- In this part: discretizations on uniform grids
- Efficient solution schemes based on wavelets
- Convergence and optimal complexity

Literature:

[BK] C. Burstedde, A. Kunoth, Fast iterative solution of elliptic control problems in wavelet discretization, Journal of Computational and Applied Mathematics 196 (2006), 299-319.

## Optimization Problems: First Order Necessary Conditions

Constrained minimization problem

| | | | | |
|---|---|---|---|---|
| $\inf_{(y,u)\in\mathcal{Y}\times\mathcal{U}}$ | $\mathcal{J}(y,u)$ | $\mathcal{J}:\mathcal{Y}\times\mathcal{U}\to\mathbb{R}$ | $\mathcal{Y},\mathcal{U},\mathcal{Q}$ | Hilbert spaces |
| subject to | $\mathcal{K}(y,u)=0$ | $\mathcal{K}:\mathcal{Y}\times\mathcal{U}\to\mathcal{Q}'$ | | control $u\in\mathcal{U}$, state $y\in\mathcal{Y}$ |

Assumption on $\mathcal{K}$: for given $u\in\mathcal{U}$, there exists unique state $y\in\mathcal{Y}$

Solution approach: compute zeroes of first order Fréchet derivatives of Lagrangian functional

$\mathcal{L}(y,u,p) := \mathcal{J}(y,u) + \langle\mathcal{K}(y,u),p\rangle_{\mathcal{Q}'\times\mathcal{Q}}$ $\qquad \mathcal{L}:\mathcal{Y}\times\mathcal{U}\times\mathcal{Q}\to\mathbb{R}$ $\qquad$ costate/adjoint $p\in\mathcal{Q}$

$$\rightsquigarrow\quad \delta\mathcal{L}(y,u,p) := \begin{pmatrix}\mathcal{L}_y(y,u,p)\\ \mathcal{L}_u(y,u,p)\\ \mathcal{L}_p(z,u,p)\end{pmatrix} = 0 \qquad\Longleftrightarrow\qquad \begin{pmatrix}\mathcal{J}_y(y,u)+\langle\mathcal{K}_y(y,u),p\rangle_{\mathcal{Q}'\times\mathcal{Q}}\\ \mathcal{J}_u(y,u)+\langle\mathcal{K}_u(y,u),p\rangle_{\mathcal{Q}'\times\mathcal{Q}}\\ \mathcal{K}(y,u)\end{pmatrix} = 0$$

Special case: $\mathcal{J}$ quadratic in $y,u$ $\qquad\mathcal{K}$ linear in $y,u$

$\Longrightarrow$ necessary conditions for optimality are sufficient

$\rightsquigarrow$ linear (Karush-Kuhn-Tucker (KKT) or saddle point) system

$$\begin{pmatrix}\mathcal{L}_{yy} & \mathcal{L}_{yu} & \mathcal{K}_y^*\\ \mathcal{L}_{uy} & \mathcal{L}_{uu} & \mathcal{K}_u^*\\ \mathcal{K}_y & \mathcal{K}_u & 0\end{pmatrix}\begin{pmatrix}y\\ u\\ p\end{pmatrix} = g \quad\Longleftrightarrow:\quad \begin{pmatrix}\mathcal{A} & \mathcal{B}^*\\ \mathcal{B} & 0\end{pmatrix}\begin{pmatrix}(y,u)^T\\ p\end{pmatrix} = g \quad\Longleftrightarrow:\quad \mathcal{G}\,q = g$$

$\langle\mathcal{C}^*q,r\rangle := \langle q,\mathcal{C}r\rangle$

$\mathcal{A},\mathcal{B}$ linear, continuous; $\mathcal{A}$ invertible on $\ker\mathcal{B}$; im $\mathcal{B}=\mathcal{Q}'\Longrightarrow$ $\mathcal{G}$ boundedly invertible

# Control problems constrained by elliptic Neumann problem

Linear–Quadratic Elliptic Control Problems: Neumann Problem with Distributed Control

Given $y_*$, $f$, $\omega > 0$

$$
\begin{aligned}
\text{minimize} \quad J(y,u) \quad &= \tfrac{1}{2}\|y - y_*\|^2_{H^{1-s}(\Omega)} + \tfrac{\omega}{2}\|u\|^2_{(H^{1-t}(\Omega))'} \\
\text{subject to} \quad -\Delta y + y \quad &= f + u \qquad \text{in } \Omega \subset \mathbb{R}^d \\
\frac{\partial y}{\partial n} \quad &= 0 \qquad\qquad \text{on } \partial\Omega
\end{aligned}
\tag{1}
$$

$0 \le s \le 1$ smoothness parameter for state $y$

$0 \le t$ smoothness parameter for control $u$

$A : H^1(\Omega) \to (H^1(\Omega))'$

weak formulation employing $\langle Av, w\rangle := \int_\Omega (\nabla v \cdot \nabla w + vw)dx$

nontrivial solution for $y_* \not\equiv A^{-1}f$

$$
\begin{aligned}
\text{minimize} \quad J(y,u) \quad &= \tfrac{1}{2}\|y - y_*\|^2_{H^{1-s}(\Omega)} + \tfrac{\omega}{2}\|u\|^2_{(H^{1-t}(\Omega))'} \\
\text{subject to} \qquad Ay \quad &= f + u
\end{aligned}
\tag{2}
$$

Discretization of (continuous) control problem in (infinite) wavelet coordinates
$\rightsquigarrow$

- Modelling: Cost functional with Sobolev norms $\sim$ weighted sequence norms
- Numerical Analysis: Optimal preconditioning
- (in next part) Numerical Solution by Adaptive Scheme: Linear elliptic PDE as constraints
  Iterative scheme – A–posteriori error estimates – convergence
  – convergence rate and optimal complexity estimates
  regularity theory in Besov spaces – wavelet–best $N$–term approximation

Standard Discretizations

Elliptic PDE: $A : Y \to Y'$ isomorphism $\rightsquigarrow$ (finite–dimensional) discretization $\rightsquigarrow$ iterative solvers, efficient preconditioners

control problem: Evaluation of fractional Sobolev norms ?

New Paradigm

(I) mapping property for $A : Y \to Y'$

(II) transformation into equivalent $\infty$–dimensional well–posed $\ell_2$ problem

(III) restriction to uniform grids: CG method for control with inner iterations to update state

$\rightsquigarrow$ set up control problem in wavelet coordinates

$H$ Hilbert space on domain $\Omega \subset \mathbb{R}^d$ with $\|\cdot\|_H$ $\qquad$ $H'$ dual space for $H$ with $\langle\cdot,\cdot\rangle$

$\Psi := \{\psi_\lambda : \lambda \in \mathbb{I}\} \subset H$ Wavelets $\qquad$ $\mathbb{I}$ (infinite) index set

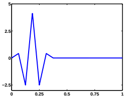
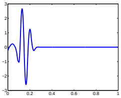
(NE) $\Psi$ Riesz basis for $H$

$$v \in H: \quad v = \mathbf{v}^T\Psi := \sum_{\lambda\in\mathbb{I}} \langle v, \tilde{\psi}_\lambda\rangle\, \psi_\lambda \qquad \text{such that} \qquad \|v\|_H \sim \|\mathbf{v}\|_{\ell_2(\mathbb{I})}$$

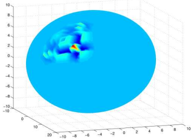
(L) Locality $\qquad \mathrm{diam}\,(\mathrm{supp}\,\psi_\lambda) \sim 2^{-|\lambda|}$ $\qquad$ $|\lambda|$ resolution

$\psi_\lambda$ centered around $2^{-|\lambda|}k$

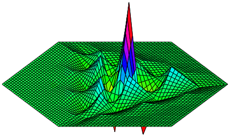
(CP) Vanishing moments $\langle v, \psi_\lambda\rangle \lesssim 2^{-|\lambda|(\frac{d}{2}+\tilde{m})}\, \|v^{(\tilde{m})}\|_{L_\infty(\mathrm{supp}\ \psi_\lambda)}$ for some $\tilde{m}$

[Dahmen, Kunoth, Urban '99]



[Dahmen, Schneider '99], [Kunoth, Sahner '06]



[Harbrecht, Schneider '00]

## Modelling in Wavelet Coordinates: Norms in Cost Functional

$$J(y,u) = \tfrac{1}{2}\|y - y_*\|^2_{H^{1-s}(\Omega)} + \tfrac{\omega}{2}\|u\|^2_{(H^{1-t}(\Omega))'} \qquad 0 \le s \le 1,\ 0 \le t$$

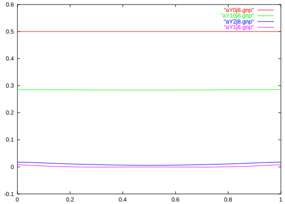
Example: $\min J(y,u)$ subject to $\begin{cases} -y'' + y = 1 + u & \text{in } (0,1) \qquad y_* = 0 \qquad \omega = 1 \\ \frac{dy}{dn} = 0 & \text{at } 0,1 \end{cases}$

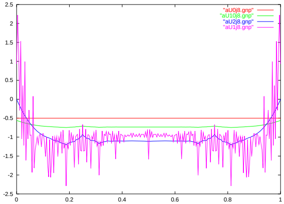
Representer in wavelet coordinates

(without Riesz operators for $\|v\|_Z^2 \sim \|\mathbf{R}_Z^{1/2}\mathbf{v}\|$ and $\|\cdot\|_U \qquad \|\cdot\| = \|\cdot\|_{\ell_2}$):

$$\mathbf{J}(\mathbf{y},\tilde{\mathbf{u}}) = \tfrac{1}{2}\|\mathbf{D}^{-s}(\mathbf{y} - \mathbf{y}_*)\|^2 + \tfrac{\omega}{2}\|\mathbf{D}^{t}\,\tilde{\mathbf{u}}\|^2 \qquad 0 \le s \le 1,\ 0 \le t$$

$\rightsquigarrow \qquad \min \mathbf{J}(\mathbf{y},\tilde{\mathbf{u}})$ subject to $\mathbf{A}\mathbf{y} = \mathbf{f} + \tilde{\mathbf{u}}$




state $y$          control $u$

| | | | | |
|---|---|---|---|---|
| — (red) | $s = 1$ | $L_2$ | aY/U0n8.gnp | $t = s$ |
| — (green) | $s = 0.9$ | $H^{1/10}$ | aY/U10n8.gnp | |
| — (blue) | $s = 1/2$ | $H^{1/2}$ | aY/U2n8.gnp | |
| — (magenta) | $s = 0$ | $H^1$ | aY/U1n8.gnp | |

Minimize

$$\mathbf{J}(\mathbf{y}, \tilde{\mathbf{u}}) \;=\; \tfrac{1}{2}\,\|\mathbf{R}_Z^{1/2}\mathbf{D}^{-s}(\mathbf{y} - \mathbf{y}_*)\|^2 + \tfrac{\omega}{2}\,\|\mathbf{R}_U^{1/2}\mathbf{D}^{t}\,\tilde{\mathbf{u}}\|^2 \qquad 0 \le s \le 1, 0 \le t \tag{3}$$

subject to

$$\mathbf{A}\mathbf{y} = \mathbf{f} + \tilde{\mathbf{u}} \tag{4}$$

$$\mathbf{A} : \ell_2 \to \ell_2 \text{ automorphism} \qquad \|\cdot\| := \|\cdot\|_{\ell_2}$$

Necessary (and Sufficient Conditions) for Optimality

$$\mathbf{Lagr}(\mathbf{y}, \mathbf{u}, \mathbf{p}) \;:=\; \mathbf{J}(\mathbf{y}, \mathbf{u}) + \left\langle \mathbf{p},\; \mathbf{A}\mathbf{y} - (\mathbf{f} + \mathbf{D}^{-t}\,\mathbf{u}) \right\rangle \qquad \text{and} \qquad \delta\mathbf{Lagr} = 0 \;\rightsquigarrow$$

$$\boxed{\begin{aligned} \mathbf{A}\mathbf{y} &= \mathbf{f} + \mathbf{D}^{-t}\,\mathbf{u} \\ \mathbf{A}^T\mathbf{p} &= -\,\mathbf{D}^{-s}\mathbf{R}_Z\mathbf{D}^{-s}(\mathbf{y} - \mathbf{y}_*) \qquad (5) \\ \omega\mathbf{R}_U\mathbf{u} &= \mathbf{D}^{-t}\,\mathbf{p} \end{aligned}} \qquad \Longleftrightarrow \qquad \mathbf{Q}\,\mathbf{u} = \mathbf{g} \qquad (6)$$

$$\mathbf{Q} : \ell_2 \to \ell_2 \quad \text{automorphism}$$

where

$$\mathbf{Q} \;:=\; \mathbf{D}^{-t}\mathbf{A}^{-T}\mathbf{D}^{-s}\mathbf{R}_Z\mathbf{D}^{-s}\mathbf{A}^{-1}\mathbf{D}^{-t} + \omega\mathbf{R}_U \qquad \text{symmetric positive definite}$$

$$\mathbf{g} \;:=\; \mathbf{D}^{-t}\mathbf{A}^{-T}\mathbf{D}^{-s}\mathbf{R}_Z\mathbf{D}^{-s}(\mathbf{y}_* - \mathbf{A}^{-1}\mathbf{f})$$

$\mathbf{Q}$ should not be realized by setting up and inverting $\mathbf{A}$ explicitly !

Condensed form (6) useful for deriving a convergent numerical scheme — but realization done through extended form (5)

Recall: on uniform grids, stiffness matrix $\mathbf{A}$ should not be explicitly set up in wavelet basis $\rightsquigarrow$ use fast wavelet transform instead

## A Nested Iteration-Inexact CG (NIICG) Algorithm

In this chapter from now on: uniform grids with highest grid level $J$ (amount of unknowns here shortly $N$) and $j_0$ coarsest level of resolution;
consider coupled system (5) and condensed system (6)

### A Basic Conjugate Gradient (CG) Method

Consider linear system

$$\mathbf{M}\mathbf{q} = \mathbf{z}, \quad \mathbf{M} \in \mathbb{R}^{N\times N} \text{ symmetric positive definite}, \quad c_{\mathbf{M}}\|\mathbf{v}\| \le \|\mathbf{M}\mathbf{v}\| \le C_{\mathbf{M}}\|\mathbf{v}\|, \quad \mathbf{v} \in \mathbb{R}^N, \tag{7}$$

with given right hand side $\mathbf{z} \in \mathbb{R}^N$, $\|\cdot\| := \|\cdot\|_{\ell_2(\Delta_J)}$, constants $0 < c_{\mathbf{M}} \le C_{\mathbf{M}} < \infty$;

denote residual using an approximation $\tilde{\mathbf{q}}$ to $\mathbf{q}$ for (7) by $\mathrm{RES}\,(\tilde{\mathbf{q}}) := \mathbf{M}\tilde{\mathbf{q}} - \mathbf{z}$

Idea: employ a basic conjugate gradient (CG) method that iteratively computes approximate solution $\mathbf{q}_K$ to (7) with given initial vector $\mathbf{q}_0$ and given tolerance $\varepsilon > 0$ such that

$$\|\mathbf{M}\mathbf{q}_K - \mathbf{z}\| = \|\mathrm{RES}\,(\mathbf{q}_K)\| \le \varepsilon, \tag{8}$$

where $K$ is number of iterations
(later: $\varepsilon$ specified depending on discretization tolerance for (7)

Scheme $\mathrm{CG}$ below contains routine

$\mathrm{APPLY}\,(\eta_k, \mathbf{M}, \mathbf{d}_k)$ (9):
for $\mathbf{M} = \mathbf{A}, \mathbf{A}^T$ is simply matrix–vector multiplication $\mathbf{M}\mathbf{d}_k$;
otherwise, it approximately computes $\mathbf{M}\mathbf{d}_k$ up to tolerance $\eta_k = \eta_k(\varepsilon)$ depending on $\varepsilon$

$$\mathrm{CG}\,[\varepsilon, \mathbf{q}_0, \mathbf{M}, \mathbf{z}] \to \mathbf{q}_K \tag{10}$$

(I) SET $\mathbf{d}_0 := \mathbf{z} - \mathbf{M}\mathbf{q}_0$ AND $\mathbf{r}_0 := -\mathbf{d}_0$. LET $k = 0$.

(II) WHILE $\|\mathbf{r}_k\| > \varepsilon$

$$\begin{array}{lcllcl}
\mathbf{m}_k & := & \mathrm{APPLY}\,(\eta_k(\varepsilon), \mathbf{M}, \mathbf{d}_k) & \alpha_k & := & \dfrac{(\mathbf{r}_k)^T\mathbf{r}_k}{(\mathbf{d}_k)^T\mathbf{m}_k} \\
\mathbf{q}_{k+1} & := & \mathbf{q}_k + \alpha_k\mathbf{d}_k & \mathbf{r}_{k+1} & := & \mathbf{r}_k + \alpha_k\mathbf{m}_k \\
\beta_k & := & \dfrac{(\mathbf{r}_{k+1})^T\mathbf{r}_{k+1}}{(\mathbf{r}_k)^T\mathbf{r}_k} & \mathbf{d}_{k+1} & := & -\mathbf{r}_{k+1} + \beta_k\mathbf{d}_k \\
k := k+1 & & & & &
\end{array}$$

(III) SET $K := k - 1$.

Note: Routine $\mathrm{CG}$ computes residual up to stopping criterion $\varepsilon$;
error in solution is multiplied by $\|\mathbf{M}^{-1}\| = c_{\mathbf{M}}^{-1}$ $\leadsto$

$$\text{error:} \quad \|\mathbf{q} - \mathbf{q}_K\| = \|\mathbf{M}^{-1}(\mathbf{z} - \mathbf{M}\mathbf{q}_K)\| \le \|\mathbf{M}^{-1}\|\,\|\mathrm{RES}\,(\mathbf{q}_K)\| \le \varepsilon\, c_{\mathbf{M}}^{-1} \tag{11}$$

Next: design routine CG for condensed system (6) involving

APPLY for $\mathbf{M}$ replaced by $\mathbf{Q}$ ( $= \mathbf{D}^{-t}\mathbf{A}^{-T}\mathbf{D}^{-s}\mathbf{R}_Z\mathbf{D}^{-s}\mathbf{A}^{-1}\mathbf{D}^{-t} + \omega\mathbf{R}_U$) (symmetric positive definite)

and right hand side $\mathbf{z}$ replaced by $\mathbf{g}$ ( $= \mathbf{D}^{-t}\mathbf{A}^{-T}\mathbf{D}^{-s}\mathbf{R}_Z\mathbf{D}^{-s}(\mathbf{y}_* - \mathbf{A}^{-1}\mathbf{f})$)

## Approximate Right Hand Sides

Approximate computation of $\mathbf{g}$ by applying interior cg iterations up to stopping criterion $\zeta$:

$$\textsc{Rhs}\,[\zeta, \mathbf{A}, \mathbf{f}, \mathbf{y}_*] \to \mathbf{g}_\zeta \tag{12}$$

$$\begin{aligned}
&\text{(I)} \quad \mathrm{CG}\,[\tfrac{c_\mathbf{A}}{2C}\,\tfrac{c_\mathbf{A}}{C^2 C_0^2}\,\zeta, \mathbf{0}, \mathbf{A}, \mathbf{f}] \to \mathbf{g_1}\\
&\text{(II)} \quad \mathrm{CG}\,[\tfrac{c_\mathbf{A}}{2C}\,\zeta, \mathbf{0}, \mathbf{A}^\mathbf{T}, -\mathbf{D}^{-\mathbf{s}}\mathbf{R_Z}\mathbf{D}^{-\mathbf{s}}(\mathbf{g_1} - \mathbf{y}_*)] \to \mathbf{g_2}\\
&\text{(III)} \quad \mathbf{g}_\zeta := \mathbf{D}^{-t}\mathbf{g}_2.
\end{aligned}$$

Note: tolerances used within the two conjugate gradient methods depend on constants $c_\mathbf{A}$, $C$, $C_0$ from mapping property of $\mathbf{A}$, bounds $C$ of diagonal operators $\mathbf{D}^{-s}, \mathbf{D}^{-t}$ and bounds $C_0$ for $L_2$ Riesz operator.

Note also: the additional factor $c_\mathbf{A}(CC_0)^{-2}$ in stopping criterion in step (I) in comparison to step (II) is in general smaller than one

$\rightsquigarrow$ primal system needs to be solved more accurately than adjoint system in step (II)

### Proposition

The result $\mathbf{g}_\zeta$ of $\textsc{Rhs}\,[\zeta, \mathbf{A}, \mathbf{f}, \mathbf{y}_*]$ satisfies upon completion

$$\|\mathbf{g}_\zeta - \mathbf{g}\| \le \zeta. \tag{13}$$

Proof: See step by step definition of different right hand sides and employ bounds on operators. □

For computation of approximation $\mathbf{m}_\eta$ to matrix–vector product $\mathbf{Qd}$, employ this routine which needs in last step an appropriate approximation for $\mathbf{g}$ (with $\mathbf{D}_U^{-1} = \mathbf{D}^{-t}, \mathbf{D}_Z^{-1} = \mathbf{D}^{-s}$)

$$\text{APPLY}\,[\eta, \mathbf{Q}, \mathbf{d}] \to \mathbf{m}_\eta \tag{14}$$

$$\text{(I)}\quad \text{CG}\,[\tfrac{c_{\mathbf{A}}}{3C}\,\tfrac{c_{\mathbf{A}}}{C^2C_0^2}\,\eta, \mathbf{0}, \mathbf{A}, \mathbf{f} + \mathbf{D}_{\mathbf{U}}^{-1}\mathbf{d}] \to \mathbf{y}_\eta$$

$$\text{(II)}\quad \text{CG}\,[\tfrac{c_{\mathbf{A}}}{3C}\,\eta, \mathbf{0}, \mathbf{A}^{\mathbf{T}}, -\mathbf{D}_{\mathbf{Z}}^{-1}\mathbf{R}_{\mathbf{Z}}\mathbf{D}_{\mathbf{Z}}^{-1}(\mathbf{y}_\eta - \mathbf{y}_*)] \to \mathbf{p}_\eta$$

$$\text{(III)}\quad \mathbf{m}_\eta := \mathbf{g}_{\eta/3} + \omega\mathbf{R}_U\mathbf{d} - \mathbf{D}_U^{-1}\mathbf{p}_\eta.$$

Note: tolerances differ only slightly from those the routine RHS, although the ratio between the tolerances in step (I) and (II) is the same, namely, $c_{\mathbf{A}}(CC_0)^{-2}$ (reason can be seen in proof of following result)

Proposition The result $\mathbf{m}_\eta$ of APPLY $[\eta, \mathbf{Q}, \mathbf{d}]$ satisfies

$$\|\mathbf{m}_\eta - \mathbf{Qd}\| \le \eta. \tag{15}$$

Proof: Confirm that choice of the stopping criteria in steps (I) and (II) indeed yields (15): Denote by $\mathbf{y_d}$ exact solution of primal equation in (5) with $\mathbf{d}$ in place of $\mathbf{u}$ on right hand side, and by $\mathbf{p_d}$ exact solution of adjoint equation in (5) with $\mathbf{y_d}$ on the right hand side
$\rightsquigarrow$ step (III) and $\mathbf{Qu} - \mathbf{g} = \omega\mathbf{R}_U\mathbf{u} - D^{-t}\mathbf{p}$ combined with error bounds on Riesz operators and diagonal matrices yield

$$\begin{aligned} \|\mathbf{m}_\eta - \mathbf{Qd}\| &= \|\mathbf{g}_{\eta/3} - \mathbf{g} + \omega\mathbf{R}_U\mathbf{d} - \mathbf{D}_U^{-1}\mathbf{p}_\eta \;- (\mathbf{Qd} - \mathbf{g})\| \\ &\le \tfrac{1}{3}\eta + \|\omega\mathbf{R}_U\mathbf{d} - \mathbf{D}_U^{-1}\mathbf{p}_\eta - (\omega\mathbf{R}_U\mathbf{d} - \mathbf{D}_U^{-1}\mathbf{p_d})\| \\ &\le \tfrac{1}{3}\eta + C\|\mathbf{p_d} - \mathbf{p}_\eta\| \qquad (16) \end{aligned}$$

(Continuation of proof)

For $\hat{\mathbf{p}}$ exact solution of adjoint equation with $\mathbf{y}_\eta$ on right hand side, we have

$$\mathbf{p_d} - \hat{\mathbf{p}} = -\mathbf{A}^{-T}\mathbf{D}_Z^{-1}\mathbf{R}_Z\mathbf{D}_Z^{-1}(\mathbf{y_d} - \mathbf{y}_\eta) \tag{17}$$

$\rightsquigarrow$ by mapping property of **A** and by error bounds on Riesz operators and diagonal matrices we have

$$\|\mathbf{p_d} - \hat{\mathbf{p}}\| \leq \frac{C^2 C_0^2}{c_{\mathbf{A}}} \|\mathbf{y_d} - \mathbf{y}_\eta\| \leq \frac{1}{3C}\eta, \tag{18}$$

where last estimate follows by the choice of the threshold in step (I).
Combining (16) and (18) together with (13) and stopping criterion in step (II) $\rightsquigarrow$

$$\begin{aligned} \|\mathbf{m}_\eta - \mathbf{Qd}\| &\leq \tfrac{1}{3}\eta + C\left(\|\mathbf{p_d} - \hat{\mathbf{p}}\| + \|\hat{\mathbf{p}} - \mathbf{p}_\eta\|\right) \\ &\leq \tfrac{1}{3}\eta + C\left(\tfrac{1}{3C}\eta + \tfrac{1}{3C}\eta\right) = \eta \end{aligned}$$

$\square$

Note: Effect of applications of **M** in CG and more general Krylov subspace schemes with respect to convergence investigated in numerical linear algebra context by e.g. [van den Eshof, Sleijpen 2004]; for system $\mathbf{Qu} = \mathbf{g}$: difference between actually computed residual $\mathbf{r}_k$ in $\mathrm{CG}\,[\varepsilon, \mathbf{q}_0, \mathbf{Q}, \mathbf{g}]$ and $\mathrm{RES}\,(\mathbf{u}_k) = \mathbf{Qu}_k - \mathbf{g}$ can be estimated as

$$\|\mathbf{r}_k - \mathrm{RES}\,(\mathbf{u}_k)\| \leq C_{\mathbf{Q}} \sum_{i=0}^{k-1} \eta_i |\alpha_i|\, \|\mathbf{d}_i\|,$$

where $C_{\mathbf{Q}}$ is upper bound for **Q** and $\alpha_i, \mathbf{d}_i$ defined in (10) $\rightsquigarrow$ in principle, one could choose thresholds $\eta_i = \eta_i(\varepsilon)$ for inner iterations $\mathrm{APPLY}\,(\eta_i, \mathbf{Q}, \mathbf{d}_i)$ as $\eta_i = \varepsilon(|\alpha_i|\, \|\mathbf{d}_i\|)^{-1}$

## A Nested-Iteration-Inexact-Conjugate-Gradient (NIICG) Algorithm

Now combine inexact CG solver (10) for $\mathbf{Q}$ with nested iteration:

$\mathrm{NIICG}\,[\mathbf{f}, \mathbf{y}_*, J] \to \mathbf{u}^J$ (19)

(I) Initialization for coarsest level $j := j_0$

(1) Compute right hand side $\mathbf{g}^{j_0}$ by QR decomposition of $\mathbf{A}^{j_0}$ using (6)

(2) Compute solution $\mathbf{u}^{j_0}$ of (6) by QR decomposition of $\mathbf{Q}^{j_0}$

(II) While $j < J$

(1) Prolongate $\mathbf{u}^j \to \mathbf{u}_0^{j+1}$ by padding with zeros, set $j := j+1$.

(2) Compute right hand side using $\mathrm{RHS}\ [\nu\, 2^{-(m-1)j}, \mathbf{A}, \mathbf{f}^j, \mathbf{y}_*^j] \to \mathbf{g}^j$.

(3) Compute solution of (6) using $\mathrm{CG}\ [\nu\, 2^{-(m-1)j}, \mathbf{u}_0^j, \mathbf{Q}, \mathbf{g}^j] \to \mathbf{u}^j$.

Recall: step (II.3) requires multiple calls of $\mathrm{APPLY}\ [\eta, \mathbf{Q}, \mathbf{d}]$, which in turn invokes both $\mathrm{CG}\ [\ldots, \mathbf{A}, \ldots]$ as well as $\mathrm{CG}\ [\ldots, \mathbf{A}^T, \ldots]$ in each application

Note: thresholds in steps (II.2) and (II.3) chosen proportional to a-priori error estimate in energy norm (represented by $\|\cdot\|_{\ell_2}$) $2^{-(m-1)j}$ for B-spline wavelets of exactness $m$;

prolongation by padding with zeros since wavelet coefficients correspond to detail coefficients

# A Nested-Iteration-Inexact-Conjugate-Gradient (NIICG) Algorithm

## Theorem

Residual $\mathbf{Qu} - \mathbf{g}$ computed on each level $j$ up to discretization error proportional to $2^{-(m-1)j}$ and corresponding solutions are taken as initial guesses for next higher level

$\Longrightarrow$ NIICG is asymptotically optimal method: it provides the solution $\mathbf{u}^J$ up to discretization error on level $J$ in overall amount of $\mathcal{O}(N_J)$ arithmetic operations.

Note: Result follows since finite versions of $\mathbf{A}$ and $\mathbf{Q}$ have uniformly bounded condition numbers; remainder of argumentation follows as in Part II by geometric series argument

## Numerical Results

| $j$ | $\|\mathbf{r}_K^j\|$ | #O | #E | #A | #R | $\|R(\mathbf{y}^J)-\mathbf{y}^j\|$ | $\|\mathbf{y}^J-P(\mathbf{y}^j)\|$ | $\|R(\mathbf{u}^J)-\mathbf{u}^j\|$ | $\|\mathbf{u}^J-P(\mathbf{u}^j)\|$ |
|---|---|---|---|---|---|---|---|---|---|
| 3 | | | | | | 6.86e-03 | 1.48e-02 | 1.27e-04 | 4.38e-04 |
| 4 | 1.79e-05 | 5 | 12 | 5 | 8 | 2.29e-03 | 7.84e-03 | 4.77e-05 | 3.55e-04 |
| 5 | 1.98e-05 | 5 | 14 | 6 | 9 | 6.59e-04 | 3.94e-03 | 1.03e-05 | 2.68e-04 |
| 6 | 4.92e-06 | 7 | 13 | 5 | 9 | 1.74e-04 | 1.96e-03 | 2.86e-06 | 1.94e-04 |
| 7 | 3.35e-06 | 7 | 12 | 5 | 9 | 4.55e-05 | 9.73e-04 | 9.65e-07 | 1.35e-04 |
| 8 | 2.42e-06 | 7 | 11 | 5 | 10 | 1.25e-05 | 4.74e-04 | 7.59e-07 | 8.88e-05 |
| 9 | 1.20e-06 | 8 | 11 | 5 | 10 | 4.55e-06 | 2.12e-04 | 4.33e-07 | 5.14e-05 |
| 10 | 4.68e-07 | 9 | 10 | 5 | 9 | 3.02e-06 | 3.02e-06 | 2.91e-07 | 2.91e-07 |

Iteration history for two-dimensional distributed control problem with Neumann boundary conditions, $\omega = 1$, $\mathcal{Z} = H^1(\Omega)$, $\mathcal{U} = (H^{0.5}(\Omega))'$
stopping criterion for outer iteration (relative to $\|\cdot\|$ corresponding to energy norm) on level $j$ chosen proportional to $2^{-j}$
#E: maximum number of inner iterations for primal system, for adjoint system (#A) and design equation (#R)