CIME–EMS Summer School "Splines and PDEs", 2017

Adaptive Multiscale Methods for the Numerical Treatment of Systems of PDEs

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Sketch of Contents

- Elliptic and parabolic partial differential equations (PDEs) in weak form; regularity of solutions
- Control problems constrained by elliptic and parabolic PDEs
- Numerical approximations of solutions on uniform and non-uniform/adaptive grids
- Concepts of multiscale methods and adaptivity; convergence proofs and complexity estimates
- Realization of these concepts by B-spline-wavelets
- Fast solvers: multilevel preconditioning; implementation issues

Literature: see References in notes_kunoth.pdf

Problem Setup

Elliptic PDE of order 2 on domain Ω: $-\Delta u = f \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0$

Weak operator form: for given $f \in H^{-1}(\Omega)$, find $u \in H_0^1(\Omega)$ such that

$$\boxed{Au = f \quad \text{in} \quad H^{-1}(\Omega)}$$

Elliptic operator A defined by $\langle Av, w \rangle := a(v, w)$ symmetric, continuous

and coercive on $H_0^1(\Omega)$: $\quad \|Av\|_{H^{-1}(\Omega)} \sim \|v\|_{H^1(\Omega)}$

Discretization on uniform grid: $V_h \subset H_0^r(\Omega) \qquad \dim V_h < \infty \qquad \rightsquigarrow \qquad \boxed{A_h \, u_h = f_h \qquad (*)}$

$0 < h < 1$ grid size

Target:

Realize discretization error accuracy $\varepsilon \sim h^{p+1} \sim 2^{-(p+1)J}$ for grid with spacing $h \sim 2^{-J}$

Problem complexity: For $h \sim 2^{-J}$ a total of $N \sim 2^{Jd}$ unknowns

Optimal complexity for iterative solver: Minimal amount of work is $\mathcal{O}(N)$

Multilevel Preconditioner

Asymptotically optimal preconditioner: C_h such that

$$\mathrm{cond}_2(C_h A_h) \sim 1$$

and setup and application of C_h in optimal linear complexity $\mathcal{O}(N)$

Schwarz iterative schemes based on subspace corrections

$\rightsquigarrow$ Multilevel schemes yielding optimal preconditioners:

- Multiplicative schemes $\rightsquigarrow$ multigrid methods
  Brandt, Braess, Bramble, Hackbusch, Zulehner ...
  IgA: Gahalaut, Kraus, Tomar ...
- Additive schemes $\rightsquigarrow$ BPX preconditioner; wavelet discretization
  Bramble, Pasciak, Xu, Yserentant, Oswald, Dahmen, Kunoth ...

  Relevant idea from Approximation Theory: Multilevel characterization of function spaces and norm equivalences

Not optimal are preconditioners based on domain decomposition, overlapping Schwarz, hierarchical basis preconditioners... Beirao da Veiga, Cho, Pavarino. Scacci, Kleiss, Pechstein, Jüttler, Langer ...

$V_h \longleftrightarrow V_j$ uniform grid with grid spacing $h \sim 2^{-j}$ j resolution level

Multiresolution $V_{j_0} \subset V_{j_0+1} \subset \ldots \subset V_j \subset V_{j+1} \subset \ldots H_0^r(\Omega)$

$$\mathrm{clos}_{H^r(\Omega)} \left(\bigcup_{j=j_0}^{\infty} V_j \right) = H_0^r(\Omega)$$

Linear orthogonal projectors $Q_j : H_0^r(\Omega) \to V_j$ s.th. $Q_j Q_\ell = Q_j$ for $j \leq \ell \rightsquigarrow Q_j - Q_{j-1}$ projector

Corollary

(S) Φ_j uniformly stable basis for V_j: $\|\mathbf{c}\|_{\ell_2} \sim \|\mathbf{c}^T \Phi_j\|_{L_2(\Omega)}$

(J) Jackson estimate

$$\inf_{v_j \in V_j} \|v - v_j\|_{L_2(\Omega)} \lesssim 2^{-sj} \|v\|_{H^s(\Omega)} \quad v \in H^s(\Omega) \quad 0 < s \leq \delta$$

(B) Bernstein inequality $\|v_j\|_{H^s(\Omega)} \lesssim 2^{sj} \|v_j\|_{L_2(\Omega)} \quad v_j \in V_j \quad s < \tau$

$\Longrightarrow$ Norm equivalence

$$\text{(NE)} \qquad \|v\|_{H^s(\Omega)}^2 \sim \sum_{j=j_0}^{J} 2^{2sj} \|(Q_j - Q_{j-1})v\|_{L_2(\Omega)}^2 \quad v \in V_J \quad s \in (-\tilde{\sigma}, \sigma)$$

Norm Equivalence for Optimal Preconditioning

Corollary: For $H_0^r(\Omega)$ $\qquad C_J^{-1} := A_{j_0} Q_{j_0} + \sum_{j=j_0}^{J} 2^{2rj}(Q_j - Q_{j-1})$

is optimal preconditioner for $A_J : V_J \to V_J$: $\qquad \mathrm{cond}_2(C_J^{1/2} A_J C_J^{1/2}) \sim 1 \quad$ as $J \to \infty$

Realization of C_J^{-1} by wavelets:

For any $s \in (-\tilde{\sigma}, \sigma)$:
Explicit representation of difference $(Q_j - Q_{j-1})v$ in terms of wavelet basis together with diagonal $D_s := (2^{sj})_{j=j_0 \dots J}$

$\leadsto$ Fast Wavelet Preconditioner (FWT) realizes preconditioning in optimal linear complexity

[Jaffard '92], [Dahmen, Kunoth '92]

Construction of FWT preconditioner:

Multiresolution of solution space $H_0^1(\Omega) \leadsto$
nestedness $V_j \subset V_{j+1}$ implies existence of matrix $\mathbf{M}_{j,0}$ such that $\Phi_j = \mathbf{M}_{j,0}^T \Phi_{j+1}$

For some complement W_{j+1} of V_j in V_{j+1}, there exists basis called wavelet-basis Ψ_j
and matrix $\mathbf{M}_{j,1}$ such that $\Psi_j = \mathbf{M}_{j,1}^T \Phi_{j+1}$

Two-scale transforms: $\mathbf{M}_j$ performs a change of bases in the space V_{j+1}:

$$\begin{pmatrix} \Phi_j \\ \Psi_j \end{pmatrix} = \begin{pmatrix} \mathbf{M}_{j,0}^T \\ \mathbf{M}_{j,1}^T \end{pmatrix} \Phi_{j+1} =: \mathbf{M}_j^T \Phi_{j+1}$$

Construction of FWT preconditioner

Two-scale transforms: $\mathbf{M}_j$ performs a change of bases in the space V_{j+1}:

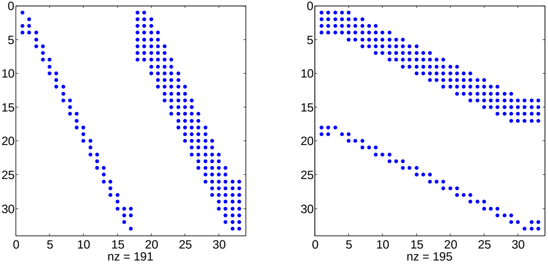
$$\begin{pmatrix}\Phi_j\\ \Psi_j\end{pmatrix} = \begin{pmatrix}\mathbf{M}_{j,0}^T\\ \mathbf{M}_{j,1}^T\end{pmatrix}\Phi_{j+1} =: \mathbf{M}_j^T\Phi_{j+1}$$

Conversely: there exists $\mathbf{G}_j$ such that reconstruction identity holds:

$$\Phi_{j+1} = \mathbf{G}_j^T\begin{pmatrix}\Phi_j\\ \Psi_j\end{pmatrix} = \mathbf{G}_{j,0}^T\Phi_j + \mathbf{G}_{j,1}^T\Psi_j \qquad \text{where } \mathbf{G}_j := \mathbf{M}_j^{-1}$$

Important for efficiency: $\mathbf{M}_j$ and $\mathbf{G}_j$ uniformly sparse

Example of the structure of the matrices $\mathbf{M}_j$ and $\mathbf{G}_j$:



Nonzero pattern of matrices $\mathbf{M}_j$ (left) and $\mathbf{G}_j$ (right) for boundary-adapted B-splines of order $m = 2$ as generators and duals of order $\tilde{m} = 4$ (providing norm equivalences (NE) for $H^s(0,1)$ for $s \in (-3/2, 3/2)$)

Basis Changes

Fix finest resolution level J and repeat function space decomposition $\rightsquigarrow$

for every $v \in V_J$:

single–scale representation $\quad v = (\mathbf{c}_J)^T \Phi_J = \sum_{k \in \Delta_J} c_{J,k} \phi_{J,k}$

multi–scale representation $\quad v = (\mathbf{c}_{j_0})^T \Phi_{j_0} + (\mathbf{d}_{j_0})^T \Psi_{j_0} + \cdots + (\mathbf{d}_{J-1})^T \Psi_{J-1}$

with respect to the multiscale or wavelet basis $\quad \Psi^J := \Phi_{j_0} \cup \bigcup_{j=j_0}^{J-1} \Psi_j =: \bigcup_{j=j_0-1}^{J-1} \Psi_j$

Both representations useful $\rightsquigarrow$

Wavelet Transform $\quad \mathbf{T}_J : \ell_2(\Delta_J) \to \ell_2(\Delta_J), \qquad \mathbf{d}^J \mapsto \mathbf{c}_J \qquad \mathbf{d}^J := (\mathbf{c}_{j_0}, \mathbf{d}_{j_0}, \ldots, \mathbf{d}_{J-1})^T$

$$\rightsquigarrow \qquad \mathbf{T}_J = \mathbf{T}_{J,J-1} \cdots \mathbf{T}_{J,j_0} \quad \text{where} \quad \mathbf{T}_{J,j} := \begin{pmatrix} \mathbf{M}_j & \mathbf{0} \\ \mathbf{0} & \mathbf{I}^{(\#\Delta_J - \#\Delta_{j+1})} \end{pmatrix} \in \mathbb{R}^{(\#\Delta_J) \times (\#\Delta_J)}$$

Theorem: $\quad \mathbf{M}_j$ (and $\mathbf{G}_j$) uniformly sparse

$\rightsquigarrow \quad \mathbf{T}_J$ (and inverse $\mathbf{T}_J^{-1}$) can be applied in $\mathcal{O}(N_J)$ arithmetic operations

(optimal complexity)

Fast Wavelet Transform (FWT)

Recall: Fast Fourier Transform (FFT) needs $\mathcal{O}(N_J \log N_J)$ arithmetic operations

$\mathbf{T}_J$ (and inverse $\mathbf{T}_J^{-1}$) should not be set up explicitly . . . . . . instead

Pyramid Scheme for Realizing Fast Wavelet Transform in $\mathcal{O}(N_J)$ Operations

$\mathbf{T}_J : \ell_2(\Delta_J) \to \ell_2(\Delta_J)$

$$\mathbf{T}_J \mathbf{d}^J = \mathbf{c}_J$$

$$\text{with } \mathbf{T}_J := \mathbf{T}_{J,J-1} \cdots \mathbf{T}_{J,j_0} \text{ and } \mathbf{T}_{J,j} := \begin{pmatrix} \mathbf{M}_j & \mathbf{0} \\ \mathbf{0} & \mathbf{I}^{(\#\Delta_J - \#\Delta_{j+1})} \end{pmatrix} \in \mathbb{R}^{(\#\Delta_J)\times(\#\Delta_J)}$$

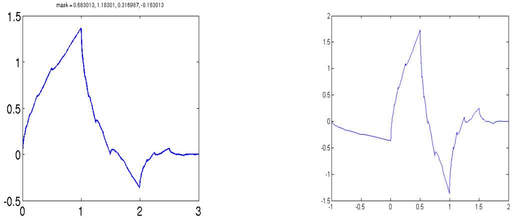
$$\begin{tikzcd}
\mathbf{c}_{j_0} \arrow[r, "\mathbf{M}_{j_0,0}"] & \mathbf{c}_{j_0+1} \arrow[r, "\mathbf{M}_{j_0+1,0}"] & \mathbf{c}_{j_0+2} \arrow[r] & \cdots & \mathbf{c}_{J-1} \arrow[r, "\mathbf{M}_{J-1,0}"] & \mathbf{c}_J \\
\mathbf{d}_{j_0} \arrow[ur, "\mathbf{M}_{j_0,1}"] & \mathbf{d}_{j_0+1} \arrow[ur, "\mathbf{M}_{j_0+1,1}"] & \mathbf{d}_{j_0+2} \arrow[ur] & \cdots & \mathbf{d}_{J-1} \arrow[ur, "\mathbf{M}_{J-1,1}"] &
\end{tikzcd}$$

j	$\kappa_2(\mathbf{T}_{\text{DKU}})$	$\kappa_2(\mathbf{T}_{\text{B}})$
4	4.743e+00	4.640e+00
5	6.221e+00	6.024e+00
6	8.154e+00	6.860e+00
7	9.473e+00	7.396e+00
8	1.023e+01	7.707e+00
9	1.064e+01	7.876e+00
10	1.086e+01	7.965e+00

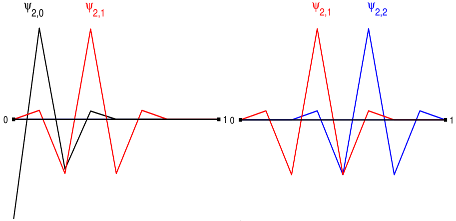
j	$\kappa_2(\mathbf{T}_{\text{DKU}})$	$\kappa_2(\mathbf{T}_{\text{B}})$
11	1.097e+01	8.011e+00
12	1.103e+01	8.034e+00
13	1.106e+01	8.046e+00
14	1.107e+01	8.051e+00
15	1.108e+01	8.054e+00
16	1.108e+01	8.056e+00

Computed spectral condition numbers for the Fast Wavelet Transform on $[0, 1]$ for different constructions of biorthogonal spline-wavelets on the interval [Dahmen, Kunoth, Urban, 1999] and [Burstedde, Dissertation, 2006]; results taken from [Pabel, Diploma Thesis, 2005]

Plots of generators Φ_j and wavelets Ψ_j



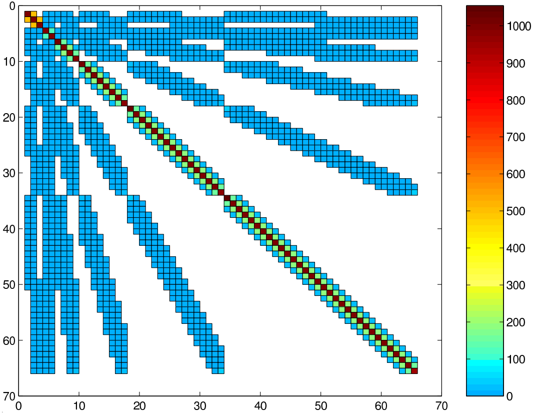
Daubechies D4 generator and wavelet (support [-1,2] and 2 vanishing moments for the wavelet) [https://www.mathematik.uni-marburg.de/∼waveletsoft/]



biorthogonal spline-wavelet for $m = 2$ and $\tilde{m} = 4$; generated by piecewise linear B-Splines as primals (providing norm equivalences (NE) for $H^s(0,1)$ for $s \in (-3/2, 3/2)$)

Preconditioning by Wavelets

Important: on uniform grids, stiffness matrix should not be explicitly set up in wavelet basis; set-up in wavelet basis leads to matrix with $\mathcal{O}(N_J \log N_J)$ entries and exhibits finger band structure



(stiffness matrix in wavelet basis for 1D problem)

[Castano '05]

Preconditioning with Fast Wavelet Transform

Application to elliptic PDE $\rightsquigarrow$

Theorem: $\mathbf{A}_J := \mathbf{D}_J^{-1}\,\mathbf{T}_J^T\,\langle \Phi_J, A\Phi_J\rangle\,\mathbf{T}_J\,\mathbf{D}_J^{-1}$

has uniformly bounded condition numbers independent of J with $\mathbf{D}_J$ diagonal matrix

Proof:

Combine mapping property of A: $\|Av\|_{H^{-1}(\Omega)} \sim \|v\|_{H^1(\Omega)}$

with norm equivalence (NE) for $v \in V_J \subset H^1(\Omega)$ in wavelet coordinates $v = \mathbf{v}^T \Psi^J$

$\|v\|_{H^1(\Omega)} \sim \|\mathbf{D}_J \mathbf{v}\|_{\ell_2}$ and similarly for dual norm

Stiffness matrix in wavelet coordinates $\langle \Psi^J, A\Psi^J\rangle = \mathbf{T}_J^T\,\langle \Phi_J, A\Phi_J\rangle\,\mathbf{T}_J$

Elliptic PDE:

$$\begin{aligned} -\Delta u + u &= f \quad \text{in } \Omega \subset \mathbb{R}^d \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

	$-\Delta + 1$		$(-\Delta + 1)^{CK}$	
j	0	1	0	1
3	229	22.3	256	27.1
4	244	23.9	263	27.9
5	255	25.0	289	30.6
6	262	25.7	301	31.9
8	271	26.6	319	33.9
10	276	27.1	330	35.0
12	278	27.3	337	35.8

space dimension $d = 1$

	$-\Delta + 1$				$(-\Delta + 1)^{CK}$			
j	0	1	4	5	0	1	3	4
3	519	78.2	76.0	49.5	256	27.8	17.3	9.64
4	627	129	128	124	308	33.4	20.9	11.8
5	646	149	149	147	372	40.4	25.3	14.3
6	664	165	165	165	416	45.1	28.2	16.0
8	681	179	179	179	480	52.1	32.6	18.4

space dimension $d = 2$

	$-\Delta + 1$		$(-\Delta + 1)^{CK}$		
j	0	9	0	1	4
3	1103	269	256	28.5	18.3
4	1917	1913	520	57.8	37.1
5	2228	2222	557	62.0	39.8
6	2459	2443	572	63.6	40.9

space dimension $d = 3$

Uniformly bounded and absolutely small spectral condition numbers $\mathrm{cond}_2(\mathbf{A}_J)$ [Burstedde '05]
Additional preconditioning transformation on lowest level using singular value decomposition of $\mathbf{A}_{j_0}$: digit at head of each column indicates number of small eigenvalues shifted upward; number 0 corresponds to no additional preconditioning; exact diagonal $(\mathrm{diag}\,\mathbf{A}_J)^{-1}$
$\mathbf{A}_{j_0} := a(\Phi_{j_0}, \Phi_{j_0}) = \mathbf{U}\mathbf{S}\mathbf{U}^T$ with orthogonal $\mathbf{U}$ and diagonal $\mathbf{S}$ containing eigenvalues;
replace $\mathbf{S}$ by $\hat{\mathbf{S}}$ with smaller range of eigenvalues and replace $\mathbf{A}_{j_0}$ by $\hat{\mathbf{A}}_{j_0} := \mathbf{U}\hat{\mathbf{S}}\mathbf{U}^T$ (can be interpreted as transformation of generator basis) [Burstedde '05, Chapter 4.3.3]

Ingredients for Efficient Numerical Solution: Nested Iteration

Recall goal: realize discretization error accuracy $\varepsilon_J \sim h^2 \sim 2^{-2J}$ for grid with spacing $h \sim 2^{-J}$ with minimal amount of work $\mathcal{O}(N)$ $\quad N \sim 2^{Jd}$ unknowns

Theorem:
Starting with coarsest level j_0, solve $\mathbf{A}_j \mathbf{y}_j = \mathbf{f}_j$ on each level j up to discretization error accuracy ε_j and prolongate result from level j to next level $j+1$ as initial guess

$\rightsquigarrow$ Optimal preconditioner + nested iteration yields method of optimal complexity $\mathcal{O}(N_J)$ to reach discretization error accuracy on finest level J

Numerical results in Part III in context of control problems ...

Excursion: Wavelets for Image Processing

Wavelets in signal and image processing . . .

- Signal or image: explicitly given object described by N data points
- Goal: data compression without loosing essential information
- Method: single-(fine-)scale $\longleftrightarrow$ multi-scale representation of object
- Change of representation by Fast Wavelet Transform in $\mathcal{O}(N)$ operations (based on locally supported functions)

  $\rightsquigarrow$ Discard small coefficients in multi-scale representation

  $\rightsquigarrow$ Data compression
- Landmark: Daubechies' construction of $L_2(\mathbb{R})$ orthonormal wavelets with compact support [1988]

Image Compression — (Old) Examples

Original (768×768 pixels, 589.824 bytes) JPEG compression (12.9:1, 45.853 bytes)
Wavelet compression: JPEG 2000 (12.9:1, 45.621 bytes) [Brislawn, FBI, Los Alamos Laboratory, 1996]

Original (left), compression 100:1 (MT-WICE (Wavelet Based Image Compression), Mevis, right)

Compression 80:1 (MT-WICE left) JPEG (right)